CSE 373: Data Structures & Algorithms
Spanning Trees and Minimum Spanning Trees

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Winter 2017

Course Logistics

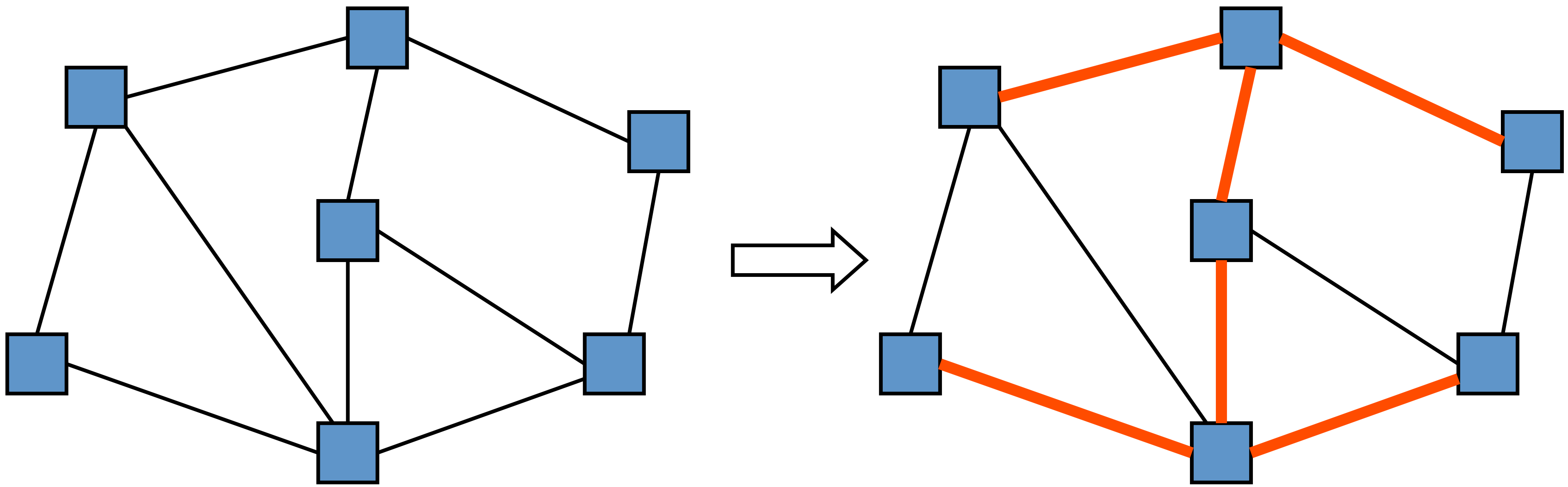
- HW4 due tonight
- HW5 out tomorrow (more graphs!)
  - coding: Dijkstra's shortest path algorithm
  - written: lots of practice with BFS, DFS, Topological Sort, and Spanning Trees (today!)

- Midterm regrades due by the end of this week

Problem Statement

Given a *connected* undirected graph **G**=(**V**,**E**), find a minimal subset of edges such that **G** is still connected

– A graph **G2**=(**V**,**E2**) such that **G2** is connected and removing any edge from **E2** makes **G2** disconnected



Observations

1. Problem not defined if original graph not connected. Therefore, we know **|E| >= |V|-1**

2. Any solution to this problem is a tree
   - Recall a tree does not need a root; just means acyclic
   - For any cycle, could remove an edge and still be connected

3. Solution not unique unless original graph was already a tree

4. A tree with **|V|** nodes has **|V|-1** edges
   - So every solution to the spanning tree problem has **|V|-1** edges

Motivation

A spanning tree connects all the nodes with as few edges as possible

In most compelling uses, we have a *weighted* undirected graph and we want a tree of least total cost

- Example: Electrical wiring for a house or clock wires on a chip
- Example: A road network if you cared about asphalt cost rather than travel time

This is the minimum spanning tree problem

  – Will do that next, after intuition from the simpler case

Two Approaches

Different algorithmic approaches to the spanning-tree problem:

1. Do a graph traversal (e.g., depth-first search, but any traversal will do), keeping track of edges that form a tree

2. Iterate through edges; add to output any edge that does not create a cycle

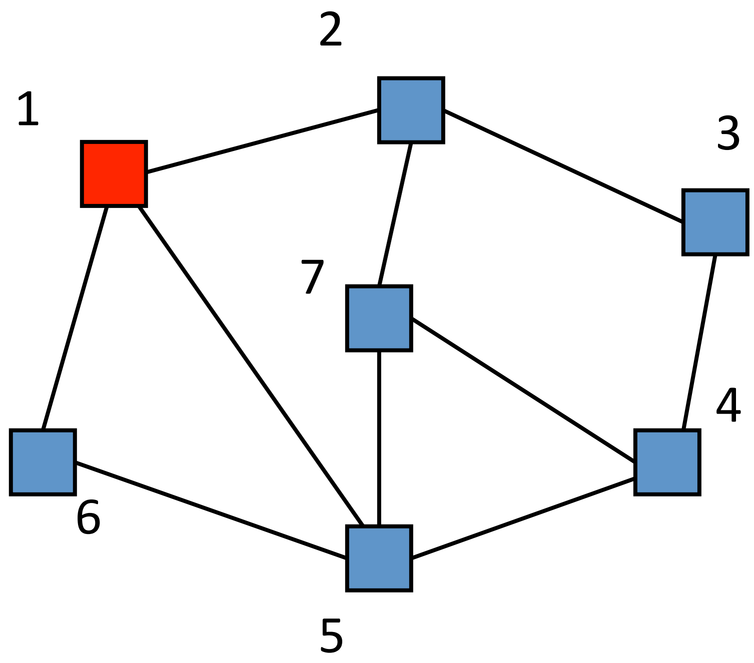
Spanning tree via DFS

```
spanning_tree(Graph G) {
  for each node v:
      v.marked = false
  dfs(someRandomStartNode)
}
dfs(Vertex a) {  // recursive DFS
  a.marked = true
  for each b adjacent to a:
    if(!b.marked) {
      add(a,b) to output
      dfs(b)
  }
}
```

Correctness: DFS reaches each node in connected graph.
We add one edge to connect it to the already visited nodes.
Order affects result, not correctness. Runtime: ***O*(|E|)**

Example

dfs(1)



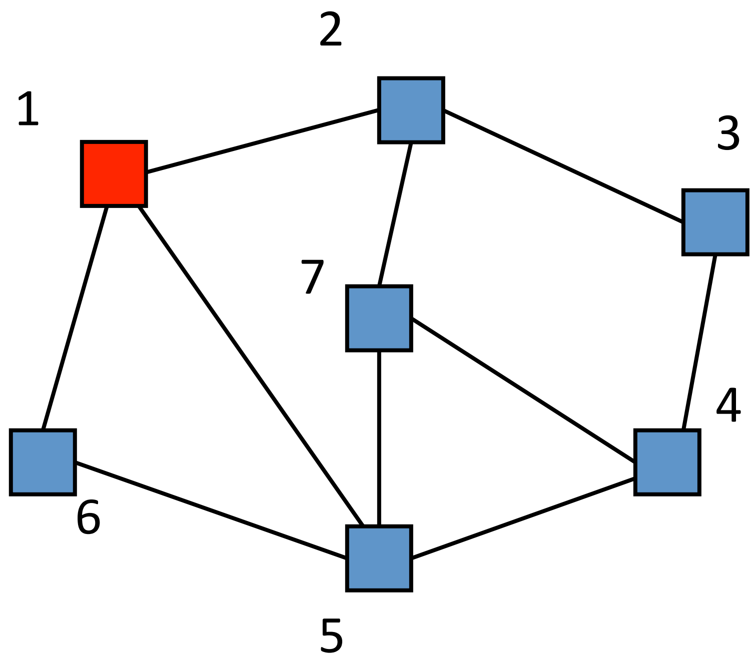
Output:

Example

dfs(1)

Pending Callstack:

- dfs(2)
- dfs(5)
- dfs(6)

