CSE 373: Data Structures and Algorithms

Lecture 6: Finishing Amortized Analysis; Dictionaries ADT; Introduction to Hash Tables

Instructor: Lilian de Greef
Quarter: Summer 2017

Today:

- Finish up Amortized Analysis
- Dictionary ADT
- Introduce Hash Tables

Reminder: No class on Monday!

Unofficial holiday – have a good 4-day weekend! ☺

🇺🇸 🎉

- Will ask you to do a ~30 minute activity to make up for last class time
- Remember that homework 2 is also due the day after we’re back

Homework 2 update:

- There was a typo (woops!) for problem 7
- The website now has a corrected version.

Amortized Analysis

How we calculate the average time!

Amortized Cost

The **amortized cost** of *n* operations is the worst-case total cost of the operations divided by *n*.

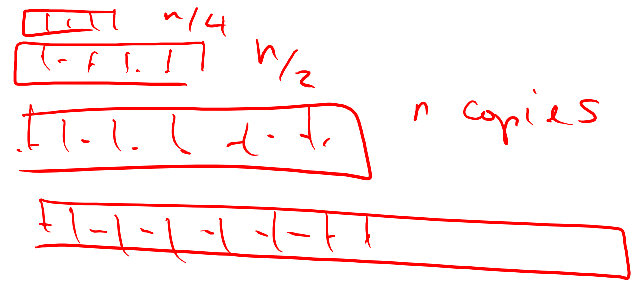
Shorthand:

If T(n) = worst-case (upper bound) of total cost

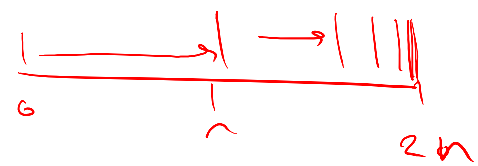
for n = number of operations

$\Rightarrow$ **Amortized Cost = T(n) / n**

Example: Array Stack



What's the amortized cost of calling `push()` *n* times if we double the array size when it's full?



n operations:

- *n* pushes at O(1) each -> total cost = n
- cost of resizing = $\underbrace{n + n/2 + n/4 + n/8 + \ldots}$ $\leq 2n$ -> total cost $\leq 2n$

-> $T(n) = n + 2n = 3n$

-> Amortized cost = $T(n)/n = 3n/n = 3$

-> Amortized Running time = O(1)

The **amortized cost** of *n* operations is the worst-case total cost of the operations divided by *n*.

Another Perspective: Paying and Saving “Currency”

A	B	C	D

1 operation costs us 1$ to the computer

A	B	C	D	E			

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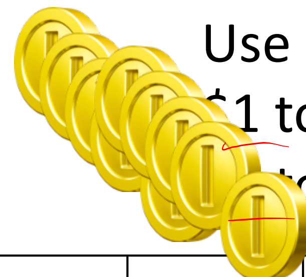
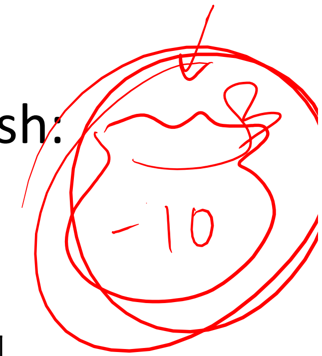
Another Perspective: Paying and Saving “Currency”

A	B	C	D



Use $2 for each push:
$1 to computer,
$1 to bank



A	B	C	D	E			

Potential Function

4

Spend our savings in the bank to resize. That way it only costs $1 to push(E)!

