CSE 373: Data Structures and Algorithms

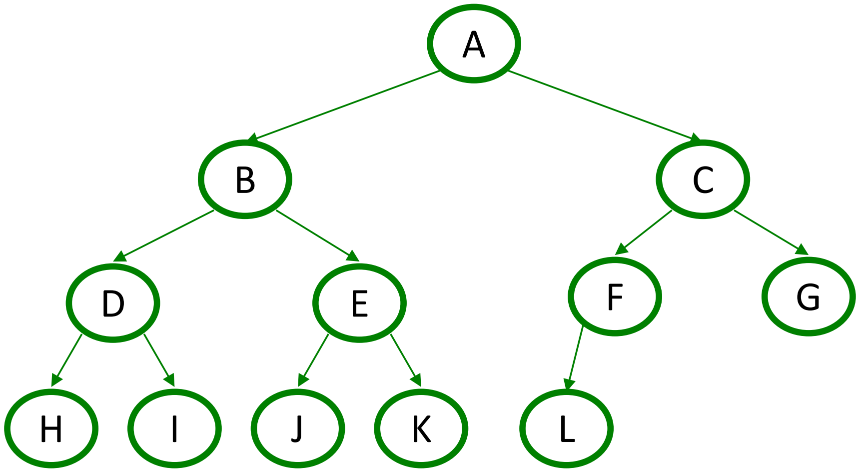
Lecture 13: Finish Binary Heaps

Instructor: Lilian de Greef
Quarter: Summer 2017

Announcements

- Midterm on Friday
  - Will start at 10:50, will end promptly at 11:50 (even if you’re late), so be early
  - Anything we’ve covered is fair game (including this lecture)
  - Only bring pencils and erasers
  - Turn off / silence and put away any devices (e.g. phone) before exam
- Section
  - Will go over solutions for select problems from practice set
  - Practice set posted on course webpage (under Sections)
  - Recommendation: do the practice problems, then use section to go over the questions you found hardest (there isn’t enough time to cover all of them)
- Homework 3 grades come out today!
- Course feedback today! (anonymous, confidential, something I have set up)

Binary Trees Implemented with an **Array**



From node `i`:

left child: `i*2`
right child: `i*2+1`
parent: `i/2`

(wasting index 0 is convenient for the index arithmetic)

0	1	2	3	4	5	6	7	8	9	10	11	12	13

Judging the array implementation

Pros:

- Non-data space: just index 0 and unused space on right
  - In conventional tree representation, one edge per node (except for root), so n-1 wasted space (like linked lists)
  - Array would waste more space if tree were not complete
- Multiplying and dividing by 2 is very fast (shift operations in hardware)
- Last used position is just index

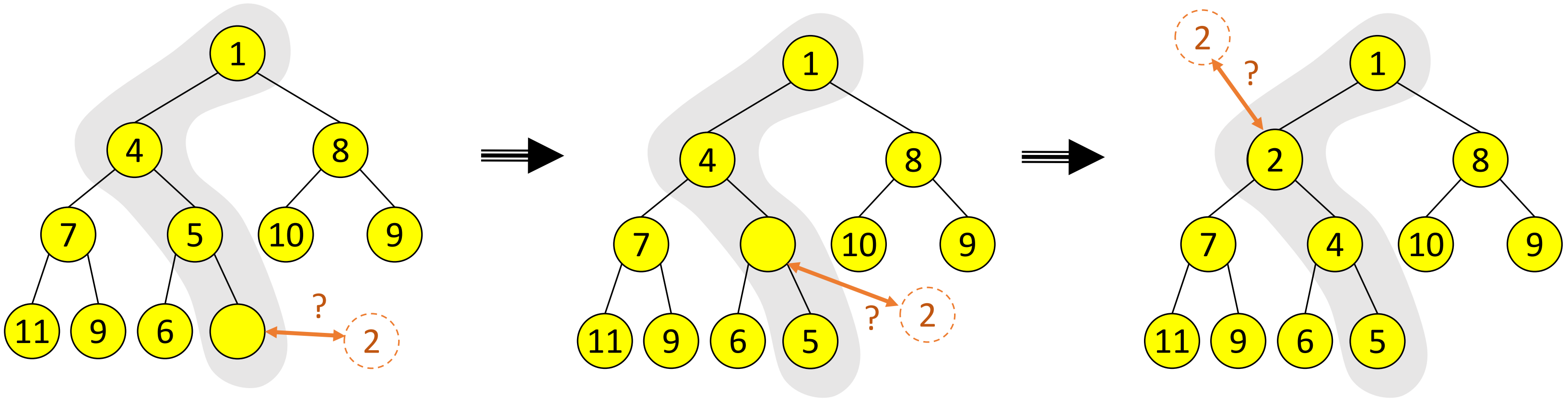
Cons:

