CSE 373: Data Structures and Algorithms

Lecture 18: Minimum Spanning Trees (Graphs)

Instructor: Lilian de Greef
Quarter: Summer 2017

Today

- Spanning Trees
  - Approach #1: DFS
  - Approach #2: Add acyclic edges
- Minimum Spanning Trees
  - Prim’s Algorithm
  - Kruskal’s Algorithm

Announcements

- Midterms
  - I brought midterms with me, can get them after class
  - Next week, will only have them at CSE220 office hours

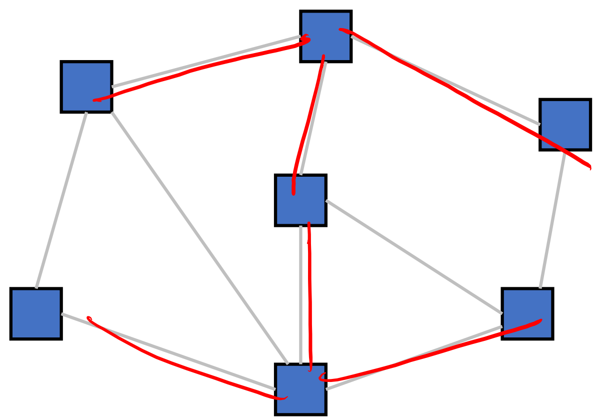
- Reminder: hw4 due on Friday!

Spanning Trees

Introductory Example

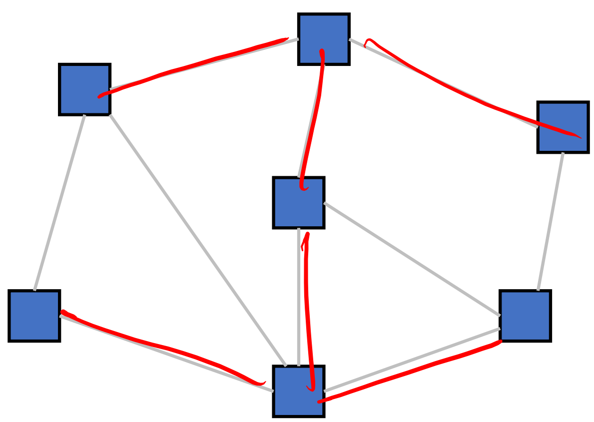
All the roads in Seattle are covered in snow.
You were asked to shovel or plow snow from roads so that Seattle drivers can travel.
Because you don't want to shovel/plow that many roads, what is the smallest set of roads to clear in order to reconnect Seattle?

Spanning Tree!

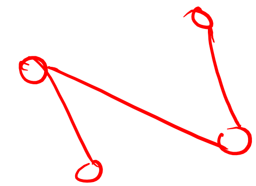


Spanning Trees

- Goal: Given a *connected* undirected graph **G**=(**V**,**E**), find a minimal subset of edges such that **G** is still connected
  - A graph **G2** = (**V**,**E2**) such that **G2** is connected and removing any edge from **E2** makes **G2** disconnected



Observations



1. Any solution to this problem is a tree
    - Recall a tree does not need a root; just means acyclic
    - For any cycle, could remove an edge and still be connected

2. Solution not unique unless original graph was already a tree

3. Problem ill-defined if original graph not connected
    - So **|E| >= |V|-1**

4. A tree with **|V|** nodes has $|V| - 1$ edges
    - So every solution to the spanning tree problem has $|V| - 1$ edges

Two Approaches

Different algorithmic approaches to the spanning-tree problem:

1. Do a graph traversal (e.g., depth-first search, but any traversal will do), keeping track of edges that form a tree

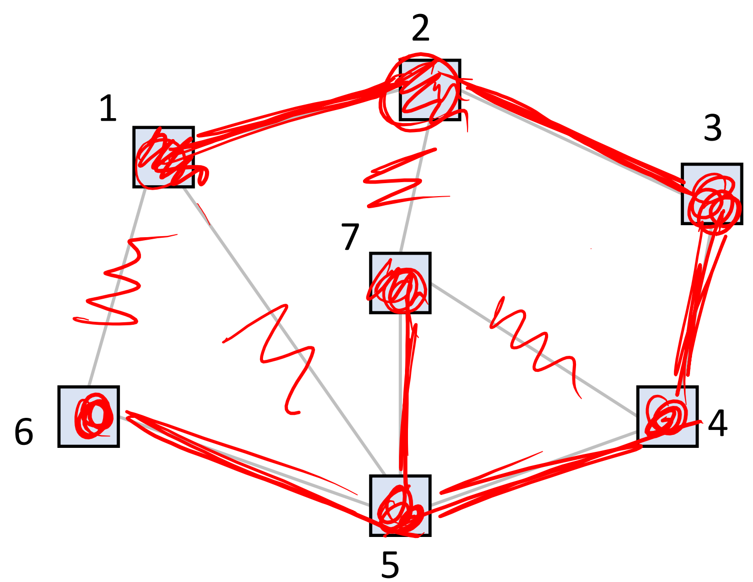
2. Iterate through edges; add to output any edge that does not create a cycle