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Example #2: Queue made of Stacks

FIFO

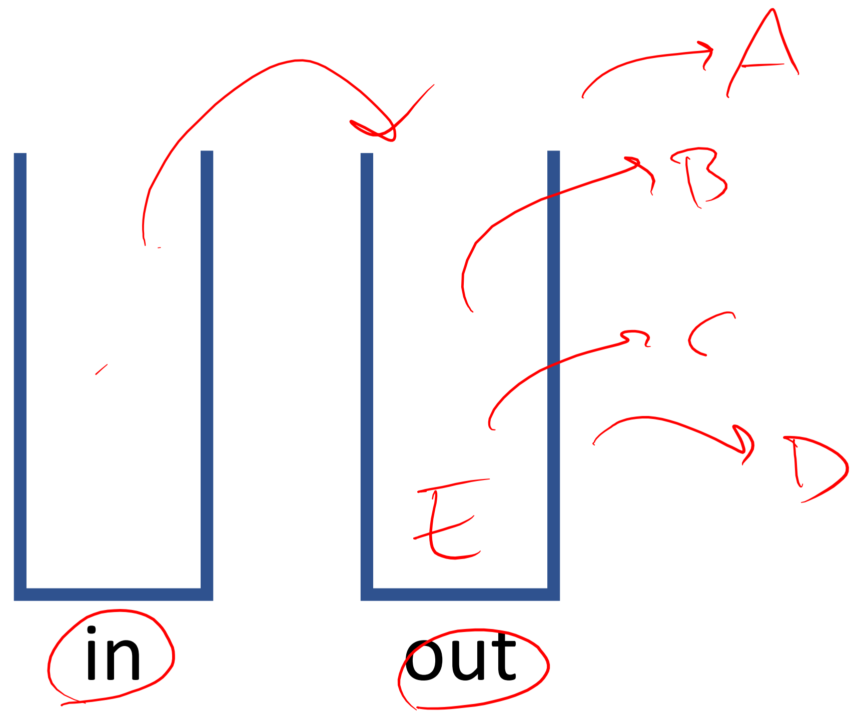
A B C D E

first

A sneaky way to implement Queue using two Stacks

Example walk-through:

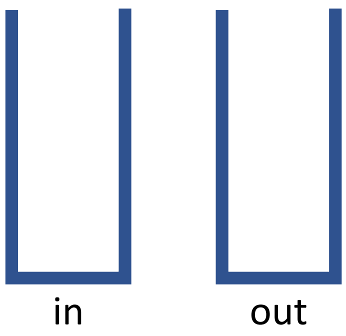
- enqueue A
- enqueue B
- enqueue C
- dequeue
- enqueue D
- enqueue E
- dequeue
- dequeue
- dequeue



Example #2: Queue made of Stacks

A sneaky way to implement Queue using two Stacks

```
class Queue<E> {
  Stack<E> in  = new Stack<E>();
  Stack<E> out = new Stack<E>();
  void enqueue(E x){ in.push(x); }
  E dequeue(){
    if(out.isEmpty()) {
      while(!in.isEmpty()) {
        out.push(in.pop());
      }
    }
    return out.pop();
  }
}
```

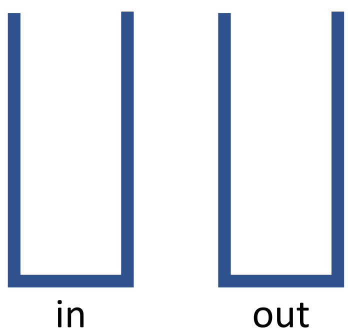


Wouldn’t it be nice to have a queue of t-shirts to wear instead of a stack (like in your dresser)?
So have two stacks

- *in*: stack of t-shirts go after you wash them
- *out*: stack of t-shirts to wear
- if *out* is empty, reverse *in* into *out*

Example #2: Queue made of Stacks (Analysis)

```
class Queue<E> {
  Stack<E> in  = new Stack<E>();
  Stack<E> out = new Stack<E>();
  void enqueue(E x){ in.push(x); }
  E dequeue(){
    if(out.isEmpty()) {
      while(!in.isEmpty()) {
        out.push(in.pop());
      }
    }
    return out.pop();
  }
}
```



Assume stack operations are (amortized) O(1).
What's the worst-case for `dequeue()`?

→ $O(n)$ (everything is in "in" "out" stack is empty)

What operations did we need to do to reach that condition (starting with an empty Queue)?

