CSE 373: Data Structures and Algorithms

Lecture 20: More Sorting

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Today: More sorting algorithms!

- Merge sort analysis
- Quicksort
- Bucket sort
- Radix sort

Divide and conquer

Very important technique in algorithm design

1. Divide problem into smaller parts

2. Independently solve the simpler parts
    - Think recursion
    - Or parallelism

3. Combine solution of parts to produce overall solution

Two great sorting methods are fundamentally divide-and-conquer
(Merge Sort & Quicksort)

Merge Sort

Merge Sort: repeatedly…

- Sort the left half of the elements
- Sort the right half of the elements
- Merge the two sorted halves into a sorted whole

To sort array from position `lo` to position `hi`:

- If range is 1 element long, it is already sorted!
- Else:
  - Sort from `lo` to `(hi+lo)/2`
  - Sort from `(hi+lo)/2` to `hi`
  - Merge the two halves together

Linked lists and big data

We defined sorting over an array, but sometimes you want to sort linked lists

One approach:

- Convert to array:
- Sort:
- Convert back to list:

Merge sort works very nicely on linked lists directly

- Heapsort and quicksort do not
- Insertion sort and selection sort do but they're slower

Merge sort is also the sort of choice for external sorting

- Linear merges minimize disk accesses
- And can leverage multiple disks to get streaming accesses

Analysis

Having defined an algorithm and argued it is correct, we should analyze its running time and space:

To sort *n* elements, we:

- Return immediately if *n*=1
- Else do 2 subproblems of size and then an merge

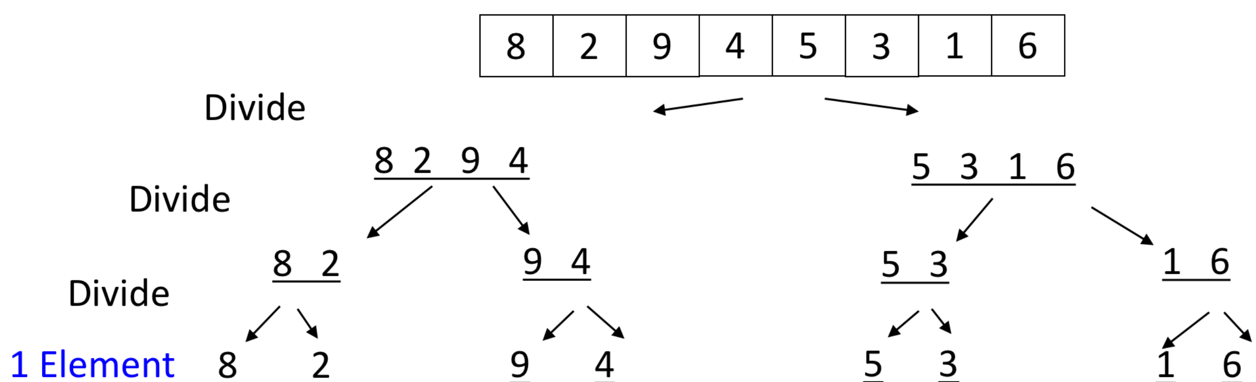
Recurrence relation:

Analysis intuitively

This recurrence is common, you just "know" it's $O(n \log n)$

Merge sort is relatively easy to intuit (best, worst, and average):

- The recursion "tree" will have height
- At each level we do a *total* amount of merging equal to



Analysis more formally

(One of the recurrence classics)

For simplicity, ignore constants (let constants be)

$T(1) = 1$

$T(n) = 2T(n/2) + n$

$= 2(2T(n/4) + n/2) + n$

$= 4T(n/4) + 2n$

$= 4(2T(n/8) + n/4) + 2n$

$= 8T(n/8) + 3n$

....

$= 2^k T(n/2^k) + kn$

We will continue to recurse until we reach the base case, i.e. T(1) for T(1), $n/2^k = 1$, i.e., $\log n = k$

So the total amount of work is $2^k T(n/2^k) + kn = 2^{\log n} T(1) + n \log n = n + n \log n$ $= O(n \log n)$

Divide-and-Conquer Sorting

Two great sorting methods are fundamentally divide-and-conquer

1. Merge Sort:
    - Sort the left half of the elements (recursively)
    - Sort the right half of the elements (recursively)
    - Merge the two sorted halves into a sorted whole

2. Quicksort:
    - Pick a “pivot” element
    - Divide elements into “less-than pivot” and “greater-than pivot”
    - Sort the two divisions (recursively on each)
    - Answer is “sorted-less-than”, followed by “pivot”, followed by ”sorted-greater-than”

