CSE 373: Data Structures and Algorithms

Lecture 20: More Sorting

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Today: More sorting algorithms!

- Merge sort analysis
- Quicksort
- Bucket sort
- Radix sort

Divide and conquer

Very important technique in algorithm design

1. Divide problem into smaller parts
2. Independently solve the simpler parts
   - Think recursion
   - Or parallelism
3. Combine solution of parts to produce overall solution

Two great sorting methods are fundamentally divide-and-conquer
(Merge Sort & Quicksort)

Merge Sort

Merge Sort: repeatedly…

- Sort the left half of the elements
- Sort the right half of the elements
- Merge the two sorted halves into a sorted whole

To sort array from position `lo` to position `hi`:

- If range is 1 element long, it is already sorted!
- Else:
  - Sort from `lo` to `(hi+lo)/2`
  - Sort from `(hi+lo)/2` to `hi`
  - Merge the two halves together

Linked lists and big data

We defined sorting over an array, but sometimes you want to sort linked lists

One approach:

- Convert to array: $O(n)$
- Sort: $O(n \log n)$ $O(n^2)$ & $O(n \log n)$
- Convert back to list: $O(n)$

Merge sort works very nicely on linked lists directly

- Heapsort and quicksort do not
- Insertion sort and selection sort do but they're slower

Merge sort is also the sort of choice for external sorting

- Linear merges minimize disk accesses
- And can leverage multiple disks to get streaming accesses

Analysis

Having defined an algorithm and argued it is correct, we should analyze its running time and space:

Base case

$n = \#$ elements

To sort n elements, we:

- Return immediately if n=1
- Else do 2 subproblems of size $\underline{n/2}$ and then an $\underline{O(n)}$ merge

Recurrence relation:

$$T(1) = c_1$$

$$T(n) = \underline{c_2 n} + \underline{2T(n/2)}$$

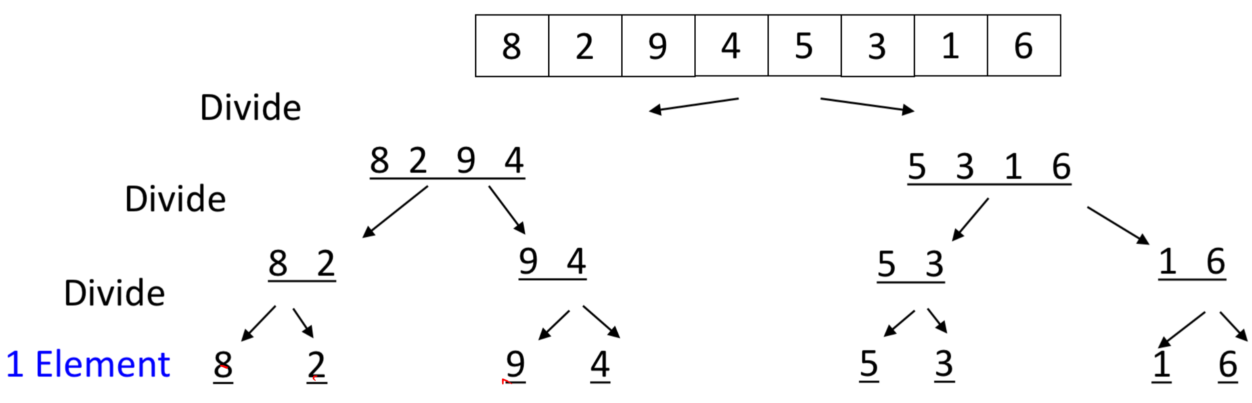
$O(n \log n)$

Analysis intuitively

This recurrence is common, you just "know" it's $O(n \texttt{ log } n)$

Merge sort is relatively easy to intuit (best, worst, and average):

- The recursion "tree" will have height $\log n$
- At each level we do a *total* amount of merging equal to n



Analysis more formally

(One of the recurrence classics)

For simplicity, ignore constants (let constants be)

$T(1) = 1$

$T(n) = 2T(n/2) + n$

$= 2(2T(n/4) + n/2) + n$

$= 4T(n/4) + 2n$

$= 4(2T(n/8) + n/4) + 2n$

$= 8T(n/8) + 3n$

....