- Same might-be-empty or might-get-full problems we saw with array-based stacks and queues (resize by doubling as necessary)

Pros outweigh cons: min-heaps almost always use array implementation

Heap `insert`:

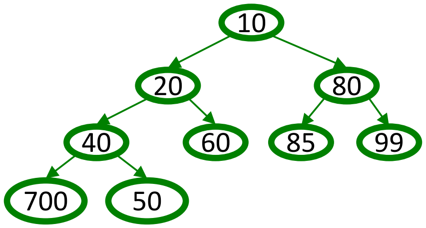
1. Put new data in new location *(preserve structure property)*
2. **Percolate up:** *(restore heap property)*
   - If higher priority than parent, swap with parent
   - Repeat until parent is more important or reached root



Semi-Pseudocode: `insert` into binary heap

```
void insert(int val) {
  if(size==arr.length-1)
    resize();
  size++;
  i=percolateUp(size,val);
  arr[i] = val;
}
```

```
int percolateUp(int hole,
                int val) {
  while(hole > 1 &&
        val < arr[hole/2])
    arr[hole] = arr[hole/2];
    hole = hole / 2;
  }
  return hole;
}
```

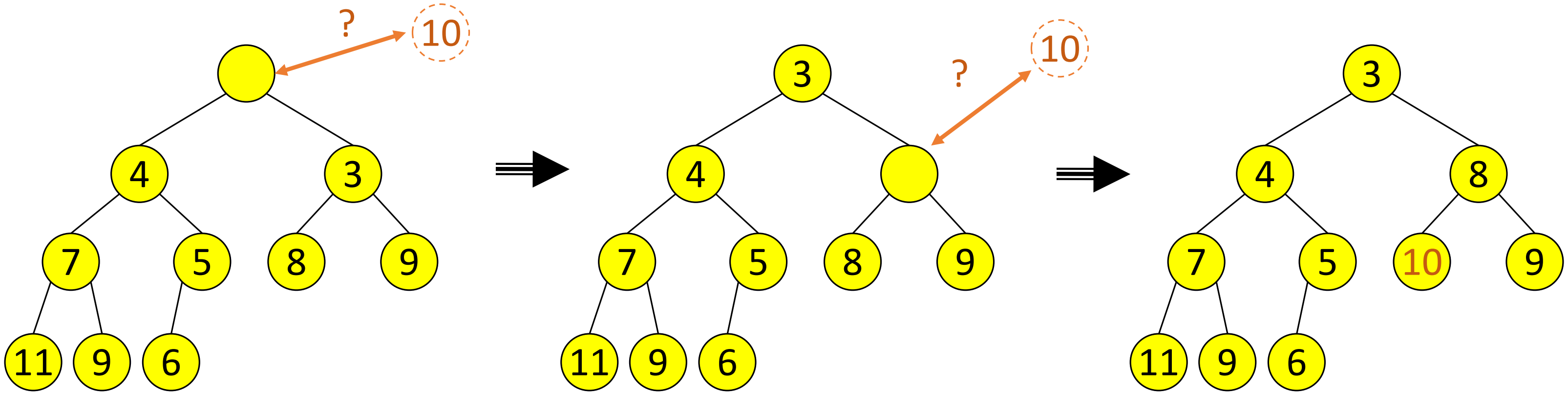


This pseudocode uses ints. In real use, you will have data nodes with priorities.

	10	**20**	**80**	**40**	**60**	**85**	**99**	**700**	**50**				
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Heap `deleteMin`:

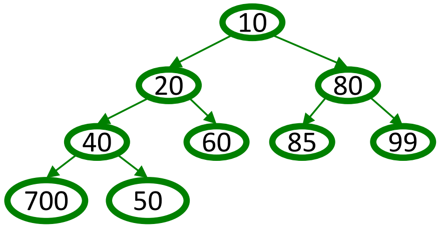
1. Remove (and later return) item at root
2. “Move” the last item in bottom row to the root *(preserve structure property)*
3. **Percolate down:** *(restore heap property)*
    - If item has lower priority, swap with the most important child
    - Repeat until both children have lower priority or we’ve reached a leaf node