Quicksort Overview

1. Pick a pivot element

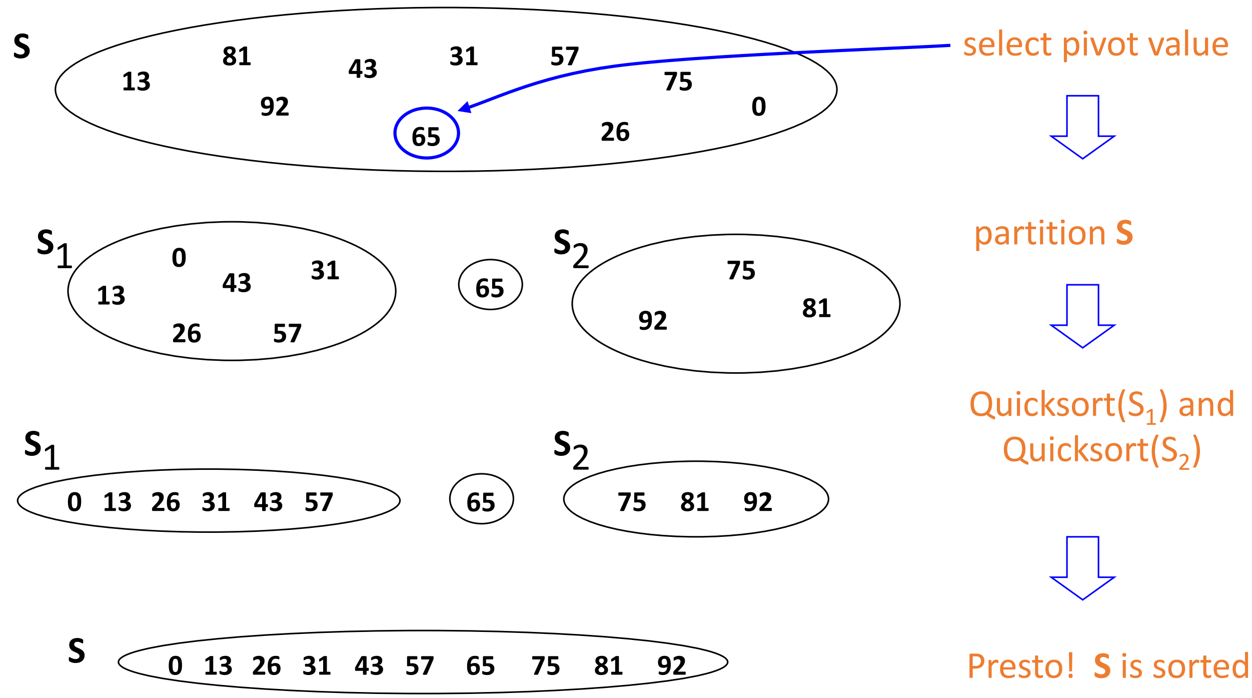
2. Partition all the data into:
   A. The elements less than the pivot
   B. The pivot
   C. The elements greater than the pivot

3. Recursively sort A and C

4. The final answer is A-B-C

(space for notes/scratch)