$= 2^k T(n/2^k) + kn$

We will continue to recurse until we reach the base case, i.e. T(1) for T(1), $n/2^k = 1$, i.e., $\log n = k$

So the total amount of work is $2^k T(n/2^k) + kn = 2^{\log n} T(1) + n \log n = n + n \log n = O(n \log n)$

Divide-and-Conquer Sorting

Two great sorting methods are fundamentally divide-and-conquer

1. Merge Sort:
   - Sort the left half of the elements (recursively)
   - Sort the right half of the elements (recursively)
   - Merge the two sorted halves into a sorted whole

2. Quicksort:
   - Pick a “pivot” element
   - Divide elements into “less-than pivot” and “greater-than pivot”
   - Sort the two divisions (recursively on each)
   - Answer is “sorted-less-than”, followed by “pivot”, followed by ”sorted-greater-than”

Quicksort Overview

1. Pick a pivot element

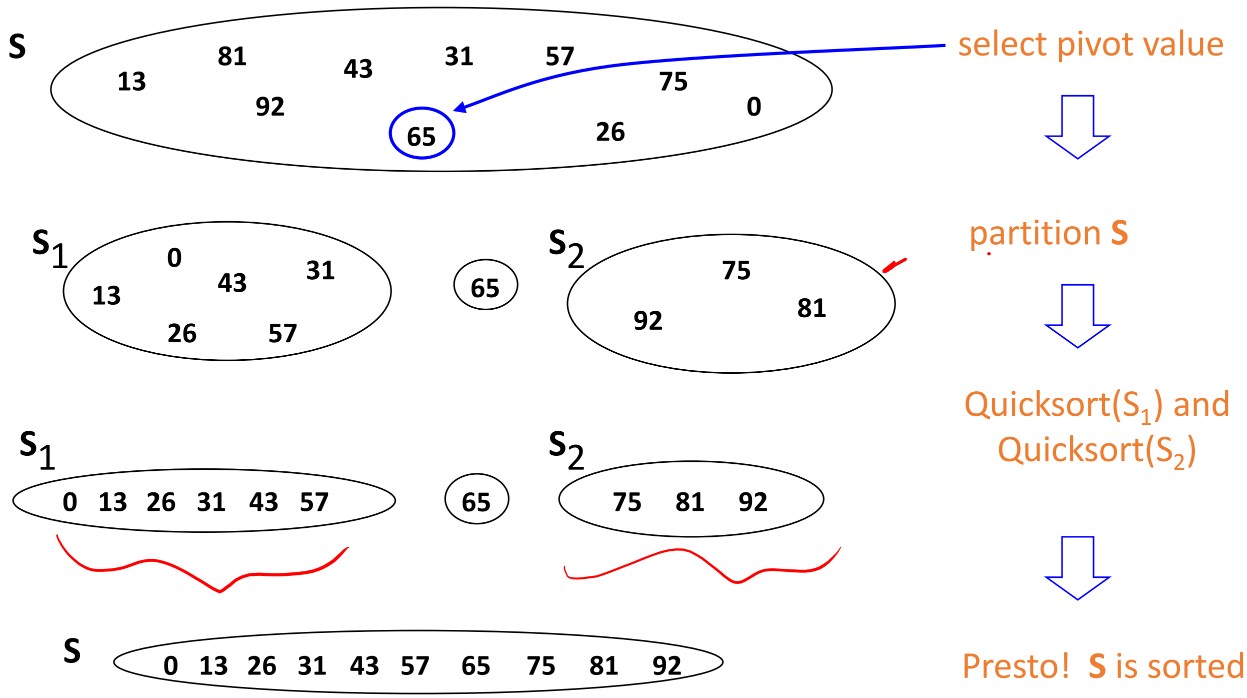
2. Partition all the data into:
   A. The elements less than the pivot
   B. The pivot
   C. The elements greater than the pivot

3. Recursively sort A and C

4. The final answer is A-B-C

Real-world example demo time!