→ n enqueues (n pushes) n

Hence, what is the amortized cost?

$$\frac{O(n)}{n} = O(1)$$

So the average time for `dequeue()` is:

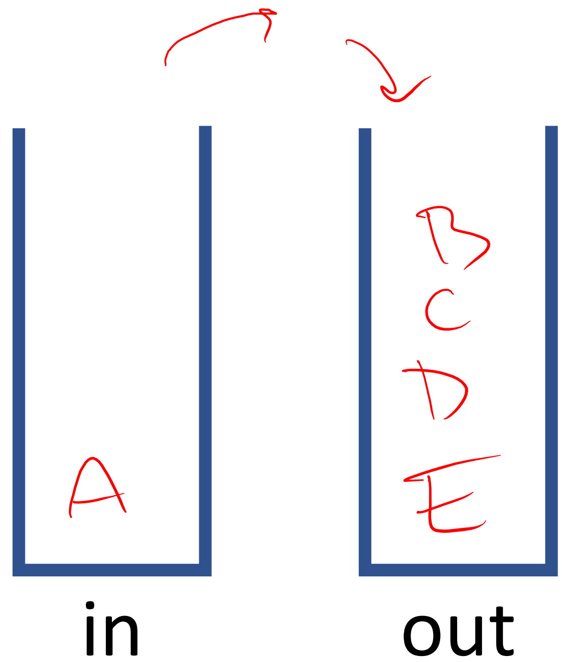
$O(1)$

Example #2: Using “Currency” Analogy

“Spend” $~~2~~ 3 for every `enqueue` – $1 to the “computer”, $~~1~~ 2 to the “bank”.

Example walk-through:

- enqueue A
- enqueue B
- enqueue C
- enqueue D
- enqueue E
- dequeue



Potential Function

Example #3: (Parody / Joke Example)

Lectures are 1 hour long, 3 times per week, so I'm supposed to lecture for 27 hours this quarter.

If I end the first 26 lectures 5 minutes early, then I'd have "saved up" 130 minutes worth of extra lecture time.

Then I could spend it all on the last lecture and can keep you here for 3 hours (bwahahahaha)!

(After all, each lecture would still be 1 hour amortized time)

Wrapping up Amortized Analysis

- In what cases do we care more about the average / amortized run time?

  Few worst-cases?

  If we want to be fast in general but occasion worst-case is not two costly

- In what cases do we care more about the worst-case run time?

  If n increases quickly?

  If wort-case is costly.

Taking a step back…

(Take a deep breath)

What have we covered so far?

- Abstract Data Types (**ADTs**)
- Two data structures
  - **Stacks** (both using arrays and linked-lists)
  - **Queues** (including **circular queues**)
- Asymptotic Analysis
  - Intuition for **Big-O**
  - Formally proving Big-O using **Inductive Proofs**
  - Calculating Big-O for recursive methods using **Recurrence Relations**
  - Big-O's cousins: **Big-Ω, Big-θ, little-o, little-ω**
  - Average running time using **Asymptotic Analysis**

Whew!

That was a *lot* of algorithm analysis.

Now shifting gears completely…
on to some new data structures!

Dictionary ADT

key, value pair

word definition

Dictionary ADT

Meaning

- set of (key, value) pairs
- can compare keys

Operations

- insert (key, value)
- delete (key)
- updateValue (key, newValue)
- find (key)

Uses of Dictionary ADT

Used to store information with some key and retrieve it efficiently – lots of programs do that!

