CSE 373: Data Structures and Algorithms

Lecture 5: Finishing up Asymptotic Analysis

Big-O, Big-Ω, Big-θ, little-o, little-ω & Amortized Analysis

Instructor: Lilian de Greef

Quarter: Summer 2017

Today:

- Announcements
- Big-O and Cousins
  - Big-Omega
  - Big-Theta
  - little-o
  - little-omega
- Average running time:
  Amortized Analysis

News about Sections

Updated times:

- **10:50** – 11:50am
- 12:00 – **1:00**pm

Bigger room!

- 10:50am section now in **THO 101**

Which section to attend:

- Last week, section sizes were unbalanced (~40 vs ~10 people)
- If you can, I encourage you to choose the 12:00 section to rebalance sizes
  - Helps the 12:00 TA's feel less lonely
  - More importantly: improves TA:student ratio in sections (better for tailoring section to your needs)

Homework 1

- Due today at 5:00pm!

- A note about grading methods:
  - Before we grade, we'll run a script on your code to replace your name with `### anonymized ###` so we won't know who you are as we grade it (to address unconscious bias).
  - It's still good practice to have your name and contact info in the comments!

Homework 2

- Written homework about asymptotic analysis (no Java this time)
- Will be out this evening
- Due Thursday, July 6th at 5:00pm
  - Because July 4th is a holiday

- A note for help on homework:
  - Note that holidays means fewer office hours
  - Remember: although you cannot share solutions, you can talk to classmates about concepts or work through non-homework examples (e.g. from section) together.
  - Give these classmates credit, write their names at the top of your homework.

Formal Definition of Big-O

Definition: f(n) is in **O(** g(n) **)** if there exist constants
c and n_0 such that f(n) ≤ ***c*** g(n) for all $n \geq n_0$

More Practice with the Definition of Big-O

Let a(n) = 10n+3n^2 and b(n) = n^2

What are some values of ***c*** and n_0
we can use to show a(n)$\in$O(b(n))?

Definition: f(n) is in **O(** g(n) **)** if there exist constants
c and n_0 such that f(n) $\leq$ ***c*** g(n) for all $n \geq n_0$

Constants and Lower Order Terms

- The constant multiplier ***c*** is what allows functions that differ only in their largest coefficient to have the same asymptotic complexity

  Example:

- Eliminate lower-order terms because

- Eliminate coefficients because
  - $3n^2$ vs $5n^2$ is meaningless without the cost of constant-time operations
  - Can always re-scale anyways
  - Do not ignore constants that are not multipliers! n^3 is not $O(n^2)$, 3^n is not $O(2^n)$

Analyzing "Worst-Case" Cheat Sheet

Basic operations take "some amount of" constant time

- Arithmetic (fixed-width)
- Assignment
- Access one Java field or array index
- *etc.*

(This is an *approximation* of reality: a very useful "lie")

Control Flow	Time Required
Consecutive statements	Sum of time of statement
Conditionals	Time of test plus slower branch
Loops	Sum of iterations * time of body
Method calls	Time of call's body
Recursion	Solve *recurrence relation*

Big-O & Big-Omega

Big-O:

f(n) is in **O(** g(n) **)** if there exist constants ***c*** and n_0 such that

f(n) $\leq$ ***c*** g(n) for all $n \geq n_0$

Big-Ω:

f(n) is in **Ω (** g(n) **)** if there exist constants ***c*** and n_0 such that

f(n) $\geq$ ***c*** g(n) for all $n \geq n_0$

Big-Theta

Big-θ:

f(*n*) is in **θ(** g(*n*) **)** if f(n) is in both O(g(*n*)) *and* Ω (g(*n*))

little-o & little-omega

little-o:

f(n) is in **o(** g(n) **)** if
constants ***c*** >0 there exists an n_0
s.t. f(n) ***c*** g(n) for all $n \geq n_0$

little-ω:

f(n) is in **ω(** g(n) **)** if
constants ***c*** >0 there exists an n_0
s.t. f(n) ***c*** g(n) for all $n \geq n_0$

Practice Time!

Let $f(n) = 75n^3 + 2$ and $g(n) = n^3 + 6n + 2n^2$
Then f(*n*) is in… (choose all that apply)

A. Big-O(g)
B. Big-Ω(g)
C. θ(g)
D. little-o(g)
E. little-ω(g)

Second Practice Time!