Think in Terms of Sets



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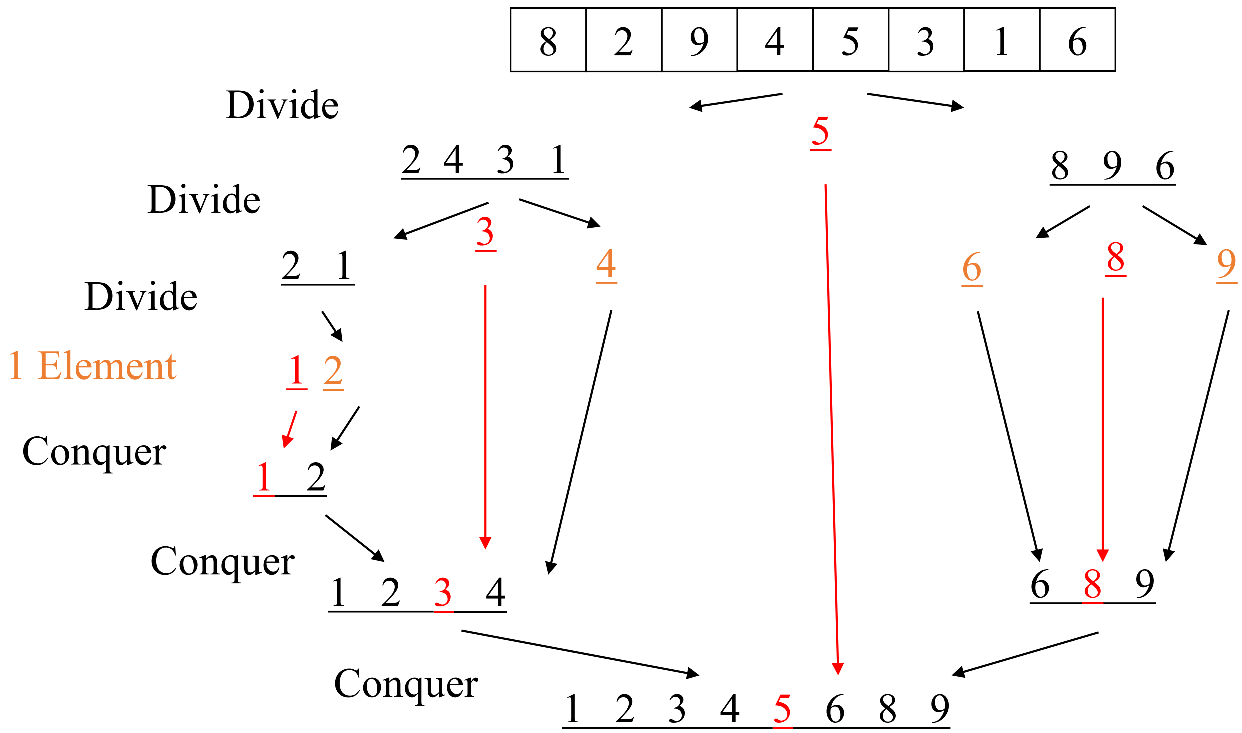
Example, Showing Recursion



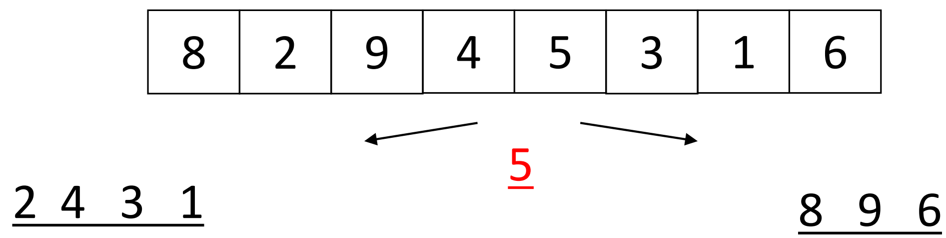
Details

Have not yet explained:

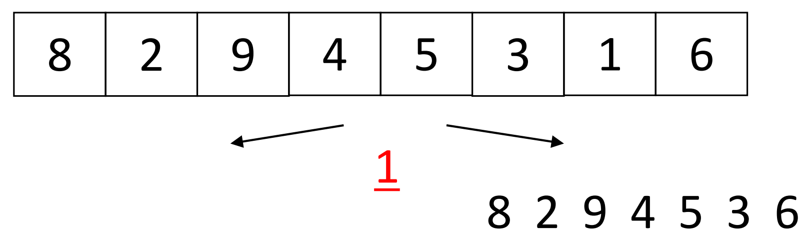
- How to pick the pivot element
  - Any choice is correct: data will end up sorted
  - But as analysis will show, want the two partitions to be about equal size
- How to implement partitioning
  - In linear time
  - In place

Pivots

- Best pivot?
  - median
  - Halve each time
  - $O(n \log n)$



- Worst pivot?
  - Greatest/least element
  - Partition of size n - 1
  - $O(n^2)$