Semi-Pseudocode: `deleteMin` from binary heap

```
int deleteMin() {
  if(isEmpty()) throw…
  ans = arr[1];
  hole = percolateDown
           (1,arr[size]);
  arr[hole] = arr[size];
  size--;
  return ans;
}
```

```
int percolateDown(int hole,
                  int val) {
 while(2*hole <= size) {
  left  = 2*hole;
  right = left + 1;
  if(right > size ||
     arr[left] < arr[right])
    target = left;
  else
    target = right;
  if(arr[target] < val) {
    arr[hole] = arr[target];
    hole = target;
  } else
      break;
 }
 return hole;
}
```



	10	**20**	**80**	**40**	**60**	**85**	**99**	**700**	**50**				
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Example

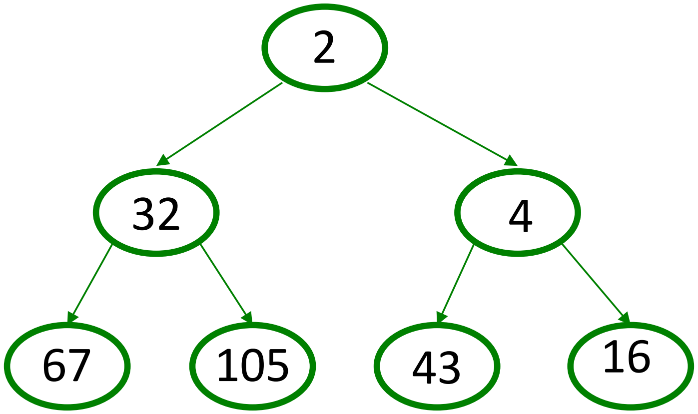
1. `insert` (in this order): 16, 32, 4, 67, 105, 43, 2
2. `deleteMin` once

0	1	2	3	4	5	6	7

Example

1. `insert` (in this order): 16, 32, 4, 67, 105, 43, 2
2. `deleteMin` once

0	1	2	3	4	5	6	7



Other operations

- `decreaseKey`: given pointer to object in priority queue (e.g., its array index), lower its priority value by *p*
  - Change priority and percolate up
- `increaseKey`: given pointer to object in priority queue (e.g., its array index), raise its priority value by *p*
  - Change priority and percolate down
- `remove`: given pointer to object in priority queue (e.g., its array index), remove it from the queue
  - `decreaseKey` with $p = \infty$, then `deleteMin`

Running time for all these operations?

Build Heap

- Suppose you have *n* items to put in a new (empty) priority queue
  - Call this operation `buildHeap`

- *n* `inserts`
  - Only choice if ADT doesn't provide `buildHeap` explicitly
  - Run time:

- Why would an ADT provide this unnecessary operation?
  - Convenience
  - Efficiency: an *O*(*n*) algorithm
  - Common issue in ADT design: how many specialized operations

`heapify` (Floyd’s Method)

1. Use *n* items to make any complete tree you want
   - That is, put them in array indices 1,…,*n*
2. Fix the heap-order property
   - Bottom-up: percolate down starting at nodes one level up from leaves, work up toward the root

`heapify` (Floyd's Method): Example

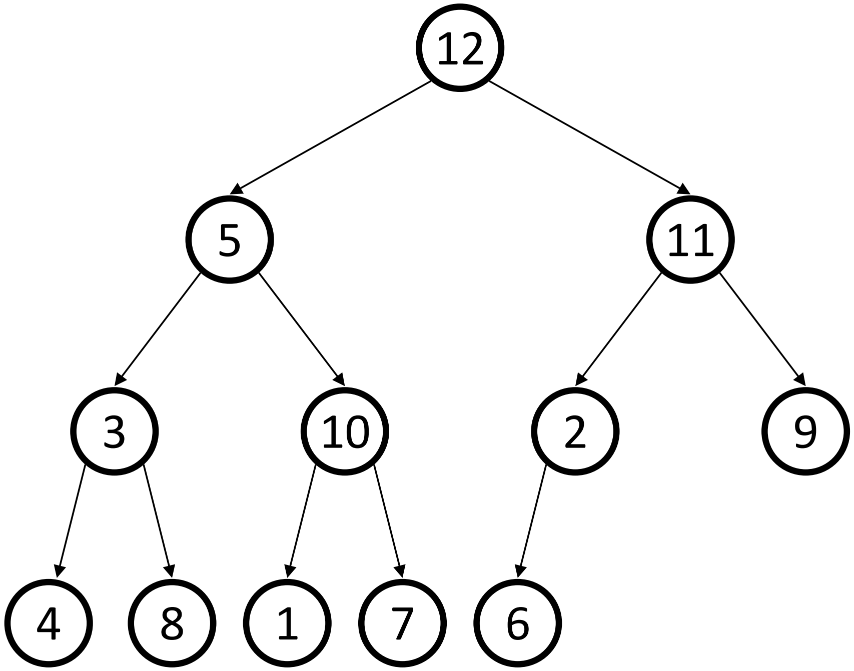
1. Use *n* items to make any complete tree you want
2. Fix the heap-order property from bottom-up

Which nodes break the heap-order property?