Let $f(n) = 3^n$ and $g(n) = n^3$
Then $f(n)$ is in… (choose all that apply)

A. Big-O(g)
B. Big-Ω(g)
C. θ(g)
D. little-o(g)
E. little-ω(g)

Big-O, Big-Omega, Big-Theta

- Which one is more useful to describe asymptotic behavior?

- A common error is to say $O(\ f(n)\ )$ when you mean $\theta(\ f(n)\ )$
  - A linear algorithm is in both O(n) and $O(n5)$
  - Better to say it is $\theta(n)$
  - That means that it is not, for example $O(\mathbf{log}\ n)$

Comments on Asymptotic Analysis

- Is choosing the lowest Big-O or Big-Theta the best way to choose the fastest algorithm?

- Big-O can use other variables (e.g. can sum all of the elements of an n-by-m matrix in O(nm))

Amortized Analysis

How we calculate the average time!

Case Study: the Array Stack

What's the **worst-case** running time of `push()`?

What's the **average** running time of `push()`?

Calculating the average: not based off of running a *single operation*,
but running *many operations in sequence*.

Technique: **Amortized Analysis**

Amortized Cost

The **amortized cost** of *n* operations is the worst-case total cost of the operations divided by *n*.

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The **amortized cost** of n operations is the worst-case total cost of the operations divided by n.

Practice:

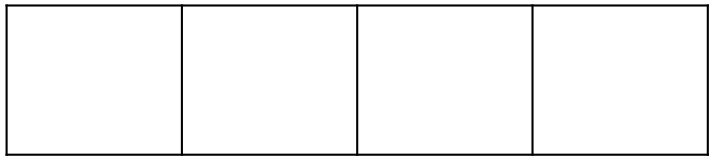
- n operations taking $O(n)$ → amortized cost =
- n operations taking $O(n^3)$ → amortized cost =
- n operations taking $O(n\ f(n))$ → amortized cost =

Example: Array Stack

What's the amortized cost of calling `push()` *n* times
if we double the array size when it's full?

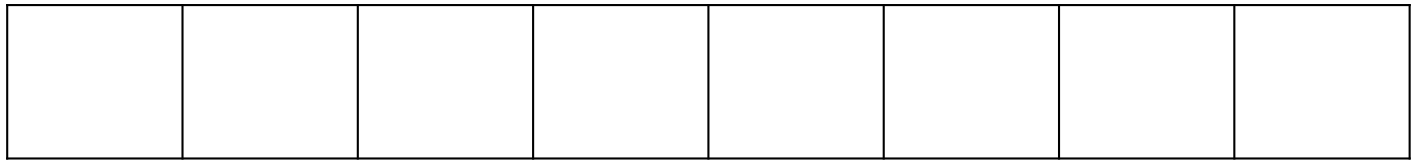
The **amortized cost** of *n* operations is the worst-case total cost of the operations divided by *n*.

Another Perspective: Paying and Saving "Currency"



1 operation costs 1$ to the computer

Use $2 for each push:
$1 to computer, $1 to bank



Potential Function

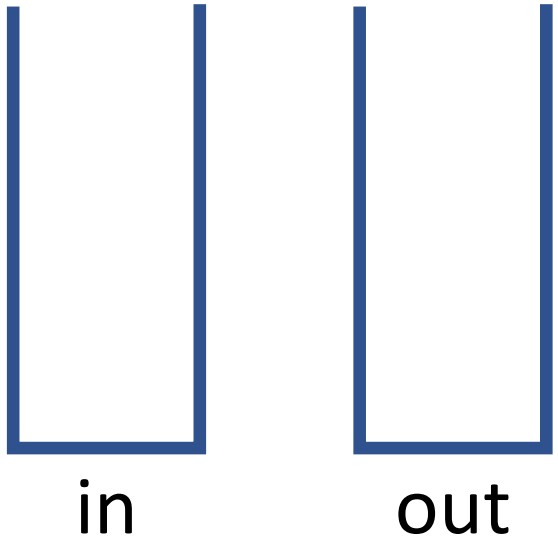
Spend our savings in the bank to resize. That way it only costs $1 extra to push(E)!