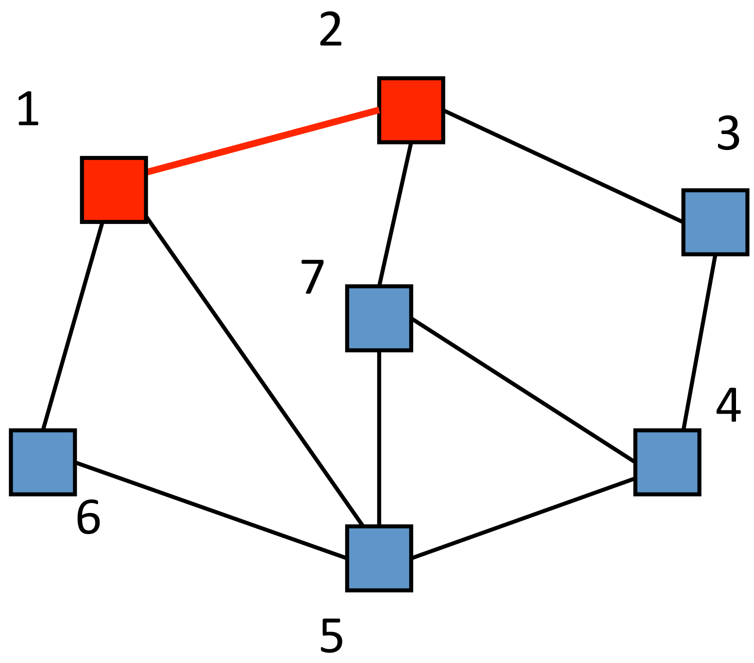


Output:

Example

dfs(2)

Pending Callstack:
- dfs(7)
- dfs(3)
- dfs(5)
- dfs(6)



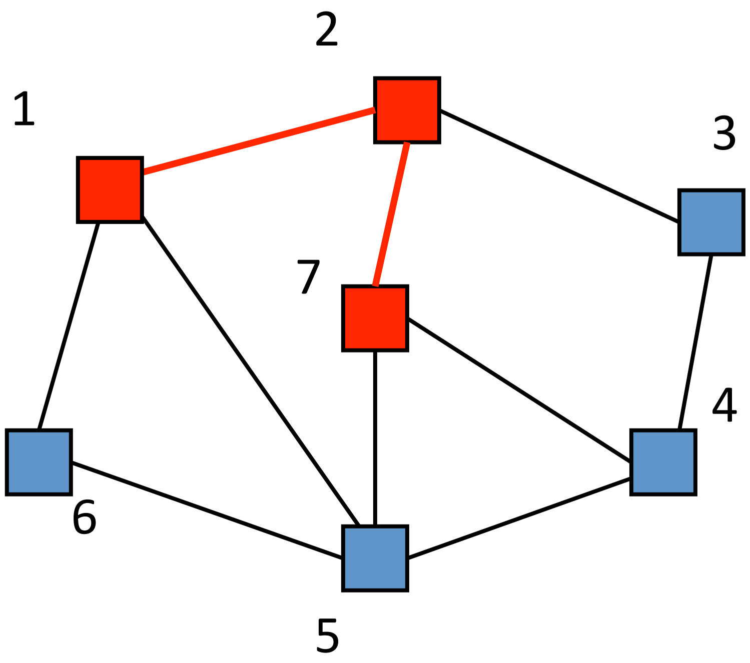
Output: (1,2)

Example

dfs(7)

Pending Callstack:

- dfs(5)
- dfs(4)
- dfs(3)
- ~~dfs(5)~~
- dfs(6)



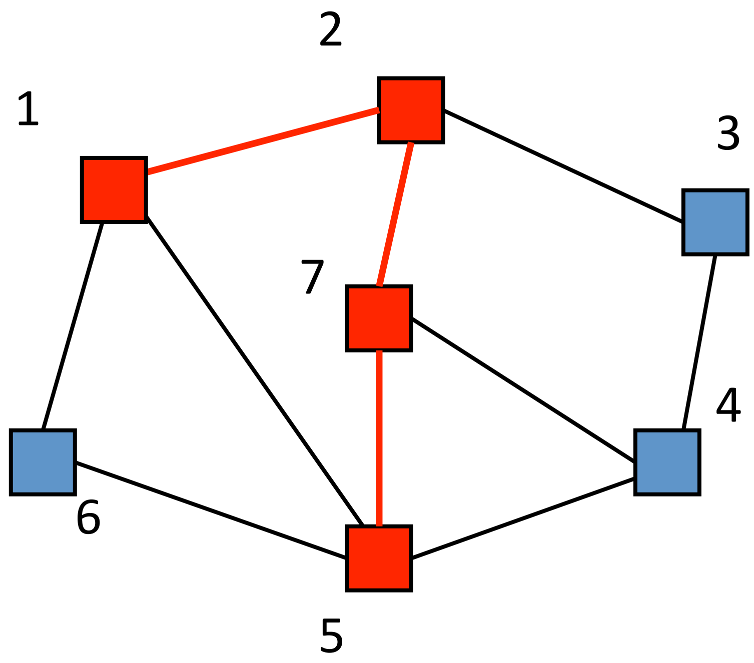
Output: (1,2), (2,7)

Example

dfs(5)

Pending Callstack:

- dfs(4)
- dfs(6)
- ~~dfs(4)~~
- dfs(3)
- ~~dfs(6)~~



Output: (1,2), (2,7), (7,5)

Example

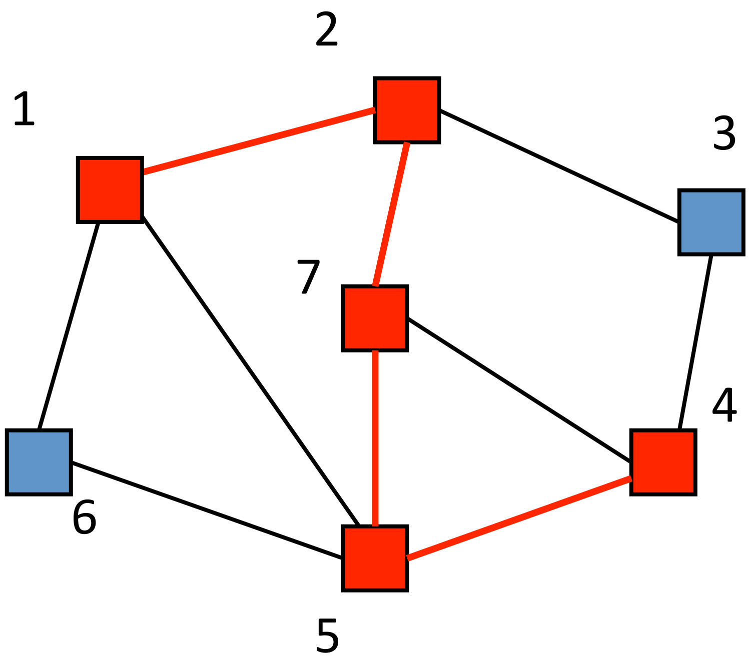
dfs(4)

Pending Callstack:

dfs(3)

dfs(6)

~~dfs(3)~~



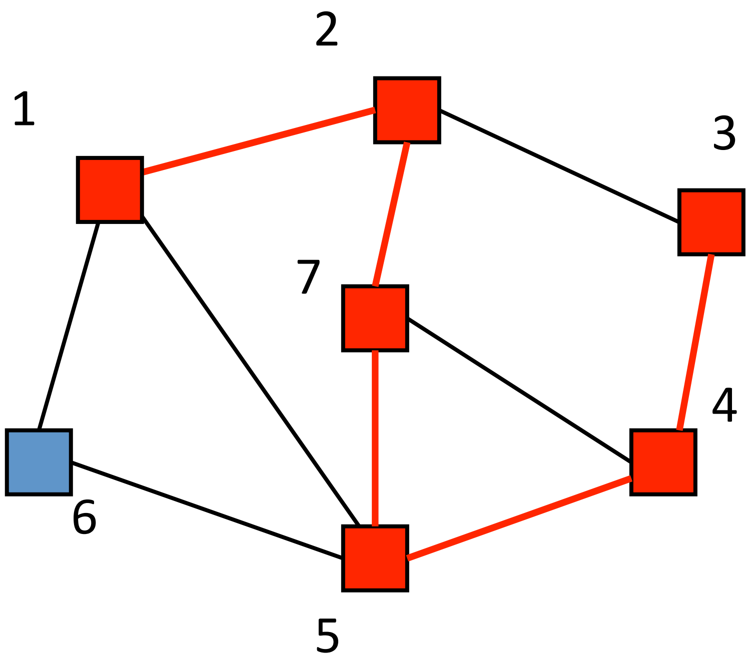
Output: (1,2), (2,7), (7,5), (5,4)

Example

dfs(3)

Pending Callstack:

dfs(6)