Potential pivot rules

While sorting `arr` from `lo` to `hi-1` …

- Pick `arr[lo]` or `arr[hi-1]`
  - Fast, but worst-case occurs with mostly sorted input
- Pick random element in the range
  - Does as well as any technique, but (pseudo)random number generation can be slow
  - Still probably the most elegant approach
- Median of 3, e.g., `arr[lo], arr[hi-1], arr[(hi+lo)/2]`
  - Common heuristic that tends to work well

Partitioning

Conceptually simple, but hardest part to code up correctly

- After picking pivot, need to partition in linear time in place

One approach (there are slightly fancier ones):

1. Swap pivot with `arr[lo]`
2. Use two fingers `i` and `j`, starting at `lo+1` and `hi-1`
3. `while (i < j)`

```
if (arr[j] > pivot) j--
else if (arr[i] < pivot) i++
else swap arr[i] with arr[j]
```

4. Swap pivot with `arr[i]` *

*skip step 4 if pivot ends up being least element

Example

- Step one: pick pivot as median of 3
  - **`lo`** = 0, **`hi`** = 10

0	1	2	3	4	5	6	7	8	9
8	1	4	9	**0**	3	5	2	7	**6**

- Step two: move pivot to the `lo` position

0	1	2	3	4	5	6	7	8	9
6	1	4	9	**0**	3	5	2	7	**8**

Example

Often have more than one swap during partition – this is a short example

Now partition in place

6	1	4	9	**0**	3	5	2	7	**8**

Move fingers

6	1	4	9	**0**	3	5	2	7	**8**

Swap

6	1	4	2	**0**	3	5	9	7	**8**

Move fingers

6	1	4	2	**0**	3	5	9	7	**8**

Move pivot

5	1	4	2	**0**	3	6	9	7	**8**

Analysis

- Best-case: Pivot is always the median

  T(0) = T(1) = 1

  T(n) = $2T(n/2) + n$ -- linear-time partition

  Same recurrence as merge sort: $O(n \log n)$

- Worst-case: Pivot is always smallest or largest element

  T(0) = T(1) = 1

  T(n) = $1T(n-1) + n$

  Basically same recurrence as selection sort: $O(n^2)$

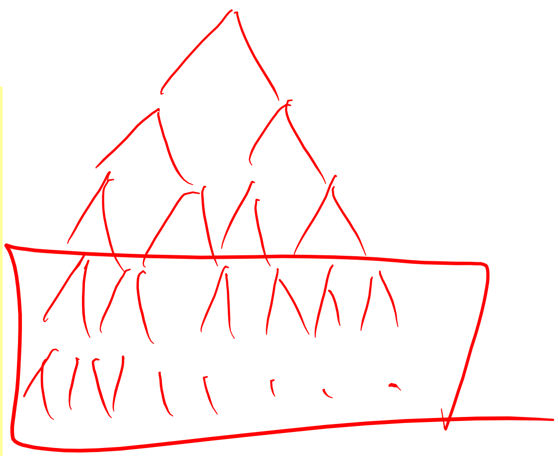
- Average-case (e.g., with random pivot)
  - O(n `log` n), not responsible for proof (in text)

Cutoffs

- For small *n*, all that recursion tends to cost more than doing a quadratic sort
  - Remember asymptotic complexity is for really large n ($n \to \infty$)
- Common engineering technique: switch algorithm below a **cutoff**
  - Reasonable rule of thumb: use insertion sort for $n < 10$
- Notes:
  - Could also use a cutoff for merge sort
  - Cutoffs are also the norm with parallel algorithms
    - Switch to sequential algorithm
  - None of this affects asymptotic complexity

Cutoff pseudocode

```
void quicksort(int[] arr, int lo, int hi)
{
  if(hi – lo < CUTOFF)
     insertionSort(arr,lo,hi);
  else
     …
}
```



Notice how this cuts out the vast majority of the recursive calls

- Think of the recursive calls to quicksort as a tree
- Trims out the bottom layers of the tree

Practice with comparison sort!

A comparison sorting algorithm is operating on an array of 8 integers. After its 4th loop or recursive call, the array looks like:

4	8	11	15	42	29	18	37

Which of these sorting algorithms can it be?