Approach #1: Using DFS (Example)

Do a graph traversal, keeping track of edges that form a tree

Stack:

f(1)
f(2)
f(3)
f(4)
f(5)
f(6) f(7)
~~f(2)~~ ~~f(4)~~



Output: (1,2), (2,3), (3,4), (4,5), (5,6), (5,7)

Approach #1: Spanning Tree via DFS

```
spanning_tree(Graph G) {
  for each node i: i.marked = false
  for some node i: f(i)
}
f(Node i) {
  i.marked = true
  for each j adjacent to i:
    if(!j.marked) {
      add(i,j) to output
      f(j) // DFS
    }
}
```

$O(|V|)$

Correctness:
DFS reaches each node.
We add one edge to connect it to the already visited nodes.
Order affects result, not correctness.

$\geq |v| - 1$

Time: $\cancel{O(|V| + |E|)}$

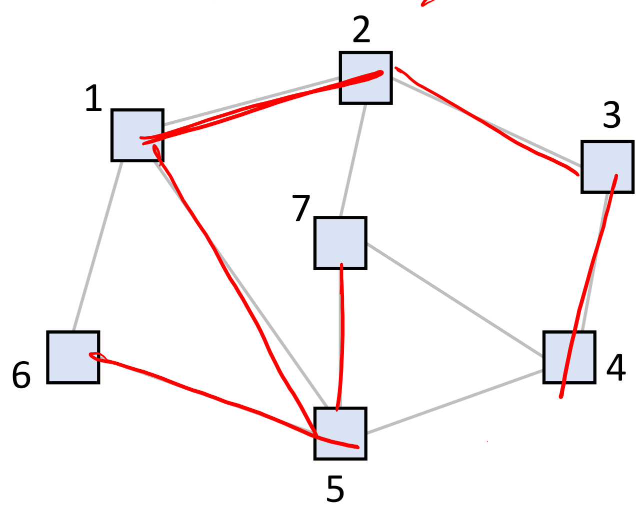
$O(|E|)$

Approach #2: Add Acyclic Edges (Example)

Iterate through edges; add to output any edge that does not create a cycle

Edges in some arbitrary order:

(1,2), (3,4), (5,6), (5,7), (1,5), (1,6), (2,7), (2,3), (4,5), (4,7)



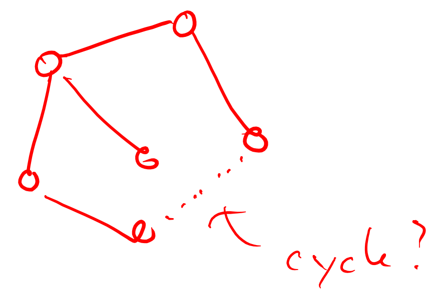
<u>Output:</u> (1,2), (3,4), (5,6), (5,7), (1,5), (2,3)

Approach #2: Add Acyclic Edges

Iterate through edges; output any edge that does not create a cycle.

Correctness (hand-wavy):

- Goal is to build an acyclic connected graph
- When we add an edge, it adds a vertex to the tree
    - Else it would have created a cycle
- The graph is connected, so we reach all vertices



Efficiency:

- Depends on how quickly you can detect cycles

DFS to check $O(|V|)$

- (Not covered: there is a way to detect these cycles at *almost* average O(1))

Summary So Far

The **spanning-tree** problem – two approaches:

- Add nodes to partial tree approach (DFS)
- Add acyclic edges approach

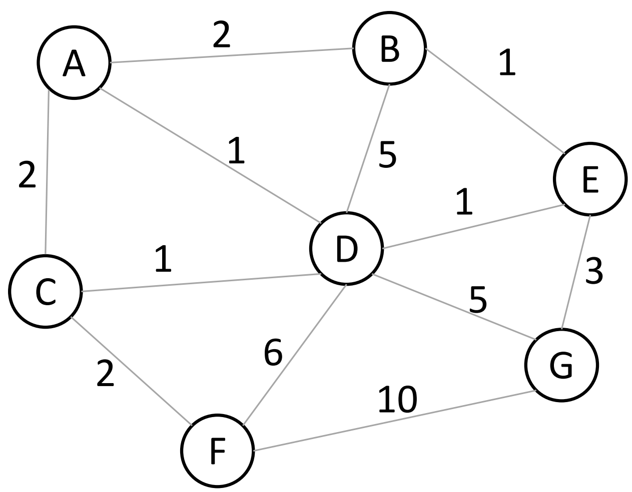
More compelling: we have a *weighted* undirected graph and we want a spanning tree with minimum total weight