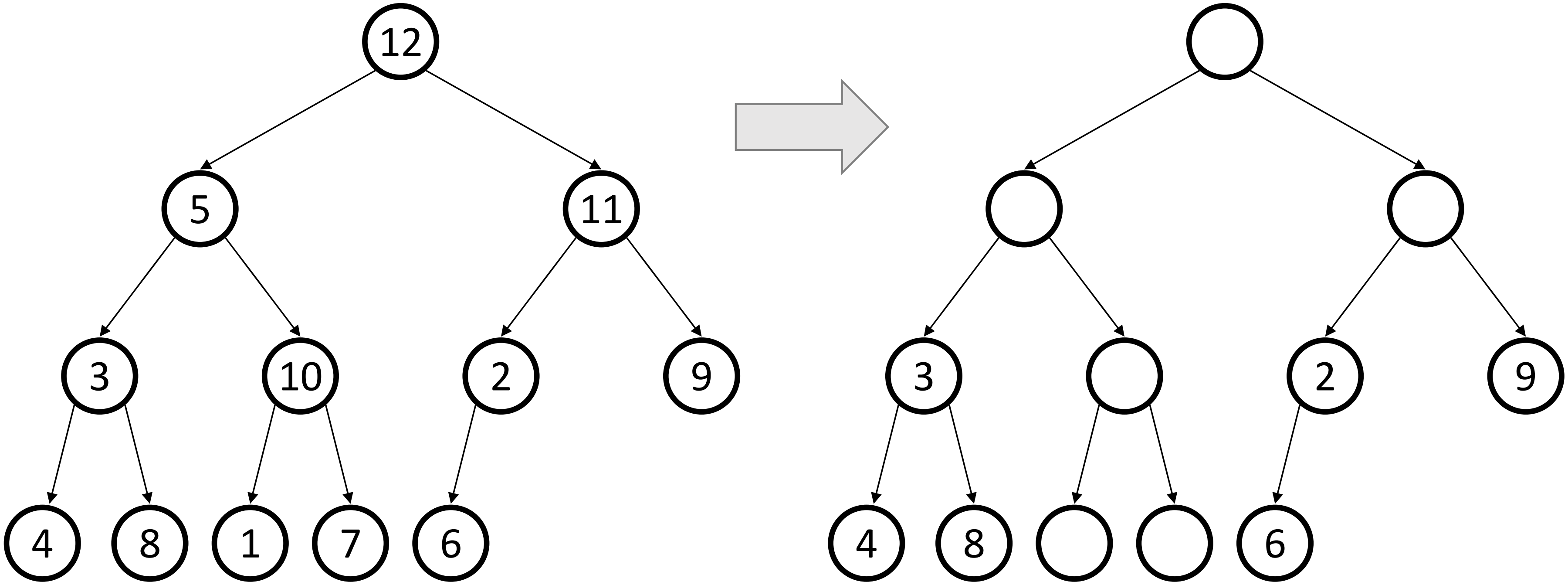
Why work from the bottom-up to fix them?

Why start at one level above the leaf nodes?