Examples:

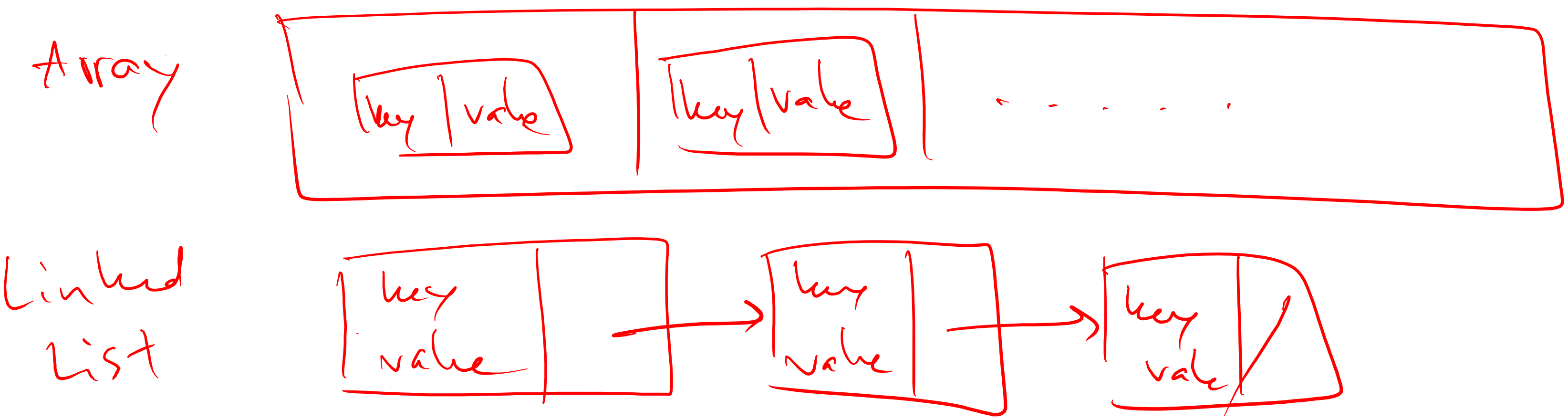
- Contacts in a phone (name: number, email)
- Orca/Husky cards (account number: balance)
- Genome maps (DNA sequence: location on genome)
- Lilian's database of your grades (student ID: assignments, grades)
- Networks (router tables), Operating Systems (page tables), Compilers (symbol tables), Databases
- … and so much more!

Possibly the most widely used ADT!

Motivating Hash Tables

Creative thinking time: how could you implement a dictionary using what you know so far (namely, linked-lists and arrays)?

e.g. map names (key) to phone numbers (value)



Motivating Hash Tables

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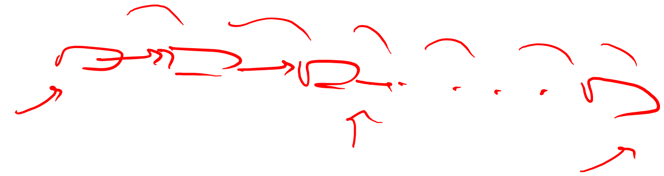
Motivating Hash Tables



Running times for Dictionary operations with *n* (key, value) pairs:

	`insert`	`find`	`delete`
Array	average $O(1)$ worst-case $O(n)$	$O(n)$	$O(n)$
Linked-List	$O(1)$	$O(n)$	$O(n)$
sorted array	$O(n)$	$O(\log n)$	$O(n)$
"Magic Array"	$O(1)$	$O(1)$	$O(1)$

“Magic Array”

computeIndex(Jon) → 3
array[3]

Use key to compute array index for an item in O(1) time

Example: phone contacts (name, number)

name → `index` = $\boxed{\texttt{computeIndex}}$(name) → `array[index]` = (name, number)

(Jon, 123-4567) → index = 3 → array[3] = (Jon, 123-4567)
(Gregor, 765-4321) → index = 1 → array[1] = (Gregor, 765-4321)
(Cersei, 111-1111) → index = 1 →

0	1	2	3	4	5
	Gregor 765-4321		Jon 123-4567		

“Magic Array”

Use key to compute array index for an item in O(1) time

Example: phone contacts (name, number)

name → `index` = `computeIndex(`name`)` → `array[index]` = (name, number)

key

What would be important about the indices from `computeIndex`?