a.k.a. the minimum – **spanning-tree** problem

Minimum Spanning Trees

Introductory Example: version 2

All the roads in Seattle are covered in snow.
You were asked to shovel or plow snow from roads so that Seattle drivers can travel.
~~Because you don't want to shovel/plow that many roads, what is the smallest set of roads to clear in order to reconnect Seattle?~~
Because you want to do the *minimum* amount of effort, what is the shortest *total distance* to clear in order to reconnect Seattle?



Minimum Spanning Tree!

Minimum Spanning Tree: Example Uses

How to most efficiently lay out…

- Telephone lines
- Electrical power lines
- Hydraulic pipes
- TV cables
- Computer networks (like the Internet!)

Minimum Spanning Tree Algorithms

The **minimum-spanning-tree** problem

- Given a weighted undirected graph, give a spanning tree of minimum weight
- Same two approaches, with minor modifications, will work

Algorithm for Unweighted Graph	Similar Algorithm for Weighted Graph
BFS for shortest path	Dijkstra's Algorithm (shortest path)
DFS for spanning tree	Prim's Algorithm (minimum spanning tree)
Adding acyclic edges approach for spanning tree	Kruskal's Algorithm (minimum spanning tree)

Prim's Algorithm: Idea

Idea: Grow a tree by adding an edge from the "known" vertices to the "unknown" vertices. *Pick the edge with the smallest weight that connects "known" to "unknown."*

Greedy Algorithm!

Recall Dijkstra "picked edge with closest known distance to source"

- That is not what we want here
- Otherwise identical (!)

Prim's Algorithm: Pseudocode

1. For each node `v`, set `v.cost = ∞ and v.known = false`

   can pick ~~any~~ node at start (cost = 0)

2. Choose any node `v`
   a) Mark `v` as known
   b) For each edge `(v,u)` with weight `w`, set `u.cost=w` and `u.prev=v`

3. While there are unknown nodes in the graph
   a) Select the unknown node `v` with lowest cost
   b) Mark `v` as known and add `(v, v.prev)` to output
   c) For each edge `(v,u)` with weight `w`,

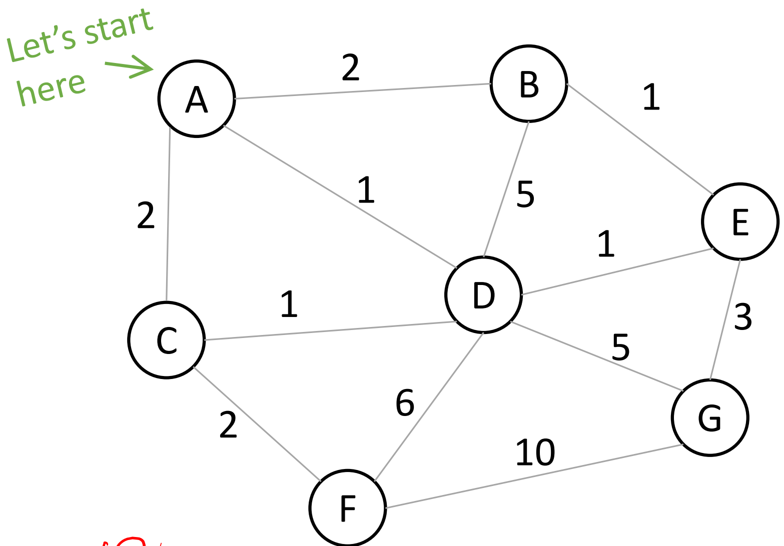
```
if(w < u.cost) {
   u.cost = w;
   u.prev = v;
}
```

not adding

(also if it does not add a cycle)

Practice Time!

Using Prim's Algorithm starting at vertex A, what's the minimum spanning tree?