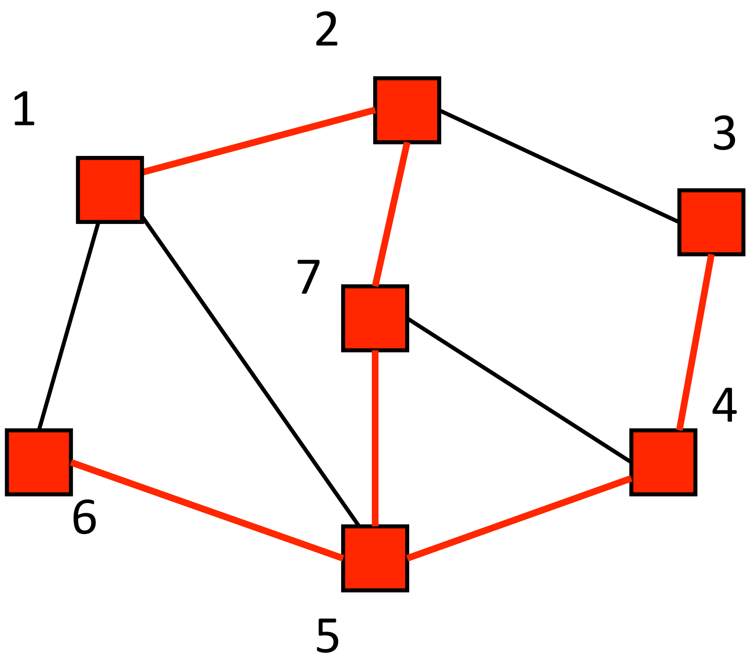
Output: (1,2), (2,7), (7,5), (5,4), (4,3)

Example

dfs(6)

Pending

Callstack:

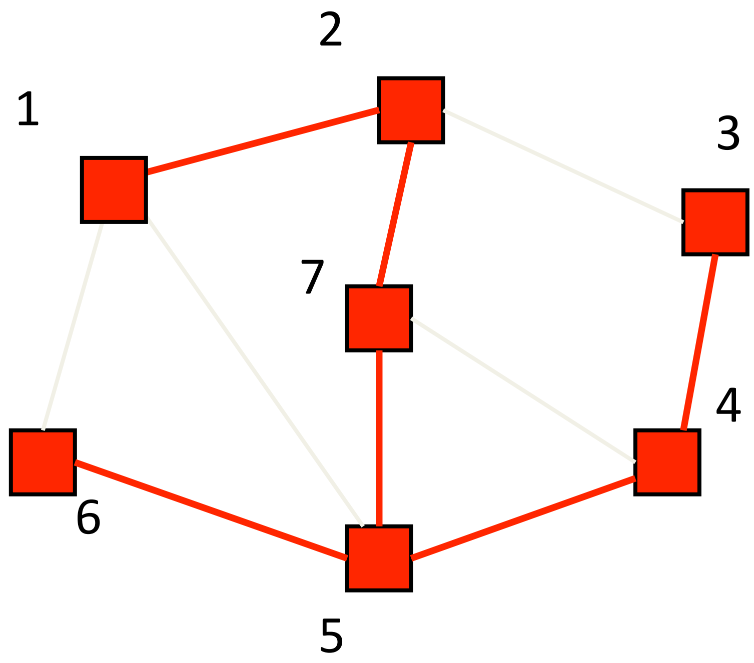


Output: (1,2), (2,7), (7,5), (5,4), (4,3), (5,6)

Example

Bubble up the recursive callstack.

Ignore each edge that would have been considered, but now is adjacent to a vertex already marked true.



Output: (1,2), (2,7), (7,5), (5,4), (4,3), (5,6)

Second Approach

Iterate through edges; output any edge that does not create a cycle

Correctness (hand-wavy):

- Goal is to build an acyclic connected graph
- When we add an edge, it adds a vertex to the tree
- The graph is connected, so we reach all vertices

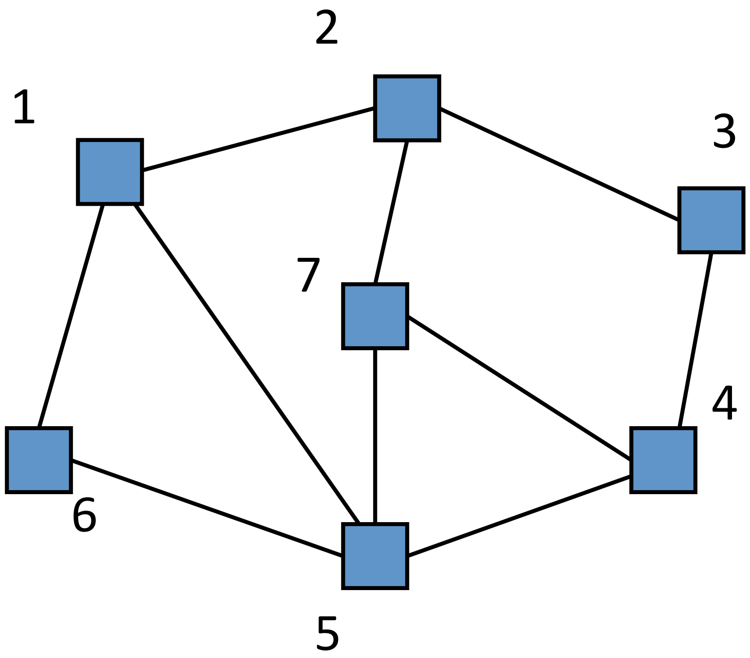
Efficiency:

- Depends on how quickly you can detect cycles
- Reconsider after the example

Example

Edges in some arbitrary order:

(1,2), (3,4), (5,6), (5,7),(1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

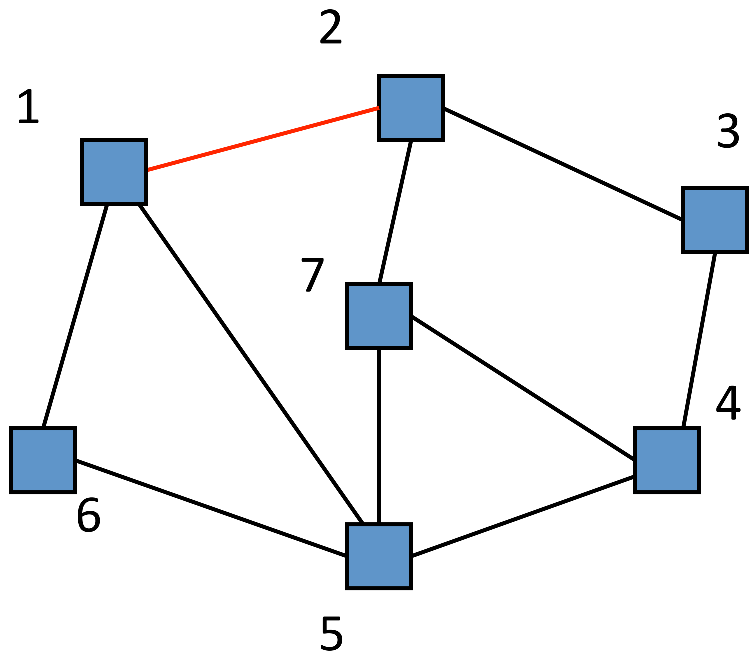


Output:

Example

Edges in some arbitrary order:

(1,2), (3,4), (5,6), (5,7),(1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

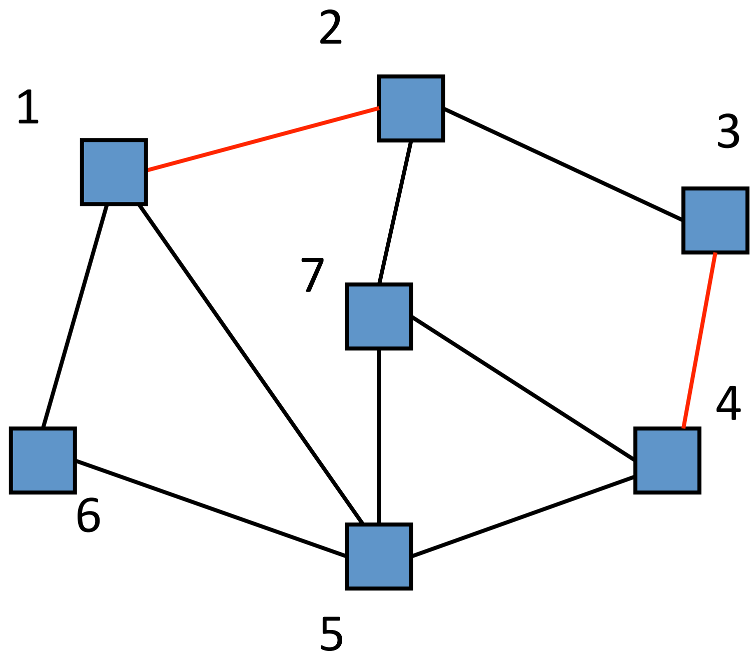


Output: (1,2)