A) Heapsort

B) Merge sort

C) Insertion sort

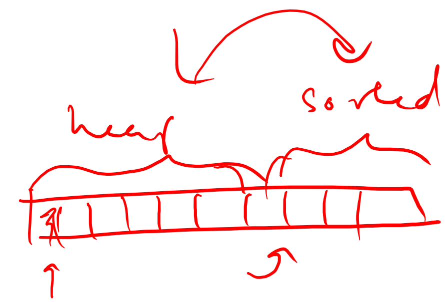
D) Quicksort using Median of 3 ← no cutoff

Practice with comparison sort!

A comparison sorting algorithm is operating on an array of 8 integers. After its 4th loop or recursive call, the array looks like:

4	8	11	15	42	29	18	37

i



Which of these sorting algorithms can it be?

A) Heapsort — maxHeap (deleteMax)

B) Merge sort

C) Insertion sort

D) Quicksort using Median of 3 ← A[low], A[high-1], A[(low+high)/2]

How Fast Can We Sort?

- Heapsort & mergesort have $O(n \texttt{ log } n)$ worst-case running time
- Quicksort has $O(n \texttt{ log } n)$ average-case running time
- These bounds are all tight, actually $\Theta(n \texttt{ log } n)$
- Comparison sorting in general is $\Omega(n \texttt{ log } n)$
  - An amazing computer-science result: proves all the clever programming in the world cannot comparison-sort in linear time

lower bound!

~~O(n)~~

The Big Picture

Surprising amount of juicy computer science: 2-3 lectures…

Simple algorithms: $O(n^2)$	Fancier algorithms: $O(n \log n)$	Comparison lower bound: $\Omega(n \log n)$	Specialized algorithms: $O(n)$	Handling huge data sets
Insertion sort Selection sort Shell sort …	Heap sort Merge sort Quick sort (avg) …		Bucket sort Radix sort	External sorting

How???

- Change the model – assume more than "compare(a,b)"

Bucket Sort (a.k.a. BinSort)

- If all values to be sorted are *known* to be integers between 1 and *K* (or any small range):
  - Create an array of size *K*
  - Put each element in its proper bucket (a.k.a. bin)
  - *If* data is only integers, no need to store more than a *count* of how times that bucket has been used
- Output result via linear pass through array of buckets

<table>
<tr><th colspan="2"><code>count</code> array</th></tr>
<tr><td>1</td><td>|||</td></tr>
<tr><td>2</td><td>|</td></tr>
<tr><td>3</td><td>||</td></tr>
<tr><td>4</td><td>||</td></tr>
<tr><td>5</td><td>|||</td></tr>
</table>

- Example:

  K=5

  input (~~5~~, ~~1~~, ~~3~~, ~~4~~, ~~3~~, ~~2~~, ~~1~~, ~~1~~, ~~5~~, ~~4~~, ~~5~~)

  output 1, 1, 1, 2, 3, 3, 4, 4, 5, 5, 5

Analyzing Bucket Sort

$n = \#$ elements
$K = \#$ buckets

- Overall: $O(n+K)$
  - Linear in n, but also linear in K
  - $\Omega(n \log n)$ lower bound does not apply because this is not a comparison sort
- Good when K is smaller (or not much larger) than n
  - We don't spend time doing comparisons of duplicates
- Bad when K is much larger than n
  - Wasted space; wasted time during linear $O(K)$ pass
- For data in addition to integer keys, use list at each bucket

Bucket Sort with Data

- Most real lists aren't just keys; we have data
- Each bucket is a list (say, linked list)
- To add to a bucket, insert in $O(1)$ (at beginning, or keep pointer to last element)

Example: spice level; scale 1-5;

1 = mild, 5 = *very* spicy

Input=

5: Habanero

3: Jalapeño

5: Ghost pepper

1: Bell pepper

count array		
1		→ Bell
2		
3		→ Jalapeño
4		
5		→ Habanero → Ghost

- Result: Bell, Jalapeño, Habanero, Ghost
- Easy to keep 'stable'; Habanero still before Ghost pepper

Interactive Visualizations

Comparison Sort (including quicksort):

- http://www.cs.usfca.edu/~galles/visualization/ComparisonSort.html

Bucket Sort:

- http://www.cs.usfca.edu/~galles/visualization/BucketSort.html
- http://www.cs.usfca.edu/~galles/visualization/CountingSort.html

Radix Sort:

- http://www.cs.usfca.edu/~galles/visualization/RadixSort.html