A) (A,B), (A,C), (A,D), (D,E), (C,F), (E,G)

B) (B,E), (C,D), (D,A), (E,D), (F, C), (G,E)

C) (B,A), (C,A), (D,A), (E,D), (F, C), (G,E)

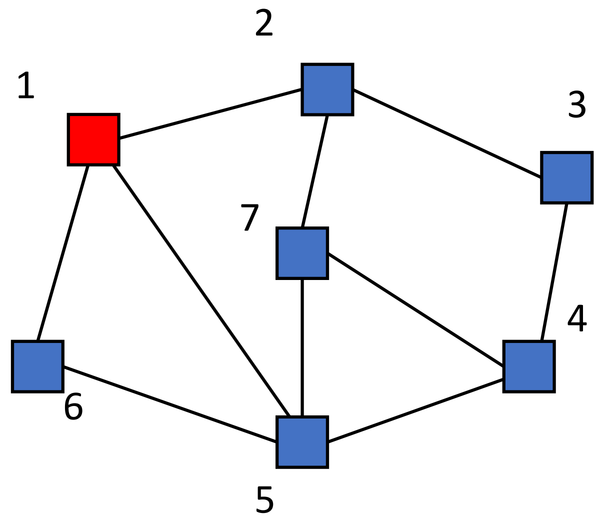
D) (B,A), (C,D), (D,A), (E,D), (F, C), (G,D)

vertex	known?	cost	prev
A			
B			
C			
D			
E			
F			
G			

Example

Stack

f(1)



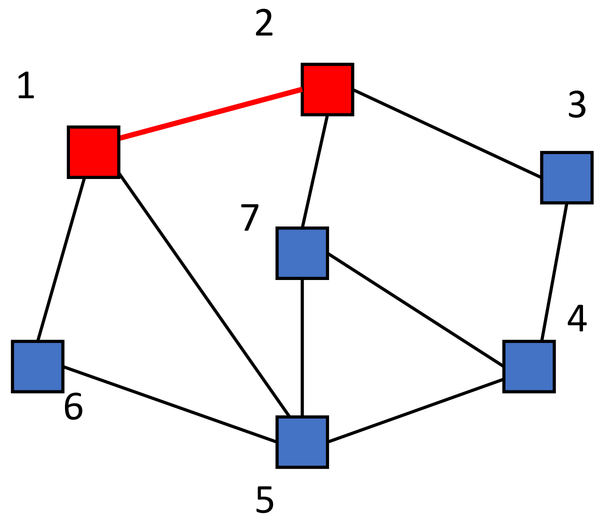
Output:

Example

Stack
  (bottom)

f(1)

f(2)



Output: (1,2)

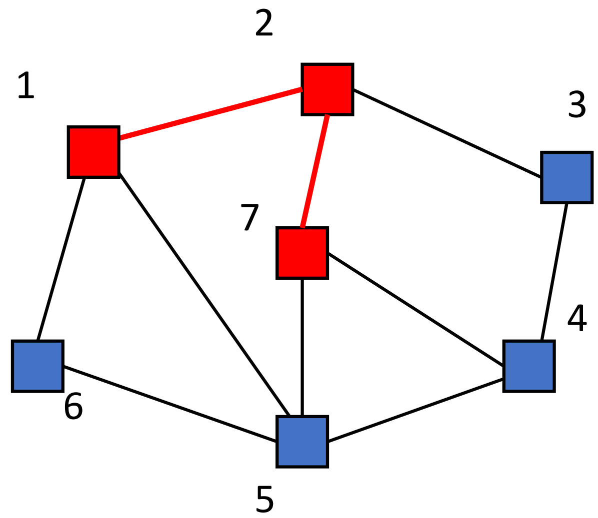
Example

Stack
(bottom)

f(1)

f(2)

f(7)



Output: (1,2), (2,7)

Example

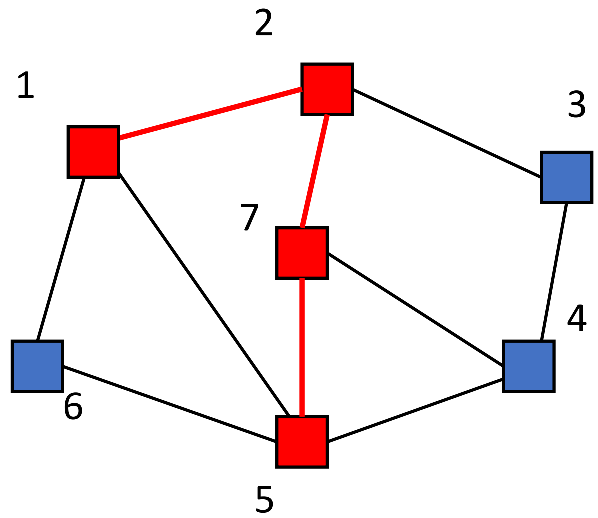
Stack
  (bottom)

f(1)

f(2)

f(7)

f(5)



Output: (1,2), (2,7), (7,5)

Example

Stack
(bottom)