Have different index for every key

~~magic!~~

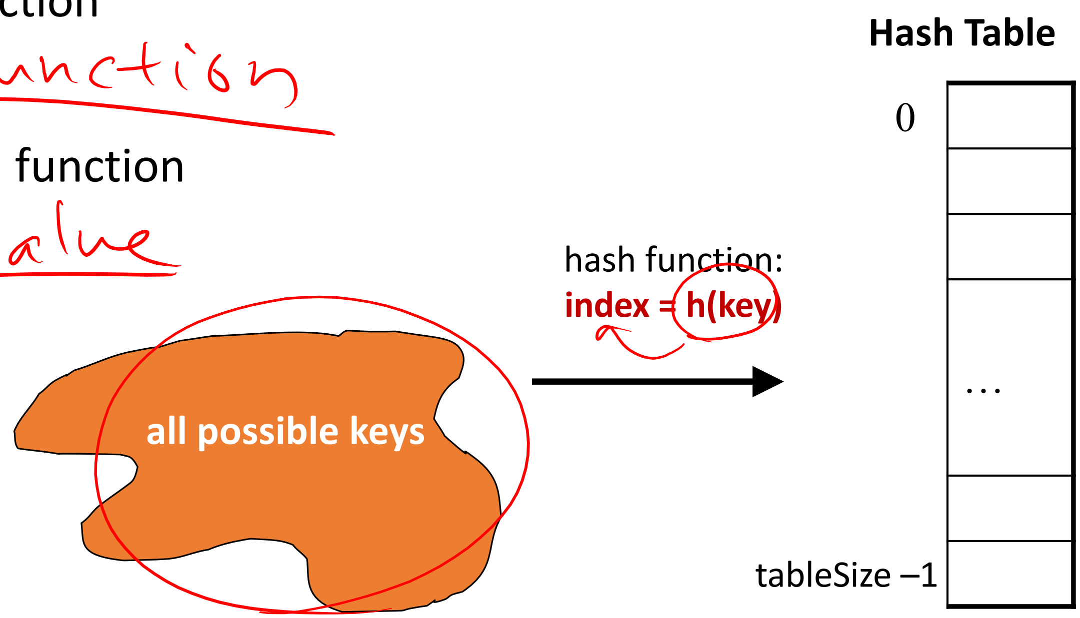
Introducing… Hash Tables!

Closest thing to our “magic array”

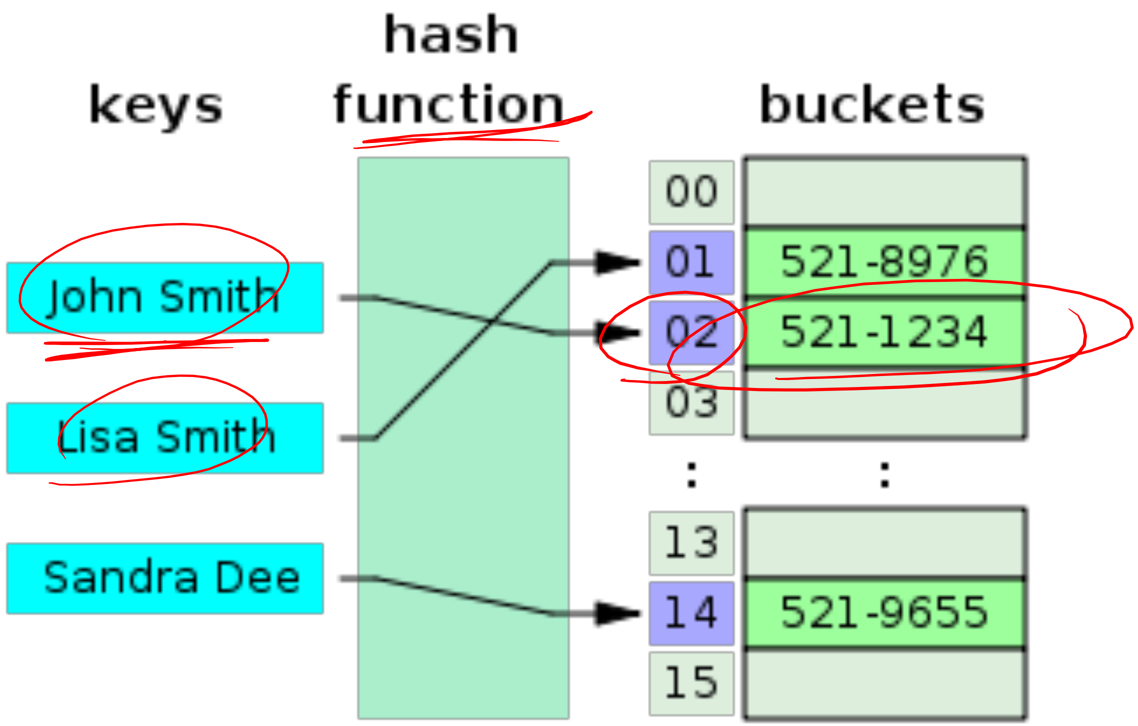
Hash Tables: closest thing to our "Magic Array"

