CSE 373: Data Structures and Algorithms

Lecture 4: Asymptotic Analysis part 3
Code Style, Recurrence Relations, Formal Big-O & Cousins

Instructor: Lilian de Greef
Quarter: Summer 2017

Code Style

Why does code style matter?

Code Style

Do

Don’t

Code Style Critique

```
import java.util.Arrays;
public boolean function(int n) {
    boolean[] p = new boolean[10000];
    Arrays.fill(p,true);
    p[0]=p[1]=false;
    for (int i=2;i<p.length;i++) {
        if(p[i]) {
            for (int j=2;i*j<p.length;j++) {
                p[i*j]=false;
            }
        }
    }
    return p[n];
}
```

Code Style Critique #2

```
// Tells you whether a number is prime.
public boolean isPrime(int n) {
    // Make an array.
    boolean[] primes = new boolean[10000];
    // Fill the array with the value "true"
    // except for the first two indices.
    Arrays.fill(primes,true);
    primes[0]=primes[1]=false;

    // Loop over the array. As you do, check
    // if the current array value is true.
    // If it is, loop over the rest of the array
    // in increments of that current value
    // and set those indices to "false".
    for (int i=2;i<primes.length;i++) {
        if (primes[i]) {
            for (int j=2;i*j<primes.length;j++) {
                primes[i*j]=false;
            }
        }
    }

    return primes[n];
}
```

Code Style Critique #3

```
// Returns whether a given number is prime.
// Assumes number is less than 10000.
public boolean isPrime(int n) {

    // Assume all numbers are prime.
    boolean[] primes = new boolean[10000];
    Arrays.fill(primes,true);

    // We know 0 and 1 are not prime.
    primes[0] = false;
    primes[1] = false;

    // Eliminate numbers that are not prime
    // using the Sieve of Eratosthenes.
    for (int i=2; i<primes.length; i++) {

        // If the current number is prime, flag
        // all of its multiples as not prime.
        if (primes[i]) {
            for (int j=2; i*j<primes.length; j++) {
                primes[i*j] = false;
            }
        }
    }

    return primes[n];
}
```

Code Style Critique #4

```java
// Constants and data members
static final int MAX_PRIME = 10000;
private boolean[] primes = new boolean[MAX_PRIME];

// An implementation of the Sieve of Eratosthenes.
// Fills our array of primes with "true" or "false"
// to match whether the index is prime.
public void fillSieve() {

    // Assume all numbers are prime.
    Arrays.fill(primes,true);

    // We know 0 and 1 are not prime.
    primes[0] = false;
    primes[1] = false;

    // Eliminate numbers that are not prime.
    for (int i=2; i<primes.length; i++) {

        // If the current number is prime, flag
        // all of its multiples as not prime.
        if (primes[i]) {
            for (int j=2; i*j<primes.length; j++) {
                primes[i*j] = false;
            }
        }
    }
}

// Returns whether a given number is prime.
// Assumes number is less than the class's maximum.
public boolean isPrime(int n) {
    return primes[n];
}
```

Recurrence Relations

How to calculate Big-O for recursive functions!

(Continued from last lecture)

Example #1: Towers of Hanoi

```java
// Prints instructions for moving disks from one
// pole to another, where the three poles are
// labeled with integers "from", "to", and "other".
// Code from rosettacode.org
public void move(int n, int from, int to, int other) {
    if (n == 1) {
        System.out.println("Move disk from pole " + from +
                           " to pole " + to);}
    else {
        move(n - 1, from, other, to);
        move(1, from, to, other);
        move(n - 1, other, to, from);
    }
}
```

Example #1: Towers of Hanoi

```
if (n == 1) {
    System.out.println("Move disk from pole " + from +
                       " to pole " + to);}



else {
    move(n - 1, from, other, to);
    move(1, from, to, other);
    move(n - 1, other, to, from);
}
```

Base Case:

Recurrence Relation:

(Example #1 continued)

(Example #1 continued)

Example #2: Binary Search

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

(Can also be done non-recursively)

```
// Requires the array to be sorted.
// Returns whether k is in array.
public boolean find(int[]arr, int k){
    return help(arr,k,0,arr.length);
}
private boolean help(int[]arr, int k, int lo, int hi) {
    int mid = (hi+lo)/2; // i.e., lo+(hi-lo)/2
    if(lo==hi)          return false;
    if(arr[mid]==k)     return true;
    if(arr[mid]< k)     return help(arr,k,mid+1,hi);
    else                return help(arr,k,lo,mid);
```