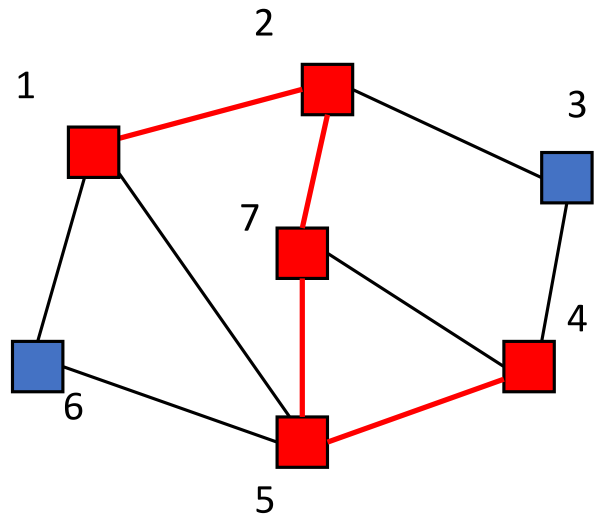
f(1)

f(2)

f(7)

f(5)

f(4)



Output: (1,2), (2,7), (7,5), (5,4)

Example

Stack
(bottom)

f(1)

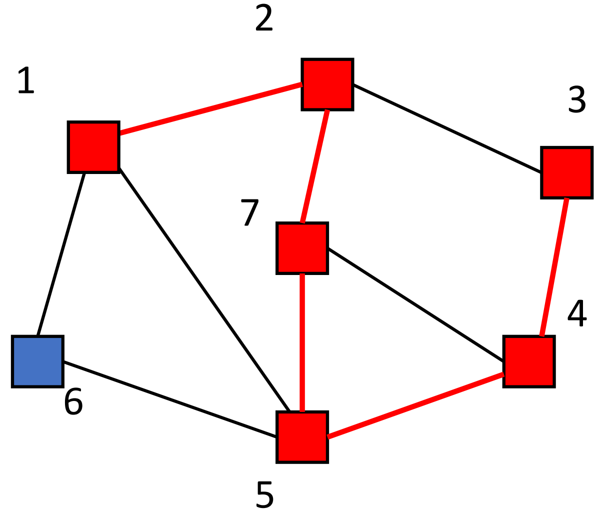
f(2)

f(7)

f(5)

f(4)

f(3)



Output: (1,2), (2,7), (7,5), (5,4),(4,3)

Example

Stack
(bottom)

f(1)

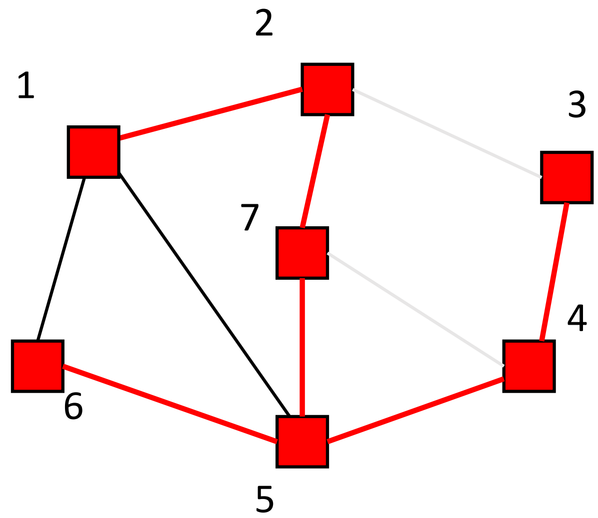
f(2)

f(7)

f(5)

f(4) f(6)

f(3)



Output: (1,2), (2,7), (7,5), (5,4), (4,3), (5,6)

Example

Stack
(bottom)

f(1)

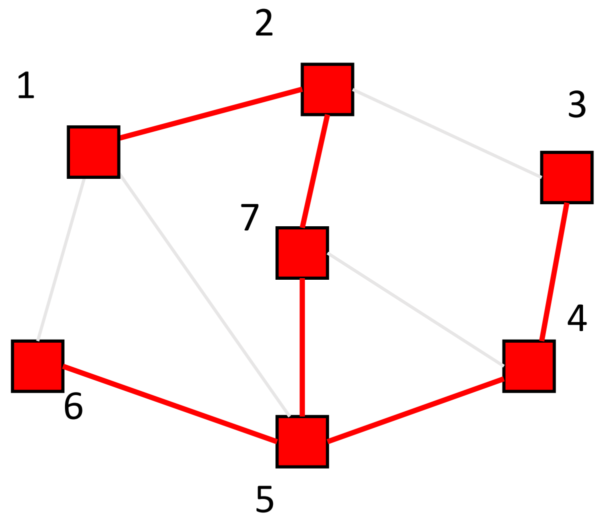
f(2)

f(7)

f(5)