Think in Terms of Sets



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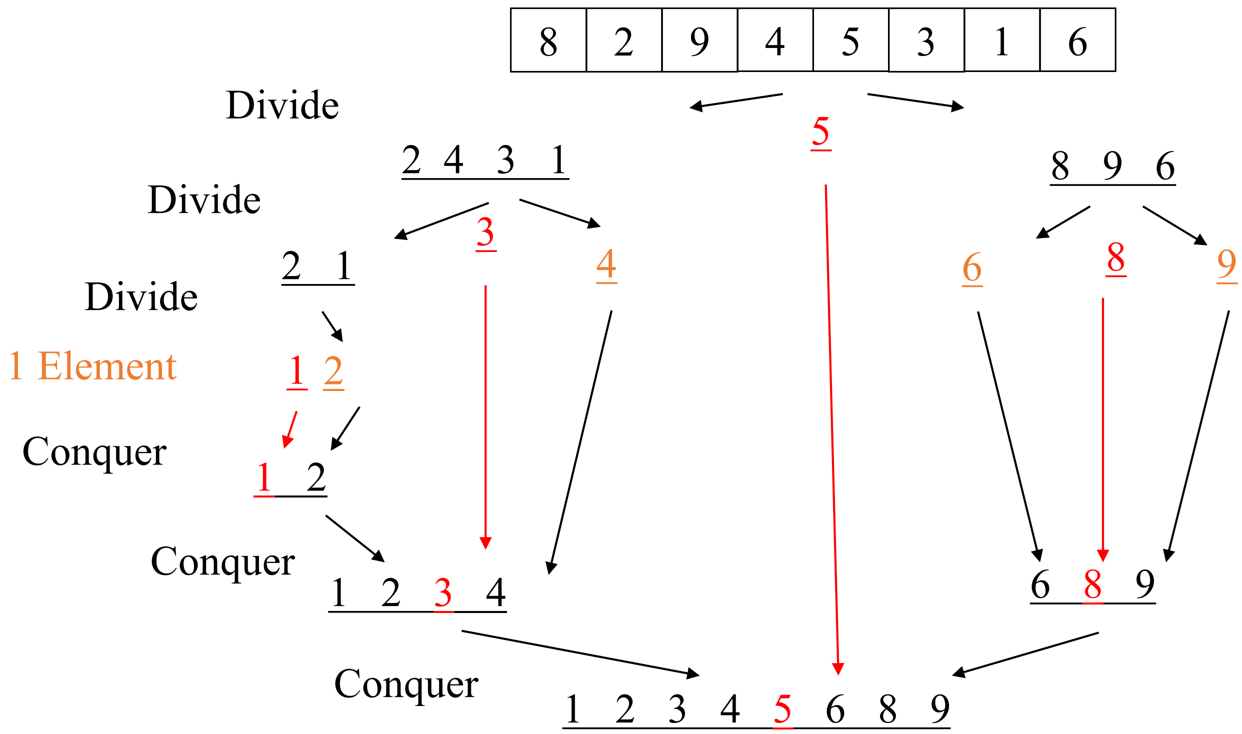
Example, Showing Recursion



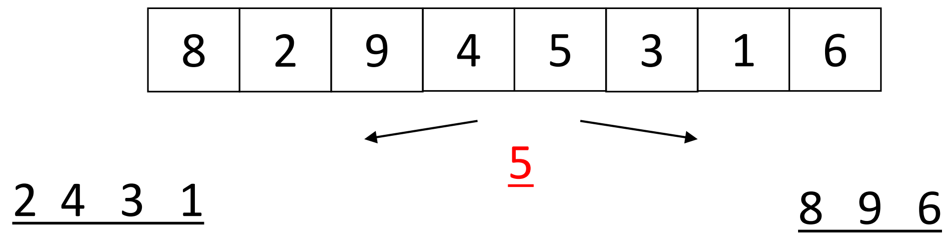
Details

Have not yet explained:

- How to pick the pivot element
  - Any choice is correct: data will end up sorted
  - But as analysis will show, want the two partitions to be about
- How to implement partitioning
  - In linear time
  - In place

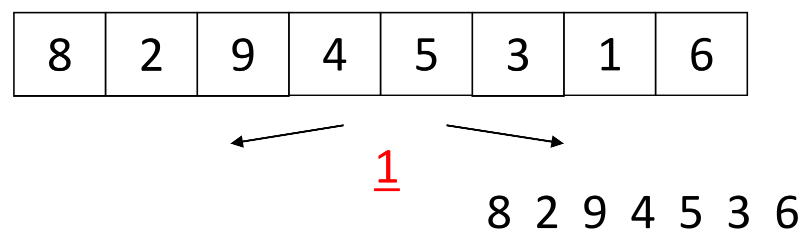
Pivots

- Best pivot?



    - Halve each time

- Worst pivot?
    - Greatest/least element
    - Partition of size n - 1



Potential pivot rules

While sorting `arr` from `lo` to `hi-1` …

- Pick `arr[lo]` or `arr[hi-1]`
  - Fast, but worst-case occurs with mostly sorted input
- Pick random element in the range
  - Does as well as any technique, but (pseudo)random number generation can be slow
  - Still probably the most elegant approach
- Median of 3, e.g., `arr[lo], arr[hi-1], arr[(hi+lo)/2]`
  - Common heuristic that tends to work well

Partitioning

Conceptually simple, but hardest part to code up correctly

- After picking pivot, need to partition in linear time in place

One approach (there are slightly fancier ones):

1. Swap pivot with `arr[lo]`
2. Use two fingers `i` and `j`, starting at `lo+1` and `hi-1`
3. `while (i < j)`

```
    if (arr[j] > pivot) j--
    else if (arr[i] < pivot) i++
    else swap arr[i] with arr[j]
```