- *Average case* O(1) `find`, `insert`, and `delete`
  (when under some often-reasonable *assumptions*)
- Our `computeIndex` function is called a hash function
- The `index` from the hash function is called a hash value

(also hash)



Hash Tables: Example Illustration



key = Cersei   h(Cersei) = same number every time

Hash Functions

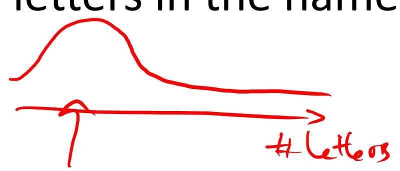
Hash functions need to…

- have uniformity (maps inputs as evenly as possible)
- be deterministic (always same hash for same key)
- O(1)

For a person's name, would it be a good hash function to…

- Use the ASCII values of first and second letter? → Joe, Joel, John . . .
  ZX WT
- Use the number of letters in the name?

1 letter rare



542 letters rare

Example Hash Function

Hash function “djb2”:

```
unsigned long
hash(unsigned char *str)
{
    unsigned long hash = 5381;
    int c;

    while (c = *str++)
        hash = ((hash << 5) + hash) + c; /* hash * 33 + c */

    return hash;
}
```

Hash Functions

- Many datatypes and Objects are hashable
- When writing a class, can make it hashable!

    Do so by implementing `hashCode` method

    ↳ want it to return (int) an index as unique as possible for any object

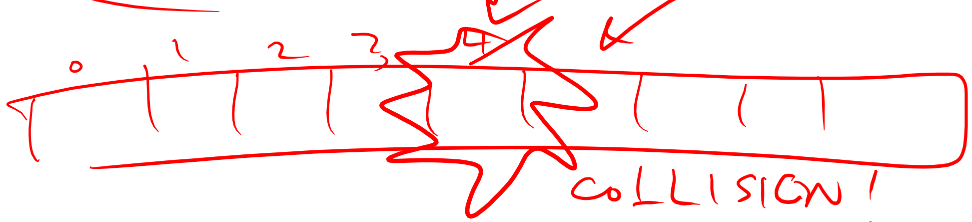
- We'll focus on `int`s and `String`s in this class.

Collisions

Happens when two elements get the same index (unavoidable in practice)

h(tyrion) = 4

h(sansa) = 4



insert(tyrion, ~)