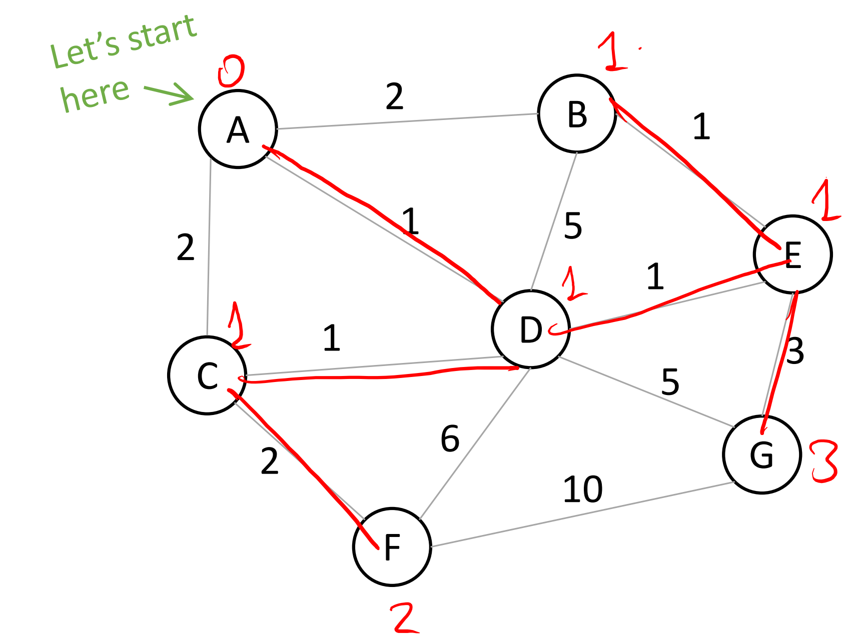
f(4) f(6)

f(3)



Output: (1,2), (2,7), (7,5), (5,4), (4,3), (5,6)

Prim's Algorithm: Example



(B,E), (C,D), ...

vertex	known?	cost	prev
A	Y	$\cancel{\infty}$ 0	
B	Y	$\cancel{\infty}$ $\cancel{2}$ 1	$\cancel{A}$ E
C	Y	$\cancel{\infty}$ $\cancel{2}$ 1	$\cancel{A}$ D
D	Y	$\cancel{\infty}$ 1	A
E	Y	$\cancel{\infty}$ 1	D
F	Y	$\cancel{\infty}$ $\cancel{6}$ 2	$\cancel{D}$ C
G	Y	$\cancel{\infty}$ $\cancel{5}$ 3	$\cancel{D}$ E

Analysis

- Correctness ??
  - A bit tricky
  - Intuitively similar to Dijkstra

- Run-time
  - Same as Dijkstra
  - $O(|E| \log |V|)$ using a priority queue
    - Costs/priorities are just edge-costs, not path-costs

Kruskal's Algorithm: Idea

Idea:

Grow a forest out of edges that do not grow a cycle, just like for the spanning tree problem.

But now consider the edges in order by weight

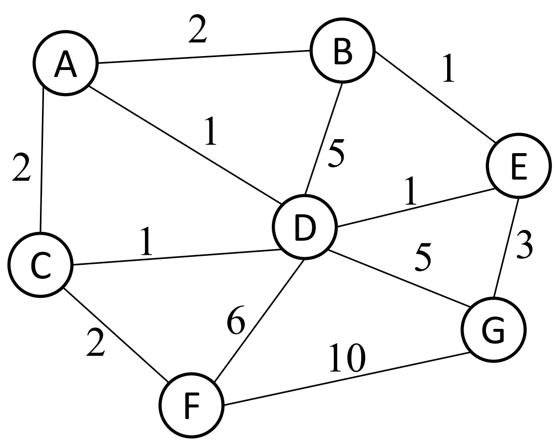
(Greedy!)

Kruskal's Algorithm: Pseudocode

1. Sort edges by weight (better: put in min-heap)
2. Each node in its own set
3. While output size < |V|-1
   - Consider next smallest edge `(u,v)`
   - If adding edge `(u,v)` doesn't introduce cycles, output `(u,v)`