4. Swap pivot with `arr[i]` *

*skip step 4 if pivot ends up being least element

Example

- Step one: pick pivot as median of 3
  - **`lo`** = 0, **`hi`** = 10

0	1	2	3	4	5	6	7	8	9
8	1	4	9	**0**	3	5	2	7	**6**

- Step two: move pivot to the `lo` position

0	1	2	3	4	5	6	7	8	9
6	1	4	9	**0**	3	5	2	7	**8**

Example

Often have more than one swap during partition – this is a short example

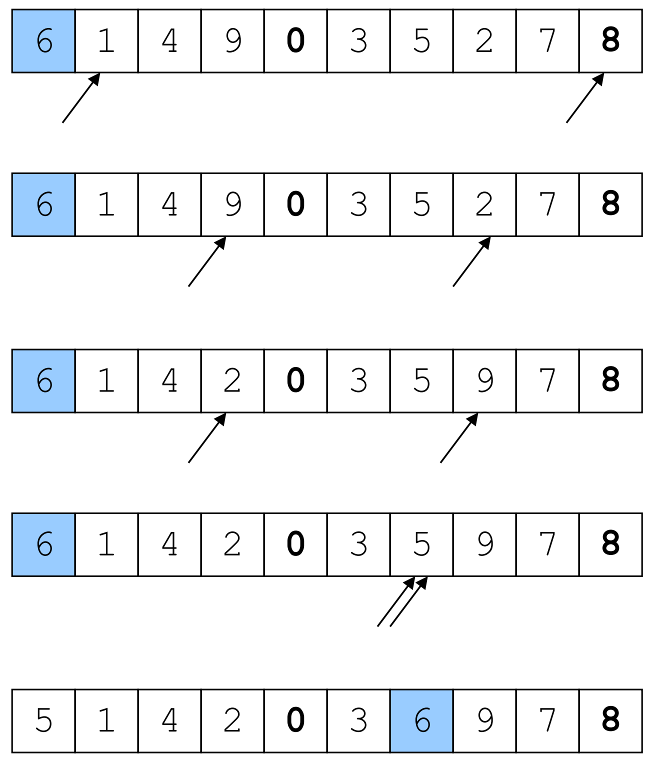
Now partition in place

Move fingers

Swap

Move fingers

Move pivot



Analysis

- Best-case: Pivot is always the median

  T(0) = T(1) = 1

  T(*n*) = -- linear-time partition

  Same recurrence as merge sort:

- Worst-case: Pivot is always smallest or largest element

  T(0) = T(1) = 1

  T(*n*) =

  Basically same recurrence as selection sort:

- Average-case (e.g., with random pivot)
  - O(*n* `log` *n*), not responsible for proof (in text)

Cutoffs

- For small n, all that recursion tends to cost more than doing a quadratic sort
  - Remember asymptotic complexity is for

- Common engineering technique: switch algorithm below a **cutoff**
  - Reasonable rule of thumb: use insertion sort for $n < 10$

- Notes:
  - Could also use a cutoff for merge sort
  - Cutoffs are also the norm with parallel algorithms
    - Switch to sequential algorithm
  - None of this affects asymptotic complexity

Cutoff pseudocode

```
void quicksort(int[] arr, int lo, int hi)
{
  if(hi – lo < CUTOFF)
     insertionSort(arr,lo,hi);
  else
     …
}
```

Notice how this cuts out the vast majority of the recursive calls

- Think of the recursive calls to quicksort as a tree
- Trims out the bottom layers of the tree

Practice with comparison sort!

A comparison sorting algorithm is operating on an array of 8 integers. After its 4^{th} loop or recursive call, the array looks like:

4	8	11	15	42	29	18	37

Which of these sorting algorithms can it be?

A) Heapsort

B) Merge sort

C) Insertion sort

D) Quicksort using Median of 3

(space for notes/scratch)

How Fast Can We Sort?

- Heapsort & mergesort have $O(n \log n)$ worst-case running time
- Quicksort has $O(n \log n)$ average-case running time
- These bounds are all tight, actually $\Theta(n \log n)$
- Comparison sorting in general is $\Omega(n \log n)$
  - An amazing computer-science result: proves all the clever programming in the world cannot comparison-sort in linear time

The Big Picture

Surprising amount of juicy computer science: 2-3 lectures…

Simple algorithms: $O(n^2)$	Fancier algorithms: $O(n \log n)$	Comparison lower bound: $\Omega(n \log n)$	Specialized algorithms: $O(n)$	Handling huge data sets
Insertion sort Selection sort Shell sort …	Heap sort Merge sort Quick sort (avg) …		Bucket sort Radix sort	External sorting

How???