Example

Edges in some arbitrary order:

(1,2), (3,4), (5,6), (5,7),(1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

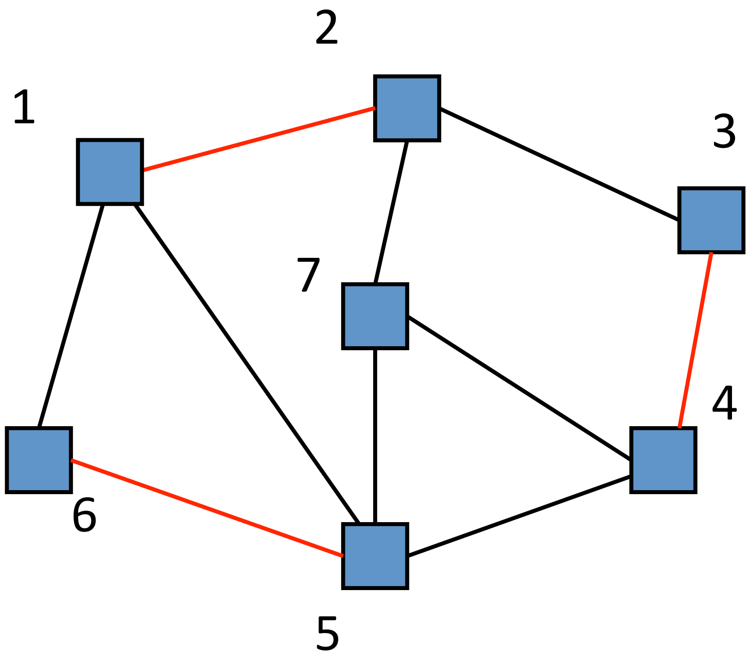


Output: (1,2), (3,4)

Example

Edges in some arbitrary order:

(1,2), (3,4), (5,6), (5,7),(1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

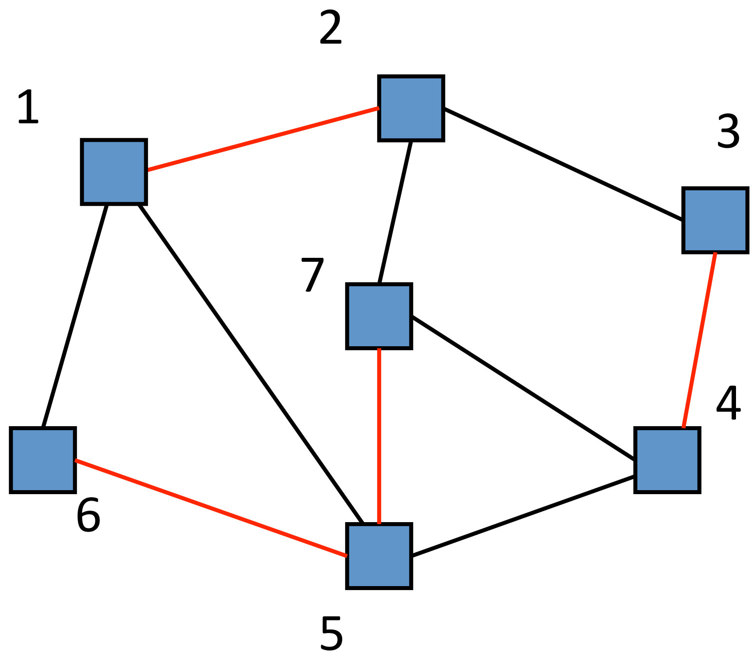


Output: (1,2), (3,4), (5,6),

Example

Edges in some arbitrary order:

(1,2), (3,4), (5,6), (5,7),(1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

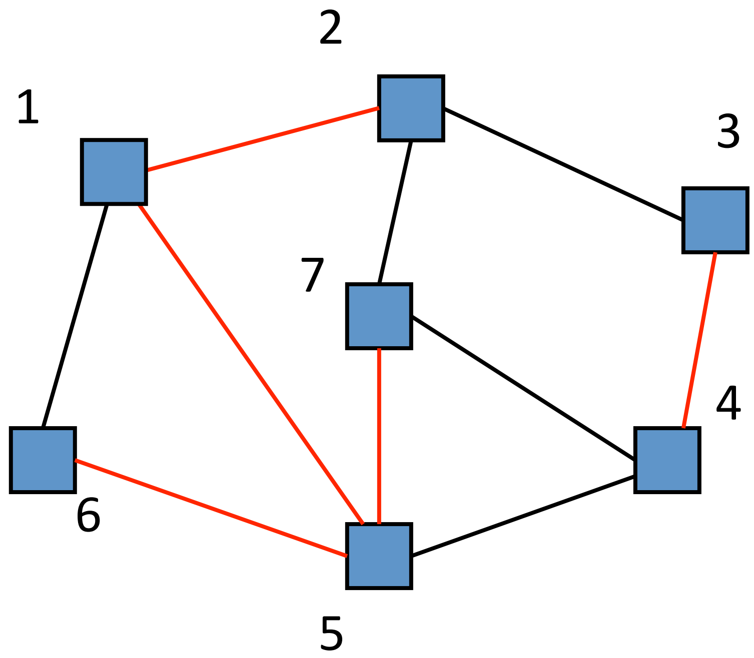


Output: (1,2), (3,4), (5,6), (5,7)

Example

Edges in some arbitrary order:

(1,2), (3,4), (5,6), (5,7), (1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

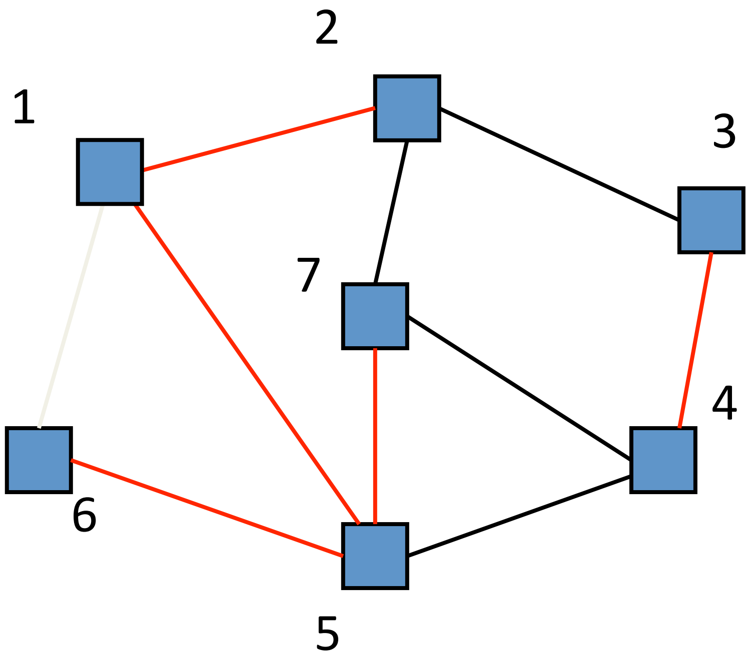


Output: (1,2), (3,4), (5,6), (5,7), (1,5)

Example

Edges in some arbitrary order:

(1,2), (3,4), (5,6), (5,7), (1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