Fast way:
sets + union find

Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

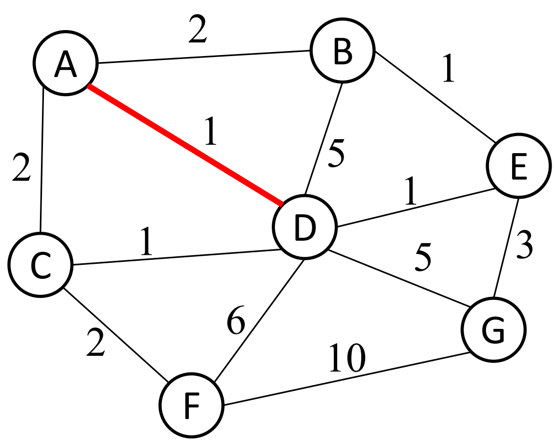
5: (D,G), (B,D)

6: (D,F)

10: (F,G)

Output:

Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

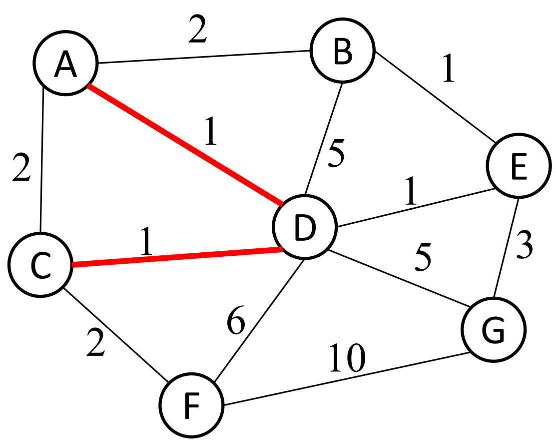
5: (D,G), (B,D)

6: (D,F)

10: (F,G)

Output: (A,D)

Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

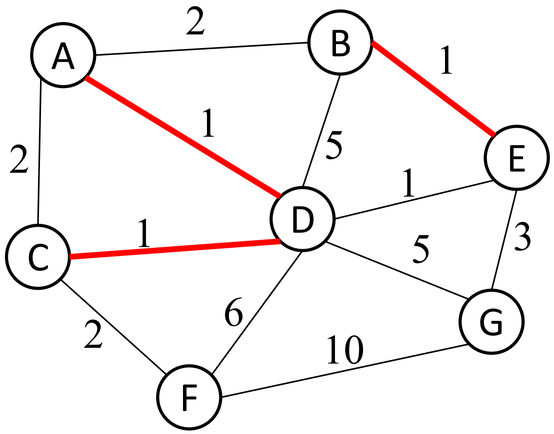
5: (D,G), (B,D)

6: (D,F)

10: (F,G)

Output: (A,D), (C,D)

Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

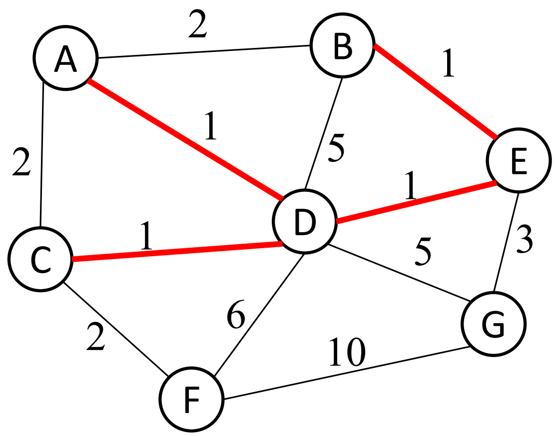
5: (D,G), (B,D)

6: (D,F)

10: (F,G)

Output: (A,D), (C,D), (B,E)

Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

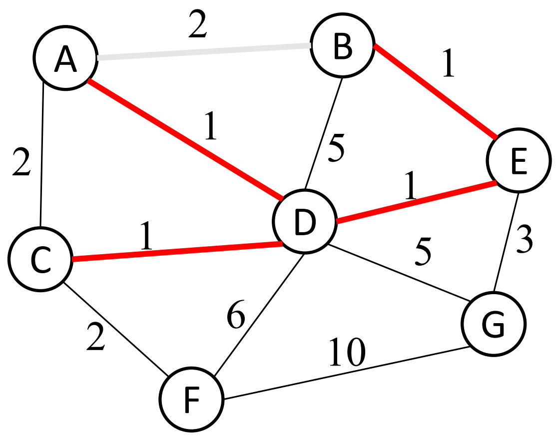
5: (D,G), (B,D)

6: (D,F)

10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E)

Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

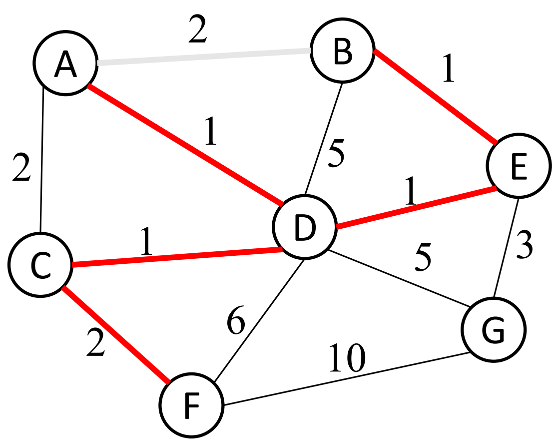
5: (D,G), (B,D)