What is the recurrence relation?

```
// Requires the array to be sorted.
// Returns whether k is in array.
public boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
private boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2;   // i.e., lo+(hi-lo)/2
   if(lo==hi)         return false;
   if(arr[mid]==k)   return true;
   if(arr[mid]< k)   return help(arr,k,mid+1,hi);
   else              return help(arr,k,lo,mid);
```

A. 2T(n-1) + 3

B. T(n-1)*T(n-1) + 3

C. T(n/2) + 3

D. T(n/2) * T(n/2) + 3

Base Case:

Recurrence Relation:

(Example #2 continued)

(Example #2 continued)

Recap: Solving Recurrence Relations

1. Determine the recurrence relation. What is the base case?
   - $T(n) = 3 + T(n/2)$ $T(1) = 3$
2. "Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.
   - $T(n) = 3 + 3 + T(n/4)$
     $= 3 + 3 + 3 + T(n/8)$
     $= \ldots$
     $= 3k + T(n/(2^k))$
3. Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case
   - $n/(2^k) = 1$ means $n = 2^k$ means $k = \log_2 n$
   - So $T(n) = 10 \log_2 n + 8$ (get to base case and do it)
   - So $T(n)$ is $O(\log n)$

Common Recurrence Relations

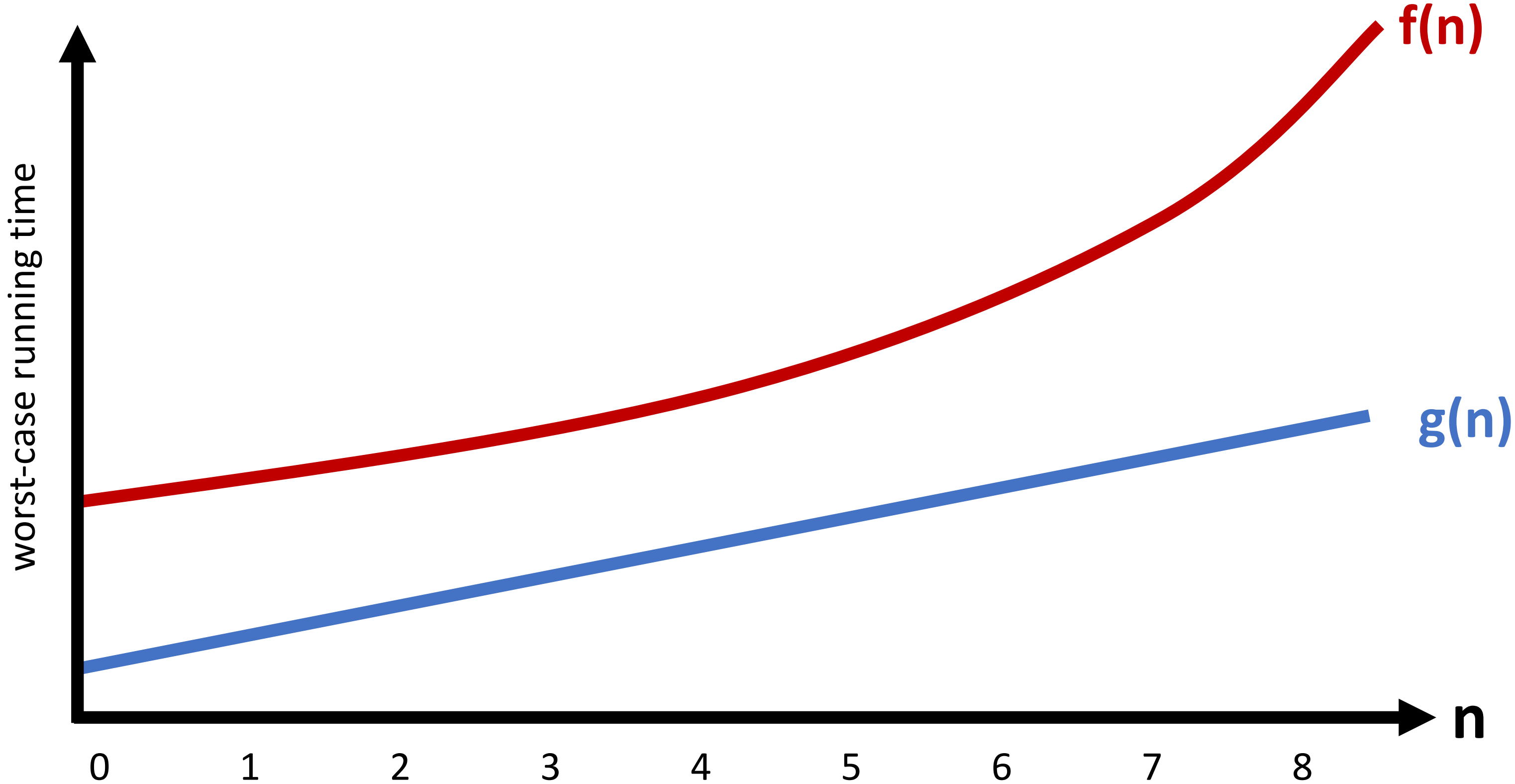
Should know how to solve recurrences but helps to recognize some common ones:

$T(n) = O(1) + T(n\text{-}1)$	linear
$T(n) = O(1) + 2T(n/2)$	linear
$T(n) = O(1) + T(n/2)$	logarithmic $O($**`log`** n)
$T(n) = O(1) + 2T(n\text{-}1)$	exponential
$T(n) = O(n) + T(n\text{-}1)$	quadratic
$T(n) = O(n) + T(n/2)$	linear (why?)
$T(n) = O(n) + 2T(n/2)$	$O($n **`log`** n)

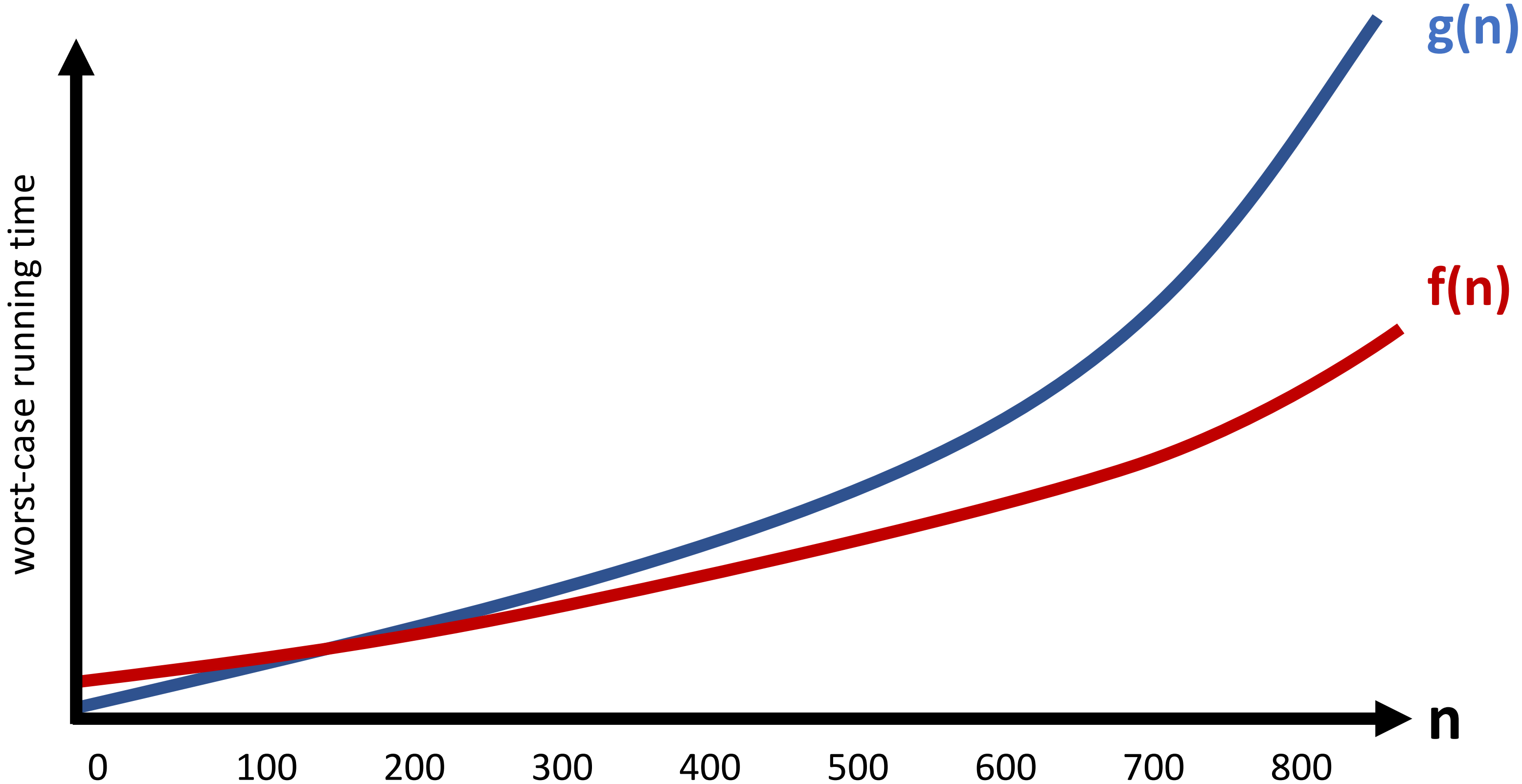
Big-O Big Picture

with its formal definition

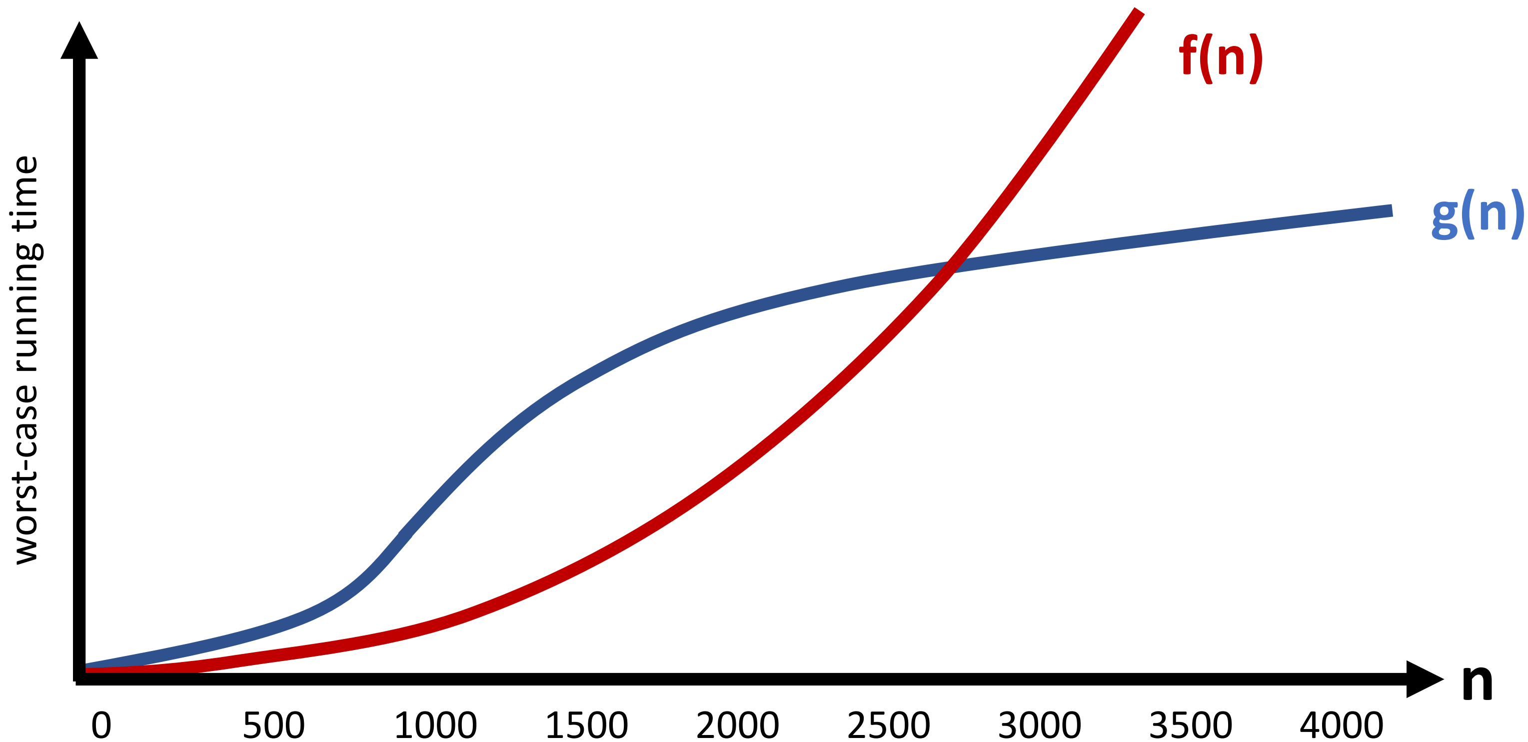
In terms of Big-O, which function has the faster asymptotic running time?