Example #2: Queue made of Stacks

A sneaky way to implement Queue using two Stacks

Example walk-through:

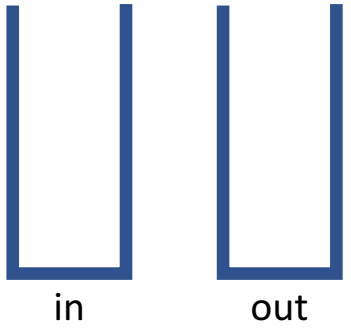
- enqueue A
- enqueue B
- enqueue C
- dequeue
- enqueue D
- enqueue E
- dequeue
- dequeue
- dequeue



Example #2: Queue made of Stacks

A sneaky way to implement Queue using two Stacks

```
class Queue<E> {
  Stack<E> in  = new Stack<E>();
  Stack<E> out = new Stack<E>();
  void enqueue(E x){ in.push(x); }
  E dequeue(){
    if(out.isEmpty()) {
      while(!in.isEmpty()) {
        out.push(in.pop());
      }
    }
    return out.pop();
  }
}
```

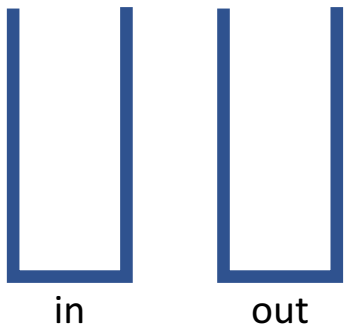


Wouldn't it be nice to have a queue of t-shirts to wear instead of a stack (like in your dresser)?
So have two stacks

- *in*: stack of t-shirts go after you wash them
- *out*: stack of t-shirts to wear
- if *out* is empty, reverse *in* into *out*

Example #2: Queue made of Stacks (Analysis)

```
class Queue<E> {
  Stack<E> in  = new Stack<E>();
  Stack<E> out = new Stack<E>();
  void enqueue(E x){ in.push(x); }
  E dequeue(){
    if(out.isEmpty()) {
      while(!in.isEmpty()) {
        out.push(in.pop());
      }
    }
    return out.pop();
  }
}
```



Assume stack operations are (amortized) O(1).
What's the worst-case for `dequeue()`?

What operations did we need to do to reach that condition (starting with an empty Queue)?

Hence, what is the amortized cost?

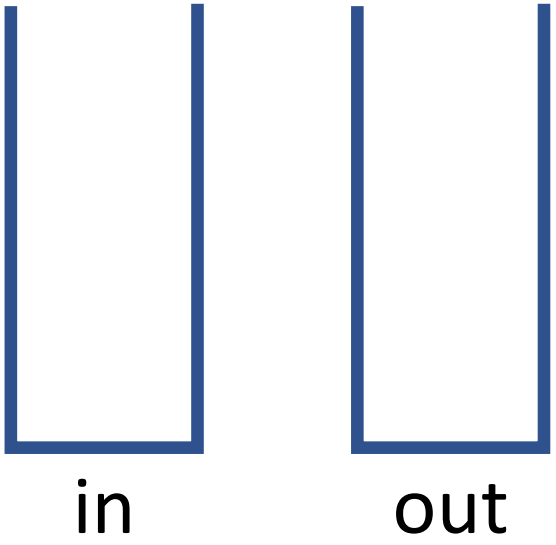
So the average time for `dequeue()` is:

Example #2: Using "Currency" Analogy

"Spend" 2 coins for every `enqueue` – 1 to the "computer", 1 to the "bank".

Example walk-through:

- enqueue A
- enqueue B
- enqueue C
- enqueue D
- enqueue E
- dequeue



Potential Function

Example #3: (Parody / Joke Example)

Lectures are 1 hour long, 3 times per week, so I'm supposed to lecture for 27 hours this quarter.

If I end the first 26 lectures 5 minutes early, then I'd have "saved up" 130 minutes worth of extra lecture time.

Then I could spend it all on the last lecture and can keep you here for 3 hours (bwahahahaha)!

(After all, each lecture would still be 1 hour amortized time)

Wrapping up Amortized Analysis

- In what cases do we care more about the average / amortized run time?

- In what cases do we care more about the worst-case run time?