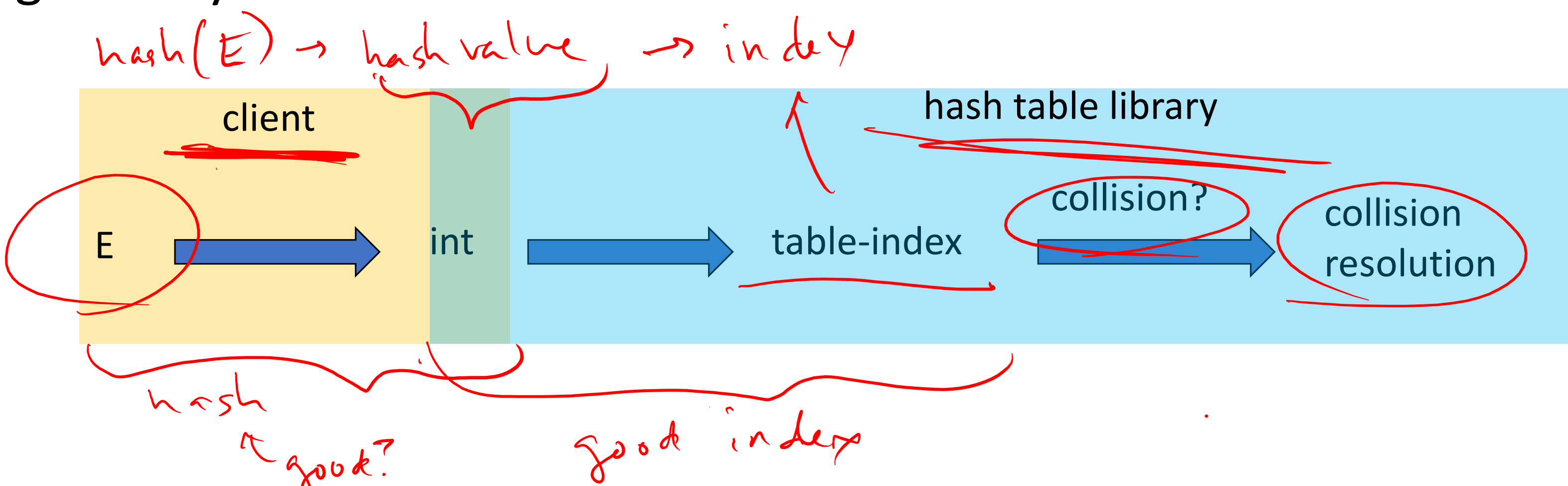
insert(sansa, ~)

$\boxed{\text{find(tyrion)}}$

Homework: come up with a strategy, write it down on paper, and bring it to class on Weds

Hash Table roles

When hash tables are a reusable library, the division of responsibility generally breaks down into two roles:



We will learn both roles, but most programmers "in the real world" spend more time as clients while understanding the library

Hash Tables

- There are ***m*** possible keys (*m* typically large, even infinite)
- We expect our table to have only ***n*** items
- ***n*** is much less than ***m*** (often written ***n << m***)

Many dictionaries have this property

- Compiler: All possible identifiers allowed by the language vs. those used in some file of one program
- Database: All possible student names vs. students enrolled
- AI: All possible chess-board configurations vs. those considered by the current player
- …

hash function → hash value → index

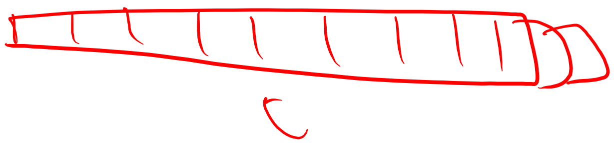
Hash Table Size

$x \% y$ = remainder of $\frac{x}{y}$

$3 \% 10 \rightarrow \frac{3}{10}$ = 0 remainder 3

- How can we keep hash values (i.e. the indices) within the table size?

e.g. tableSize = 10



index = hash(key) % table size

h(key1) = 3 → index = 3
h(key2) = 17 → index = 7
h(key3) = 7 → index = 7 } collision!

- Table size usually prime
  - Real-life data tends to have a pattern
  - "Multiples of 61" probably less likely than "multiples of 60"
  - Helpful for a collision-handling strategy we'll see next week

3
16
7

Review: Hash Tables thus far…

- The hash table is one of the most important data structures – Supports only `find`, `insert`, and `delete` efficiently average $O(1)$
  - Have to search entire table for other operations ← not as efficient
- Important to use a good hash function
- Important to keep hash table at a good size
- Side-comment: hash functions have uses beyond hash tables – Examples: Cryptography, check-sums
- Big remaining topic: Handling collision

Homework

Come up with a collision-resolution strategy, write it down on paper, and bring it to class on Wednesday

Goal: prime our brains for learning the most common collision-resolution strategies.