In terms of Big-O, which function has the faster asymptotic running time?



In terms of Big-O, which function has the faster asymptotic running time?



Take-away:

Formal Definition of Big-O

"General Idea" explanation from last week:

Mathematical upper bound describing the behavior of how long a function takes to run in terms of N. (The "shape" as $N \rightarrow \infty$)

Formal definition of Big-O:

Formal Definition of Big-O

Using the Formal Definition of Big-O

Definition: f(n) is in **O(g(n))** if there exist constants c and n_0 such that f(n) $\leq$ c g(n) for all $n \geq n_0$

To show f(n) is in **O(g(n))**, pick a c large enough to "cover the constant factors" and n_0 large enough to "cover the lower-order terms"

Example:
Let f(n) = $3n^2$+18 and g(n) = n^2

Example:
Let f(n) = $3n^2$+18 and g(n) = n^5

Practice with the Definition of Big-O

Let $f(n) = 1000n$ and $g(n) = n^2$

What are some values of ***c*** and $\boldsymbol{n_0}$
we can use to show $f(n) \in O(g(n))$?

Definition: $f(n)$ is in $\mathbf{O(} g(n) \mathbf{)}$ if there exist constants ***c*** and $\boldsymbol{n_0}$ such that $f(n) \leq c\, g(n)$ for all $n \geq n_0$

More Practice with the Definition of Big-O

Let a(n) = $10n+3n^2$ and b(n) = n^2

What are some values of ***c*** and n_0
we can use to show a(n)∈O(b(n))?

Definition: f(n) is in **O(** g(n) **)** if there exist constants ***c*** and n_0 such that f(n) ≤ ***c*** g(n) for all $n \geq n_0$

Constants and Lower Order Terms

- The constant multiplier ***c*** is what allows functions that differ only in their largest coefficient to have the same asymptotic complexity
  Example:

- Eliminate lower-order terms because

- Eliminate coefficients because
  - $3n^2$ vs $5n^2$ is meaningless without the cost of constant-time operations
  - Can always re-scale anyways
  - Do not ignore constants that are not multipliers! n^3 is not $O(n^2)$, 3^n is not $O(2^n)$

Cousins of Big-O

Big-O, Big-Omega, Big-Theta, little-o, little-omega

Big-O & Big-Omega

Big-O:

f(n) is in **O(** g(n) **)** if there exist constants c and n_0 such that f(n) c g(n) for all $n \geq n_0$

Big-Ω:

f(n) is in **Ω (** g(n) **)** if there exist constants c and n_0 such that f(n) c g(n) for all $n \geq n_0$

Big-Theta

Big- θ:

f(n) is in **θ(** g(n) **)** if f(n) is in both O(g(n)) *and* Ω (g(n))

little-o & little-omega

little-o:

f(n) is in **o(** g(n) **)** if
constants c >0 there exists an n_0
s.t. f(n) c g(n) for all $n \geq n_0$

little-ω:

f(n) is in **ω(** g(n) **)** if
constants c >0 there exists an n_0
s.t. f(n) c g(n) for all $n \geq n_0$

Big-O, Big-Omega, Big-Theta

- Which one is more useful to describe asymptotic behavior?

- A common error is to say $O(f(n))$ when you mean $\theta(f(n))$
  - A linear algorithm is in both O(n) and $O(n5)$
  - Better to say it is $\theta(n)$
  - That means that it is not, for example $O(\mathbf{log}\ n)$

Notes on Worst-Case Analysis

Analyzing “Worst-Case” Cheat Sheet

Basic operations take “some amount of” constant time

- Arithmetic (fixed-width)
- Assignment
- Access one Java field or array index
- *etc.*

(This is an *approximation* of reality: a very useful “lie”)

Control Flow	Time Required
Consecutive statements	Sum of time of statement
Conditionals	Time of test plus slower branch
Loops	Sum of iterations * time of body
Method calls	Time of call’s body
Recursion	Solve *recurrence relation*

Comments on Asymptotic Analysis

- Is choosing the lowest Big-O or Big-Theta the best way to choose the fastest algorithm?

- Big-O can use other variables (e.g. can sum all of the elements of an n-by-m matrix in O(nm))