6: (D,F)

10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E)

Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

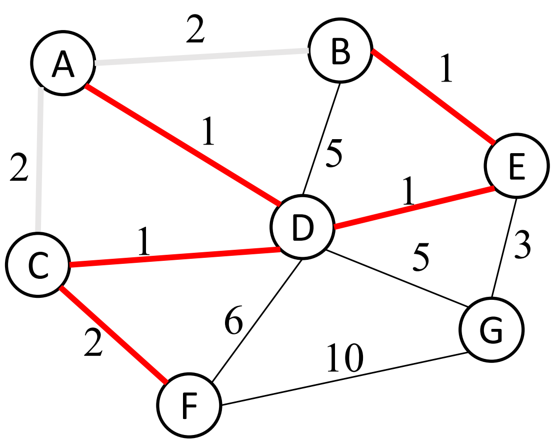
5: (D,G), (B,D)

6: (D,F)

10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E), (C,F)

Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

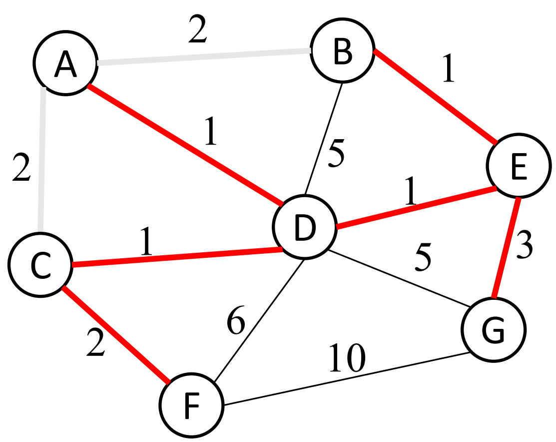
5: (D,G), (B,D)

6: (D,F)

10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E), (C,F)

Example



Edges in sorted order:
1: (A,D), (C,D), (B,E), (D,E)
2: (A,B), (C,F), (A,C)
3: (E,G)
5: (D,G), (B,D)
6: (D,F)
10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E), (C,F), (E,G)

Example

Edges in sorted order:

1: ~~(A,D)~~, ~~(C,D)~~, ~~(B,E)~~, ~~(D,E)~~

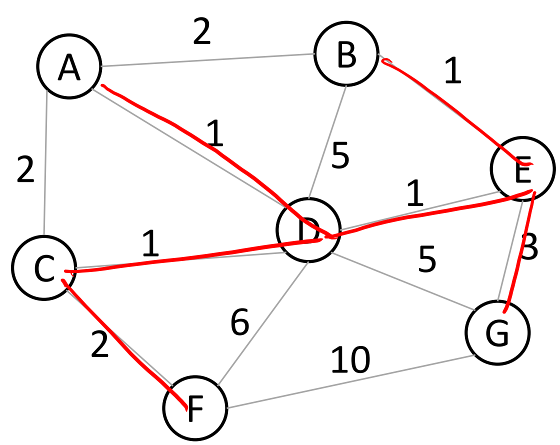
2: ~~(A,B)~~, ~~(C,F)~~, ~~(A,C)~~

3: ~~(E,G)~~

5: ~~(D,G)~~, ~~(B,D)~~

6: ~~(D,F)~~

10: ~~(F,G)~~



Output: (A,D), (C,D), (B,E), (D,E), (C,F)

Kruskal's Algorithm: Correctness (Bonus)

It clearly generates a spanning tree. Call it T_K.

Suppose T_K is *not* minimum:

- Pick another spanning tree T_{min} with *lower cost* than T_K
- Pick the smallest edge $e_1=(u,v)$ in T_K that is not in T_{min}
- T_{min} already has a path p in T_{min} from u to v
  - $\Rightarrow$ Adding e_1 to T_{min} will create a cycle in T_{min}
- Pick an edge e_2 in p that Kruskal's algorithm considered *after* adding e_1
  - (must exist: u and v unconnected when e_1 considered)
  - $\Rightarrow$ $\text{cost}(e_2) \geq \text{cost}(e_1)$
  - $\Rightarrow$ can replace e_2 with e_1 in T_{min} without increasing cost!
- Keep doing this until T_{min} is identical to T_K
  - $\Rightarrow$ T_K must also be minimal – contradiction!