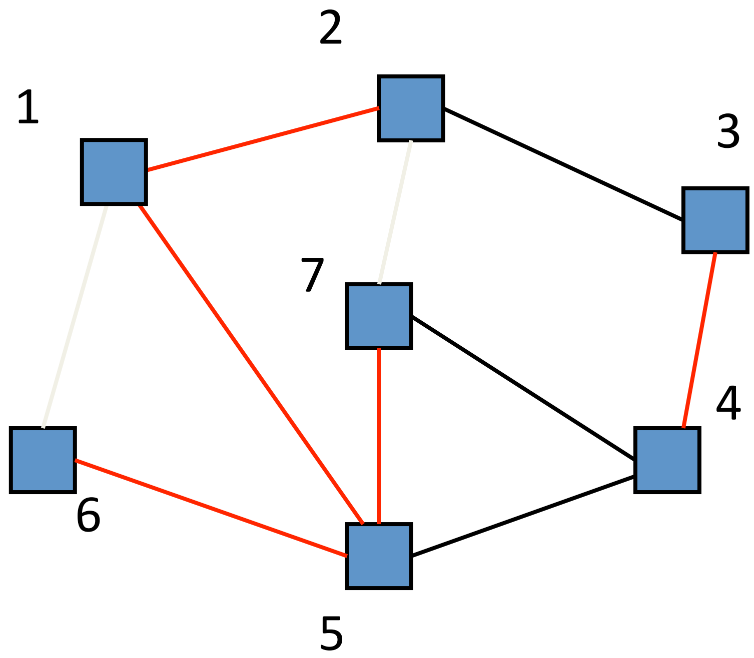


Output: (1,2), (3,4), (5,6), (5,7), (1,5)

Example

Edges in some arbitrary order:

(1,2), (3,4), (5,6), (5,7), (1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

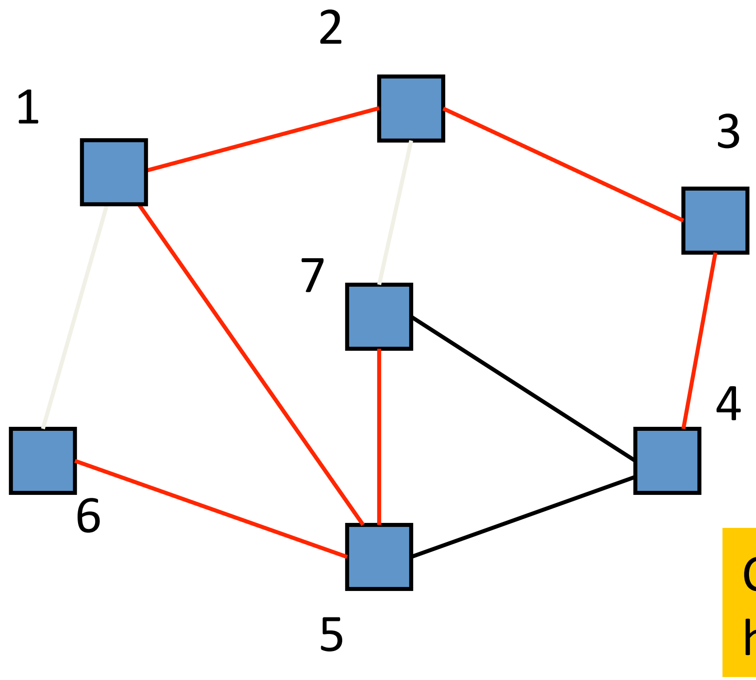


Output: (1,2), (3,4), (5,6), (5,7), (1,5)

Example

Edges in some arbitrary order:

(1,2), (3,4), (5,6), (5,7), (1,5), (1,6), (2,7), (2,3), (4,5), (4,7)



Can stop once we have |V|-1 edges

Output: (1,2), (3,4), (5,6), (5,7), (1,5), (2,3)

Cycle Detection

- To decide if an edge could form a cycle is *O*(**|V|**) because we may need to traverse all edges already in the output
- So overall algorithm would be *O*(**|V||E|**)
- But there is a faster way we know: use union-find!
  - Initially, each item is in its own 1-element set
  - Union sets when we add an edge that connects them
  - Stop when we have one set

Using Disjoint-Set

Can use a disjoint-set implementation in our spanning-tree algorithm to detect cycles:

Invariant: **`u`** and **`v`** are connected in output-so-far
iff
`u` and **`v`** in the same set

- Initially, each node is in its own set
- When processing edge **`(u,v)`**:
  - If **`find(u)`** equals **`find(v)`**, then do not add the edge
  - Else add the edge and **`union(find(u),find(v))`**
  - $O(|\mathbf{E}|)$ operations that are almost $O(1)$ amortized

Summary So Far

The spanning-tree problem

- Add nodes to partial tree approach is *O*(**|E|**)
- Add acyclic edges approach is *almost* *O*(**|E|**)
  - Using union-find

But really want to solve the minimum-spanning-tree problem

- Given a weighted undirected graph, give a spanning tree of minimum weight
- Same two approaches will work with minor modifications
- Both will be *O*(**|E|log|V|**)

MST: Getting to the Point

Algorithm #1: Prim's Algorithm

Find Minimum Spanning Trees like Dijkstra's Algorithm finds Shortest-Path.

- Both based on expanding cloud of known vertices, basically using a priority queue instead of a DFS stack

Algorithm #2: Kruskal's Algorithm

finds Minimum Spanning Trees exactly like our 2nd greedy approach to spanning tree, but process edges in cost order instead of random order

Prim's Algorithm Idea

Idea: Grow a tree by adding an edge from the "known" vertices to the "unknown" vertices. *Pick the edge with the smallest weight that connects "known" to "unknown."*

Recall Dijkstra "picked edge with closest known distance to source"

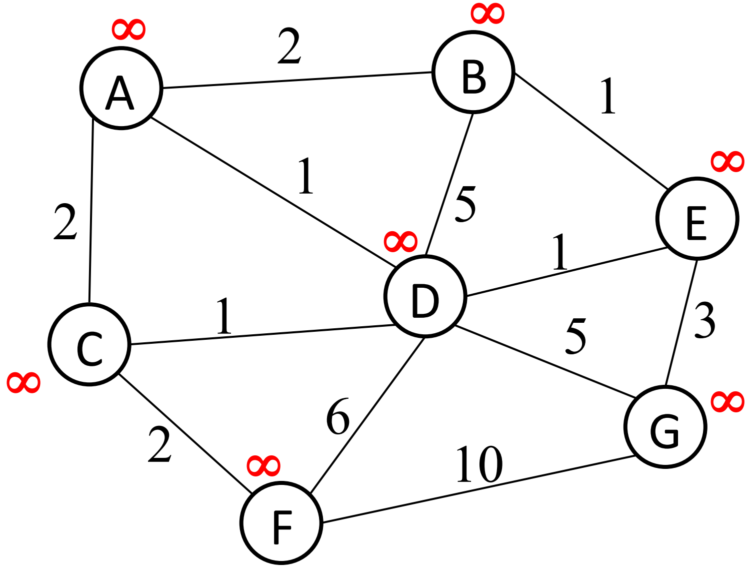
- That is not what we want here
- Otherwise identical (!)

The Algorithm

1. For each node **`v`**, set **`v.cost = ∞`** and **`v.known = false`**
2. Choose any node **`v`**
   a) Mark **`v`** as known
   b) For each edge **`(v,u)`** with weight **`w`**, set **`u.cost=w`** and **`u.prev=v`**
3. While there are unknown nodes in the graph
   a) Select the unknown node **`v`** with lowest cost
   b) Mark **`v`** as known and add **`(v, v.prev)`** to output
   c) For each edge **`(v,u)`** with weight **`w`**,

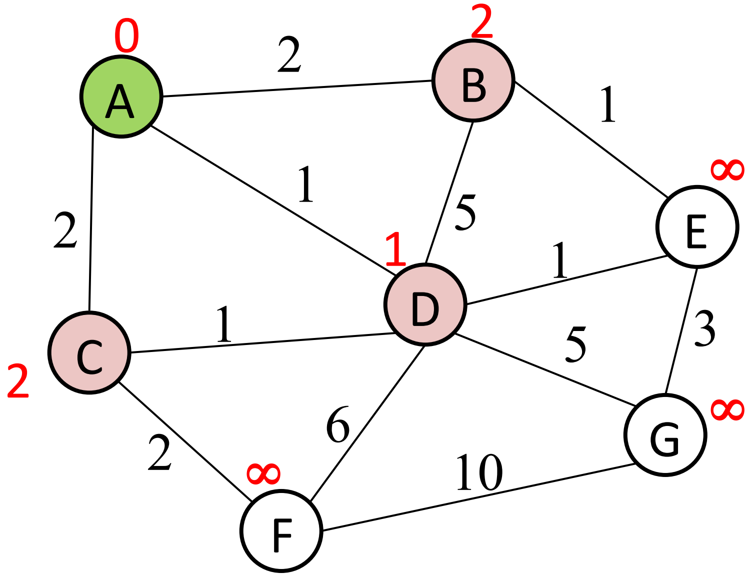
```
if(w < u.cost) {
  u.cost = w;
   u.prev = v;
}
```