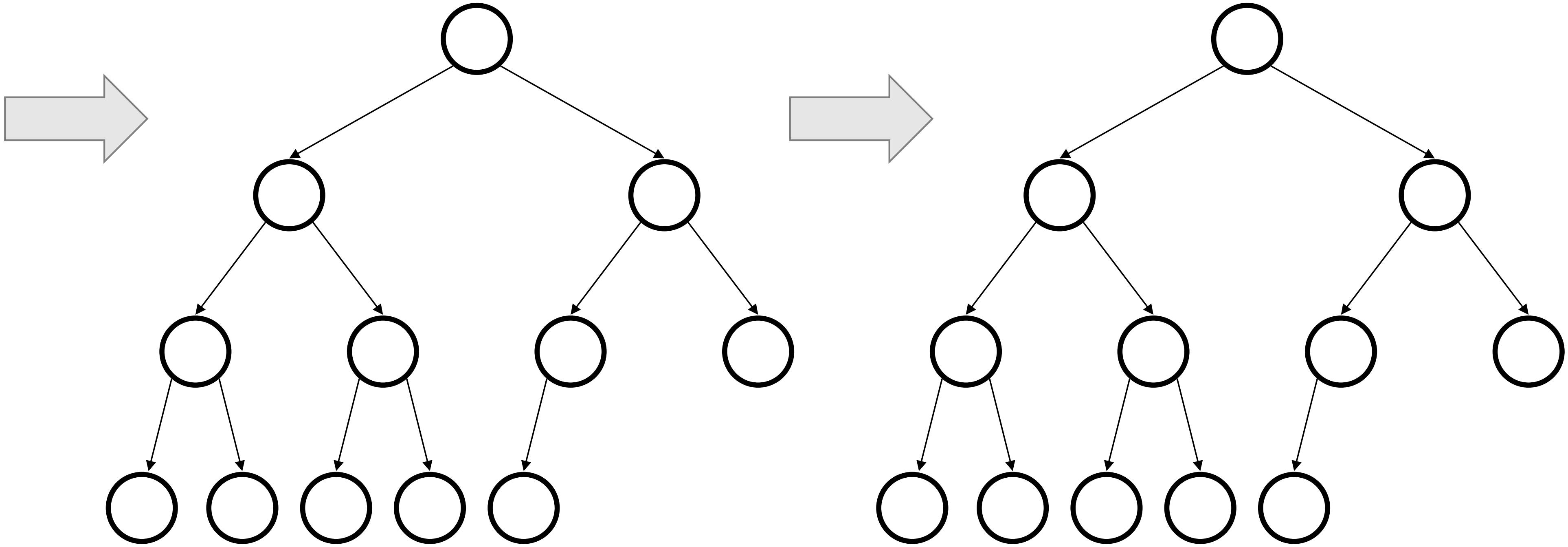
Where do we start here?



`heapify` (Floyd's Method): Example



`heapify` (Floyd's Method): Example



`heapify` (Floyd's Method)

```
void buildHeap() {
   for(int i = size/2; i>0; i--) {
     val  = arr[i];
     hole = percolateDown(i, val);
     arr[hole] = val;
   }
}
```

But is it right? … it "seems to work"

- Let's *prove* it restores the heap property
- Then let's *prove* its running time

Correctness

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Loop Invariant: For all `j` > `i`, `arr[j]` is higher priority than its children

- True initially: If `j` > `size/2`, then `j` is a leaf
  - Otherwise its left child would be at position > `size`
- True after one more iteration: loop body and `percolateDown` make `arr[i]` higher priority than children without breaking the property for any descendants

So after the loop finishes, all nodes are less than their children

Efficiency

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Easier argument: `buildHeap` is where *n* is `size`

- loop iterations
- Each iteration does one `percolateDown`, each is

This is correct, but there is a more precise ("tighter") analysis of the algorithm…

Efficiency

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Better argument: `buildHeap` is where *n* is `size`

- `size/2` total loop iterations: *O*(*n*)
- 1/2 the loop iterations `percolateDown` at most
- 1/4 the loop iterations `percolateDown` at most
- 1/8 the loop iterations `percolateDown` at most
- …
- ((1/2) + (2/4) + (3/8) + (4/16) + …) < 2 (page 4 of Weiss)

$$\sum_{i=1}^{\infty} \frac{i}{2^i} = 2$$

Lessons from `buildHeap`

- Without providing `buildHeap`, clients can implement their own that runs in worst case
- By providing a specialized operation (with access to the internal data), we can do worst case
  - Intuition: Most data is near a leaf, so better to percolate down
- Can analyze this algorithm for:
  - Correctness: Non-trivial inductive proof using loop invariant
  - Efficiency:
    - First (easier) analysis proved it was O(n `log` n)
    - Tighter analysis shows same algorithm is $O(n)$

Other branching factors for Heaps

***d*-heaps**: have *d* children instead of 2

- Makes heaps shallower

Example: 3-heap

- Only difference: three children instead of 2
- Still use an array with all positions from 1 … heapSize

Indices for 3-heap

Index	Children Indices
1	
2	
3	
4	
5	
…	…

Wrapping up Heaps

- What are heaps a data structure for?
- What is it usually implemented with? Why?
- What are some example uses?