- Change the model – assume more than "compare(a,b)"

Bucket Sort (a.k.a. BinSort)

- If all values to be sorted are *known* to be integers between 1 and *K* (or any small range):
  - Create an array of size *K*
  - Put each element in its proper bucket (a.k.a. bin)
  - *If* data is only integers, no need to store more than a *count* of how times that bucket has been used
- Output result via linear pass through array of buckets

`count` array	
1	
2	
3	
4	
5	

- Example:

  K=5

  input (5, 1, 3, 4, 3, 2, 1, 1, 5, 4, 5)

  output

Analyzing Bucket Sort

- Overall: $O(n+K)$
  - Linear in n, but also linear in K
  - $\Omega(n \log n)$ lower bound does not apply because this is not a comparison sort
- Good when K is smaller (or not much larger) than n
  - We don't spend time doing comparisons of duplicates
- Bad when K is much larger than n
  - Wasted space; wasted time during linear $O(K)$ pass
- For data in addition to integer keys, use list at each bucket

Bucket Sort with Data

- Most real lists aren’t just keys; we have data
- Each bucket is a list (say, linked list)
- To add to a bucket, insert in *O*(1) (at beginning, or keep pointer to last element)

Example: spice level; scale 1-5;

1 = mild, 5 = *very* spicy

Input=

5: Habanero

3: Jalapeño

5: Ghost pepper

1: Bell pepper

count array	
1	
2	
3	
4	
5	

- Result:
- Easy to keep ‘stable’; Habanero still before Ghost pepper

Radix sort

- Radix = “the base of a number system”
  - Examples will use 10 because we are used to that
  - In implementations use larger numbers
    - For example, for ASCII strings, might use 128
- Idea:
  - Bucket sort on one digit at a time
    - Number of buckets = radix
    - Starting with *least* significant digit
    - Keeping sort *stable*
  - Do one pass per digit
  - Invariant: After *k* passes (digits), the last *k* digits are sorted
- Aside: Origins go back to the 1890 U.S. census

Radix Sort: Example

Input:

478
537
9
721
3
38
143
67

First pass: bucket sort by one's digit

0	1	2	3	4	5	6	7	8	9

Second pass: stable bucket sort by ten's digit

0	1	2	3	4	5	6	7	8	9

Third pass: stable bucket sort by hundred's digit

0	1	2	3	4	5	6	7	8	9

Output:

Analysis

Input size: *n*
Number of buckets = Radix: *B*
Number of passes = "Digits": *P*

Work per pass is 1 bucket sort:

Total work is

Compared to comparison sorts, sometimes a win, but often not

- Example: Strings of English letters up to length 15
  - Run-time proportional to: 15*(52 + *n*)
  - This is less than *n* log n only if *n* > 33,000
  - Of course, cross-over point depends on constant factors of the implementations
    - And radix sort can have poor locality properties

Interactive Visualizations

Comparison Sort (including quicksort):

- http://www.cs.usfca.edu/~galles/visualization/ComparisonSort.html

Bucket Sort:

- http://www.cs.usfca.edu/~galles/visualization/BucketSort.html
- http://www.cs.usfca.edu/~galles/visualization/CountingSort.html

Radix Sort:

- http://www.cs.usfca.edu/~galles/visualization/RadixSort.html

Sorting massive data

- Need sorting algorithms that minimize disk/tape access time:
  - Quicksort and Heapsort both jump all over the array, leading to expensive random disk accesses
  - Merge sort scans linearly through arrays, leading to (relatively) efficient sequential disk access
- Merge sort is the basis of massive sorting
- Merge sort can leverage multiple disks

External Merge Sort

- Sort 900 MB using 100 MB RAM
  - Read 100 MB of data into memory
  - Sort using conventional method (e.g. quicksort)
  - Write sorted 100MB to temp file
  - Repeat until all data in sorted chunks (900/100 = 9 total)
- Read first 10 MB of each sorted chuck, merge into remaining 10MB
  - writing and reading as necessary
  - Single merge pass instead of *log n*
  - Additional pass helpful if data much larger than memory
- Parallelism and better hardware can improve performance
- Distribution sorts (similar to bucket sort) are also used

Last Slide on Sorting

- Simple $O(n^2)$ sorts can be fastest for small n
  - Insertion sort (latter linear for mostly-sorted)
  - Good “below a cut-off” for divide-and-conquer sorts
- $O(n \log n)$ sorts
  - Heap sort, in-place, not stable, not parallelizable
  - Merge sort, not in place but stable and works as external sort
  - Quick sort, in place, not stable and $O(n^2)$ in worst-case
    - Often fastest, but depends on costs of comparisons/copies
- $\Omega(n \log n)$ is worst-case and average lower-bound for sorting by comparisons
- Non-comparison sorts
  - Bucket sort good for small number of possible key values
  - Radix sort uses fewer buckets and more phases
- Best way to sort?