Example



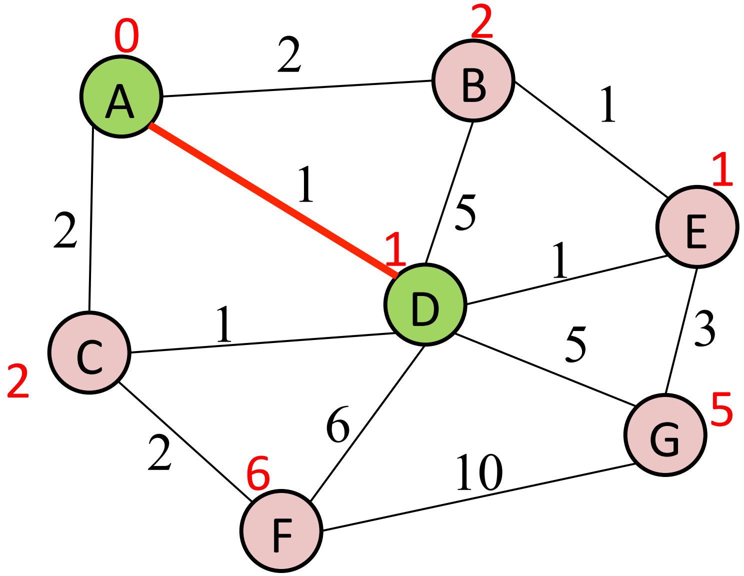
vertex	known?	cost	prev
A		∞	
B		∞	
C		∞	
D		∞	
E		∞	
F		∞	
G		∞	

Example



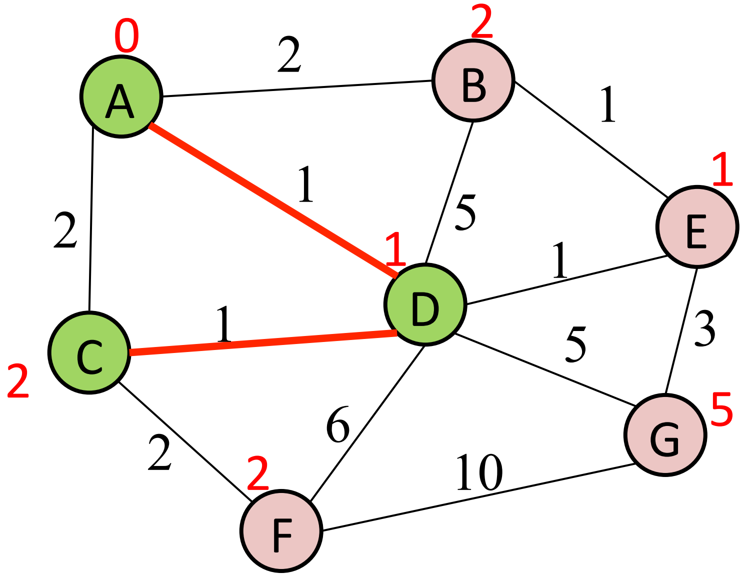
vertex	known?	cost	prev
A	Y	0	
B		2	A
C		2	A
D		1	A
E		∞	
F		∞	
G		∞	

Example



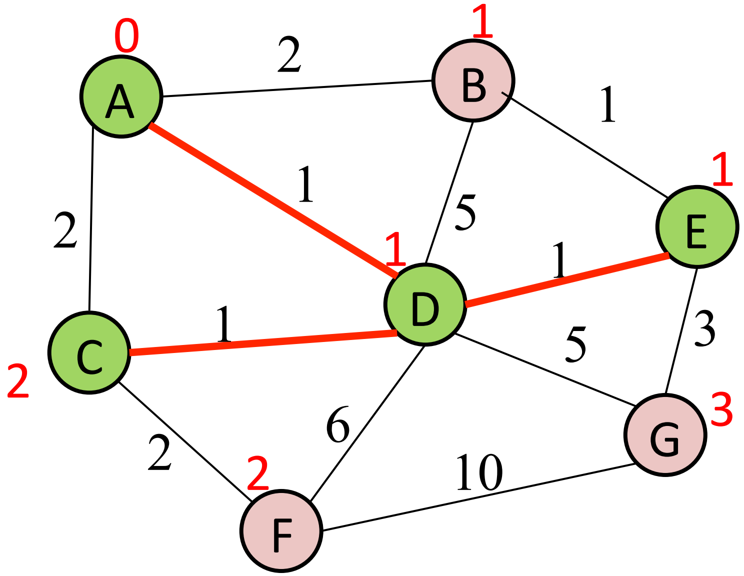
vertex	known?	cost	prev
A	Y	0	
B		2	A
C		1	D
D	Y	1	A
E		1	D
F		6	D
G		5	D

Example



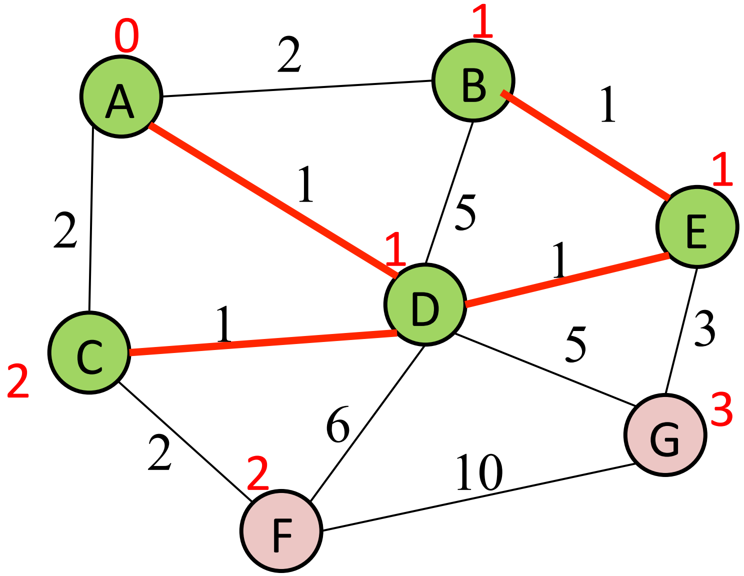
vertex	known?	cost	prev
A	Y	0	
B		2	A
C	Y	1	D
D	Y	1	A
E		1	D
F		2	C
G		5	D

Example



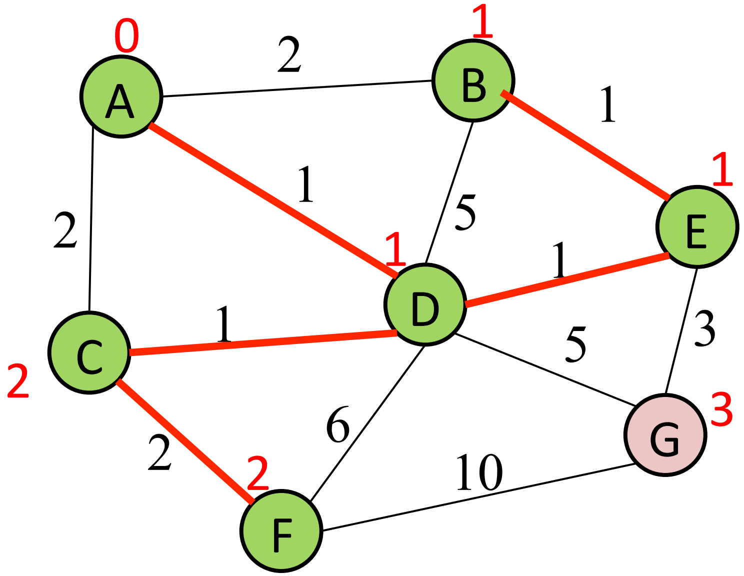
vertex	known?	cost	prev
A	Y	0	
B		1	E
C	Y	1	D
D	Y	1	A
E	Y	1	D
F		2	C
G		3	E

Example



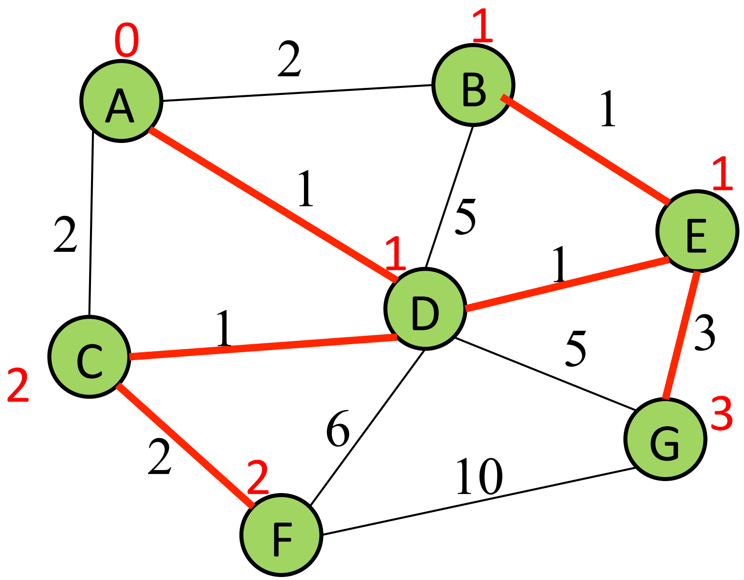
vertex	known?	cost	prev
A	Y	0	
B	Y	1	E
C	Y	1	D
D	Y	1	A
E	Y	1	D
F		2	C
G		3	E

Example



vertex	known?	cost	prev
A	Y	0	
B	Y	1	E
C	Y	1	D
D	Y	1	A
E	Y	1	D
F	Y	2	C
G		3	E

Example



vertex	known?	cost	prev
A	Y	0	
B	Y	1	E
C	Y	1	D
D	Y	1	A
E	Y	1	D
F	Y	2	C
G	Y	3	E

Prim's Analysis

- Correctness
  - A bit tricky: Intuitively similar to Dijkstra
  - Proof by contradiction. If there is an edge that is smaller connecting unknown node v to the known tree, we would have found it from the known cloud or we would be choosing it (true at every step/node v).

- Run-time
  - Same as Dijkstra
  - *O*(`|V| log|V| + |E| log|V|`) using a priority queue
    - Costs/priorities are just edge-costs, not path-costs

Kruskal's Algorithm

Idea: Grow a forest out of edges that do not grow a cycle, just like for the spanning tree problem.

- But now consider the edges in order by weight

Runtime (using sorting):

- Sort edges: O(**|E|log |E|**) (sorting is next course topic)
- Iterate through edges using union-find for cycle detection almost O(**|E|**)

Somewhat better (using a priority queue):

- Floyd's algorithm to build min-heap with edges O(**|E|**)
- Iterate through edges, using union-find for cycle detection and **`deleteMin`** to get next edge O(**|E|log|E|**)
- Not better *worst-case* asymptotically, but often stop long before considering all edges and the up front cost is cheaper

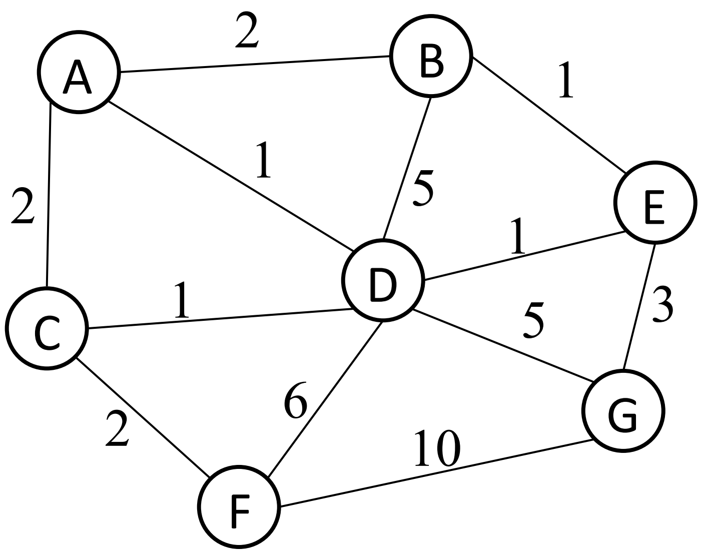
Pseudocode

1. Sort edges by weight (better: put in min-heap)
2. Each node in its own set
3. While output size **< |V|-1**
   - Consider next smallest edge **`(u,v)`**
   - if **`find(u)`** and **`find(v)`** indicates **`u`** and **`v`** are in different sets
     - output **`(u,v)`**
     - **`union(find(u),find(v))`**

Recall invariant:

`u` and **`v`** in same set iff connected in output-so-far

Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

5: (D,G), (B,D)

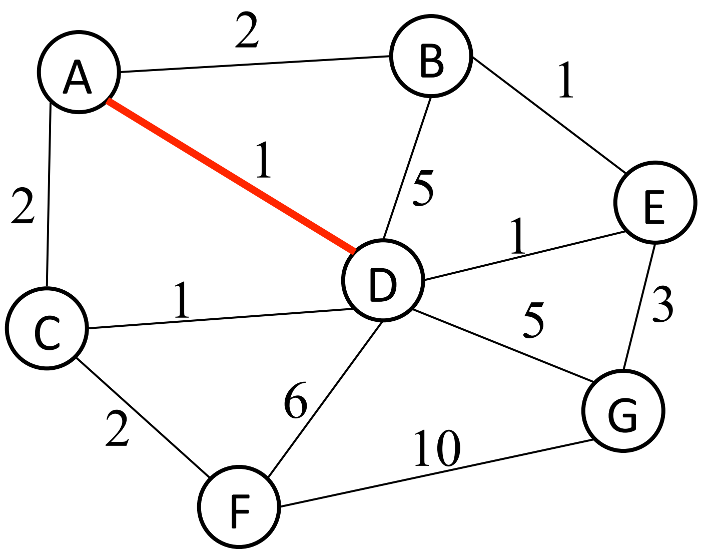
6: (D,F)

10: (F,G)

Output:

Note: At each step, the union/find sets are the trees in the forest

Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

5: (D,G), (B,D)

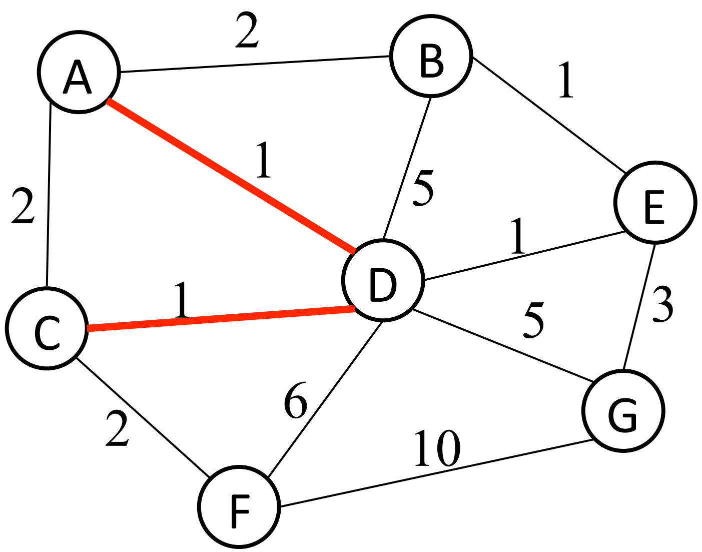
6: (D,F)

10: (F,G)

Output: (A,D)

Note: At each step, the union/find sets are the trees in the forest

Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

5: (D,G), (B,D)

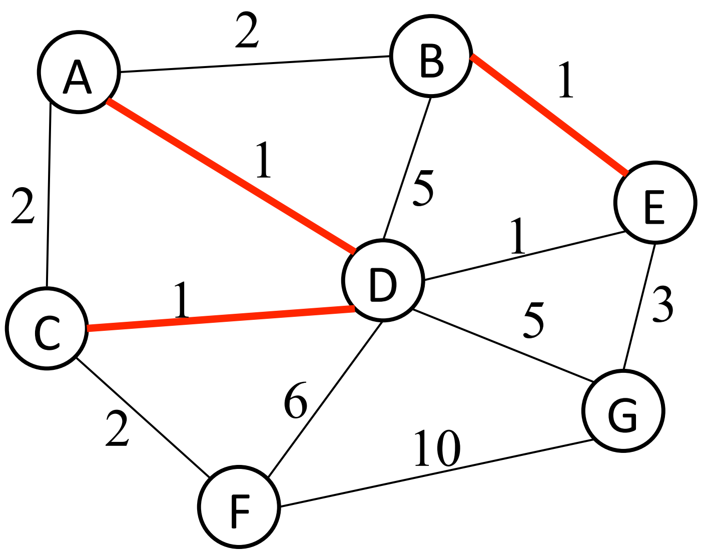
6: (D,F)

10: (F,G)

Output: (A,D), (C,D)

Note: At each step, the union/find sets are the trees in the forest

Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

5: (D,G), (B,D)

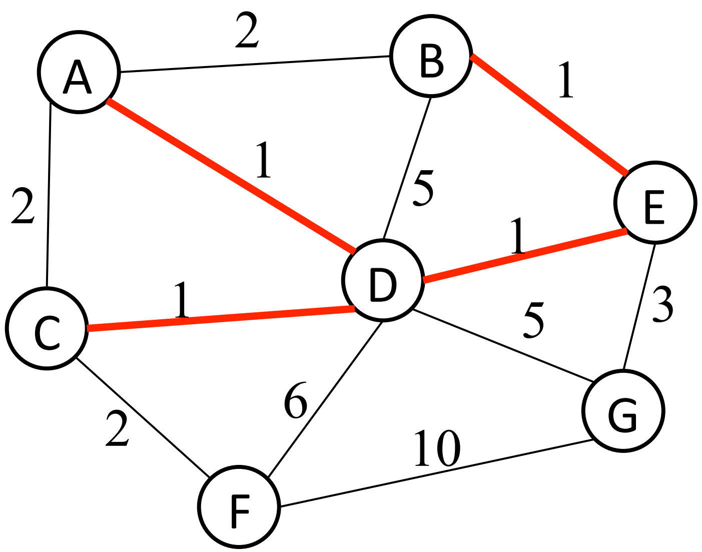
6: (D,F)

10: (F,G)

Output: (A,D), (C,D), (B,E)

Note: At each step, the union/find sets are the trees in the forest

Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

5: (D,G), (B,D)

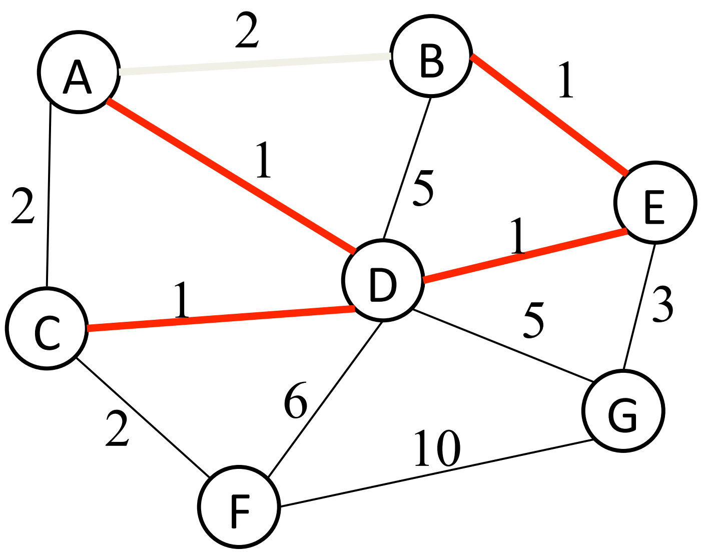
6: (D,F)

10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E)

Note: At each step, the union/find sets are the trees in the forest

Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

5: (D,G), (B,D)

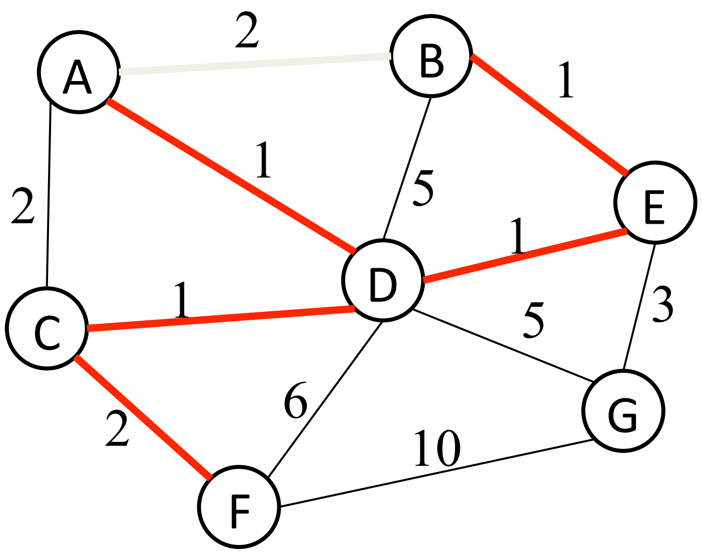
6: (D,F)

10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E)

Note: At each step, the union/find sets are the trees in the forest

Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

5: (D,G), (B,D)

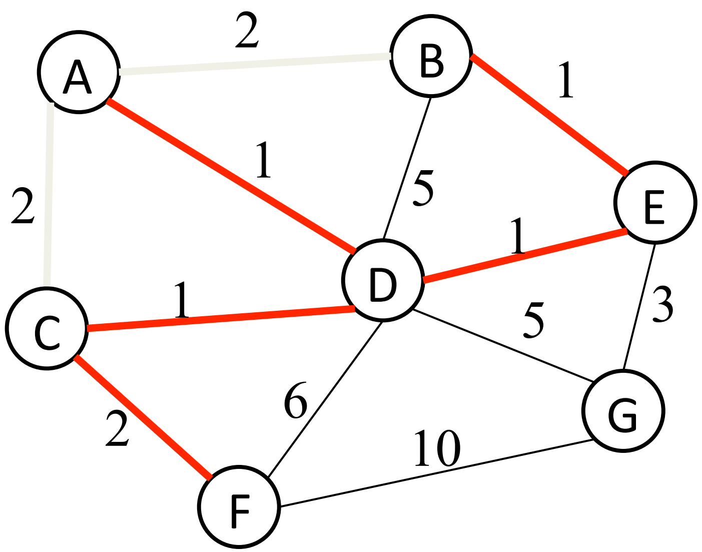
6: (D,F)

10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E), (C,F)

Note: At each step, the union/find sets are the trees in the forest

Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

5: (D,G), (B,D)

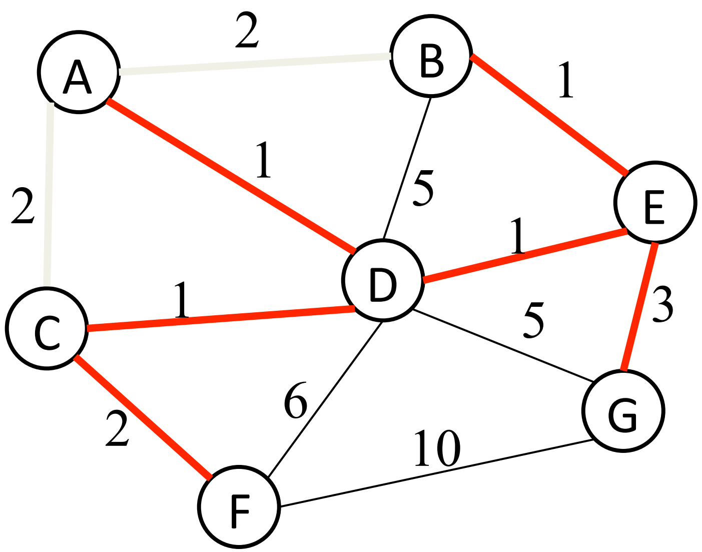
6: (D,F)

10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E), (C,F)

Note: At each step, the union/find sets are the trees in the forest

Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

5: (D,G), (B,D)

6: (D,F)

10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E), (C,F), (E,G)

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Algorithm: Correctness

It clearly generates a spanning tree. Call it T_K.

Suppose T_K is *not* minimum:

- Pick another spanning tree T_{min} with *lower cost* than T_K
- Pick the smallest edge $e_1=(u,v)$ in T_K that is not in T_{min}
- T_{min} already has a path p in T_{min} from u to v
  - ⇒ Adding e_1 to T_{min} will create a cycle in T_{min}
- Pick an edge e_2 in p that Kruskal's algorithm considered *after* adding e_1 (must exist: u and v unconnected when e_1 considered)
  - ⇒ $\text{cost}(e_2) \geq \text{cost}(e_1)$
  - ⇒ can replace e_2 with e_1 in T_{min} without increasing cost!
- Keep doing this until T_{min} is identical to T_K
  - ⇒ T_K must also be minimal – contradiction!

Today's Takeaways

- Understand Spanning Trees and some greedy algorithms (graph traversal + disjoint sets) for finding them

- Understand Minimum Spanning Trees, and the two main algorithms for finding them:
  – Prim's: like Dijkstra's, but pick the least cost edge
  – Kruskal's: