CSE 373: Data Structures and Algorithms

Lecture 4: Asymptotic Analysis part 3
Code Style, Recurrence Relations, Formal Big-O & Cousins

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Today:

- Code Style
- Recurrence Relations
- Formal Definition of Big-O
- Cousins of Big-O
    - Big-Omega
    - Big-Theta
    - little-o
    - little-omega

Code Style

Code Style

Why does code style matter?

Easier for others to read & understand

" " future you " "

" to debug

Full credit on assignments (courses)

Code Style

Do

Nice comments

Good variable names

- descriptive
- concise

Proper spacing

- indentation

Don't

over comments

Magic numbers

Boolean zen

```
if (____)
    return true
else
    return false
```

return (____)

Code Style Critique

```
import java.util.Arrays;
public boolean function(int n) {
    boolean[] p = new boolean[10000];
    Arrays.fill(p,true);
    p[0]=p[1]=false;
    for (int i=2;i<p.length;i++) {
        if(p[i]) {
            for (int j=2;i*j<p.length;j++) {
                p[i*j]=false;
            }
        }
    }
    return p[n];
}
```

space :)

:(

No comments

Ambiguous function name,

" variable names

spaces

potentially unnessary (unclear if nec.)

what's the goal here!?

Code Style Critique #2

```
// Tells you whether a number is prime.
public boolean isPrime(int n) {
    // Make an array.
    boolean[] primes = new boolean[10000];
    // Fill the array with the value "true"
    // except for the first two indices.
    Arrays.fill(primes,true);
    primes[0]=primes[1]=false;

    // Loop over the array. As you do, check
    // if the current array value is true.
    // If it is, loop over the rest of the array
    // in increments of that current value
    // and set those indices to "false".
    for (int i=2;i<primes.length;i++) {
        if (primes[i]) {
            for (int j=2;i*j<primes.length;j++) {
                primes[i*j]=false;
            }
        }
    }

    return primes[n];
}
```

???

has Comments :)

verbose comments :(

better fn name :)

Redundant comments :(

Repeats code :(

(useless, takes up space)

Code Style Critique #3

```
// Returns whether a given number is prime.
// Assumes number is less than 10000.
public boolean isPrime(int n) {

    // Assume all numbers are prime.
    boolean[] primes = new boolean[10000];
    Arrays.fill(primes,true);

    // We know 0 and 1 are not prime.
    primes[0] = false;
    primes[1] = false;

    // Eliminate numbers that are not prime
    // using the Sieve of Eratosthenes.
    for (int i=2; i<primes.length; i++) {

        // If the current number is prime, flag
        // all of its multiples as not prime.
        if (primes[i]) {
            for (int j=2; i*j<primes.length; j++) {
                primes[i*j] = false;
            }
        }
    }

    return primes[n];
}
```

Comments :)

- more concise
- explain intent

"verb ______"

convoluted :(

- 1000 could be n
- repeated do same large calculation every method call

Code Style Critique #4

```
// Constants and data members
static final int MAX_PRIME = 10000;
private boolean[] primes = new boolean[MAX_PRIME];

// An implementation of the Sieve of Eratosthenes.
// Fills our array of primes with "true" or "false"
// to match whether the index is prime.
public void fillSieve() {

    // Assume all numbers are prime.
    Arrays.fill(primes,true);

    // We know 0 and 1 are not prime.
    primes[0] = false;
    primes[1] = false;

    // Eliminate numbers that are not prime.
    for (int i=2; i<primes.length; i++) {

        // If the current number is prime, flag
        // all of its multiples as not prime.
        if (primes[i]) {
            for (int j=2; i*j<primes.length; j++) {
                primes[i*j] = false;
            }
        }
    }
}

// Returns whether a given number is prime.
// Assumes number is less than the class's maximum.
public boolean isPrime(int n) {
    return primes[n];
}
```

uses constants ☺
(instead Magic Numbers!)

Improvement:
Instead of having constant max prime, use variable to update!

1000

don't need to write hard to maintain comment

Recurrence Relations

How to calculate Big-O for recursive functions!

(Continued from last lecture)

Example #1: Towers of Hanoi

Big-O of a recursive method

```
// Prints instructions for moving disks from one
// pole to another, where the three poles are
// labeled with integers "from", "to", and "other".
// Code from rosettacode.org
public void move(int n, int from, int to, int other) {
    if (n == 1) {
        System.out.println("Move disk from pole " + from +
                           " to pole " + to);}
    else {
        move(n - 1, from, other, to);
        move(1, from, to, other);
        move(n - 1, other, to, from);
    }
}
```

Example #1: Towers of Hanoi

\# executions for method = $H(n)$

goal: closed form

Base Case: @ $n=1$

```
if (n == 1) {
    System.out.println("Move disk from pole " + from +
                       " to pole " + to);}
```

1 execution $\Rightarrow H(1) = 1$

Recursive Step:

```
else {
    move(n - 1, from, other, to);
    move(1, from, to, other);
    move(n - 1, other, to, from);
}
```

$H(n-1)$ exections

$H(1) = 1$

$H(n-1)$

All together: $H(n) = H(n-1) + H(1) + H(n-1)$

$$\boxed{H(n) = 1 + 2H(n-1)}$$

Base Case: $H(1) = 1$

Recurrence Relation: $\boxed{H(n) = 1 + 2H(n-1)}$

$(n-1)$

Expanding (plug in for $H(n)$):

1st $H(n) = 1 + 2H(n-1) = 1 + 2\left[1 + 2H((n-1)-1)\right]$

2nd $= 1 + 2 + 4H(n-2)$ plug in $n-2$ into $H(n)$

3rd $= 1 + 2 + 4 + 8H(n-3)$

$\vdots$

kth $= 2^0 + 2^1 + 2^2 + \cdots + 2^{k-1} + 2^k H(n-k)$

$$\sum_{i=0}^{k-1} 2^i = 2^{(k-1)+1} - 1$$

$$= 2^k - 1$$

(Example #1 continued)

k^{th} expansion $\quad H(n) = 2^k - 1 + 2^k H(\underbrace{n-k})$

$$\boxed{\sum_{i=0}^{x} 2^i = 2^{x+1} - 1}$$

Base case is at $H(1) = 1$

let's solve $n-k = 1$

$\Rightarrow \quad \underbrace{k = n-1}$

closed form: H(n) on one side

plug it in:

$$H(n) = 2^k - 1 + 2^k H(n-k)$$

$$= 2^k - 1 + 2^k H(n - \underbrace{[n-1]})$$

$$= 2^k - 1 + 2^k H(1)$$

$$= 2^k - 1 + 2^k (1)$$

$$H(n) = 2(2^{n-1}) - 1$$

$$\boxed{H(n) = 2^n - 1}$$

$$\boxed{O(2^n)}$$

(Example #1 continued)

Example #2: Binary Search

6 ← false

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

(Can also be done non-recursively)

```
// Requires the array to be sorted.
// Returns whether k is in array.
public boolean find(int[]arr, int k){
    return help(arr,k,0,arr.length);
}
private boolean help(int[]arr, int k, int lo, int hi) {
    int mid = (hi+lo)/2; // i.e., lo+(hi-lo)/2
    if(lo==hi)          return false;
    if(arr[mid]==k)     return true;
    if(arr[mid]< k)     return help(arr,k,mid+1,hi);
    else                return help(arr,k,lo,mid);
```

What is the recurrence relation?

```
// Requires the array to be sorted.
// Returns whether k is in array.
public boolean find(int[]arr, int k){
    return help(arr,k,0,arr.length);
}
private boolean help(int[]arr, int k, int lo, int hi) {
    int mid = (hi+lo)/2;    // i.e., lo+(hi-lo)/2
    if(lo==hi)          return false;
    if(arr[mid]==k)     return true;
    if(arr[mid]< k)     return help(arr,k,mid+1,hi);
    else                return help(arr,k,lo,mid);
```

$c \approx 3$

top half $T(n/2)$

bottom half $T(n/2)$

A. 2T(n-1) + 3

B. T(n-1)*T(n-1) + 3

C. T(n/2) + 3

D. T(n/2) * T(n/2) + 3

What is the recurrence relation?

```
// Requires the array to be sorted.
// Returns whether k is in array.
public boolean find(int[]arr, int k){
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}
private boolean help(int[]arr, int k, int lo, int hi) {
    int mid = (hi+lo)/2;    // i.e., lo+(hi-lo)/2
    if(lo==hi)          return false;
    if(arr[mid]==k)     return true;
    if(arr[mid]< k)     return help(arr,k,mid+1,hi);
    else                return help(arr,k,lo,mid);
```

$T(n) = c + T(n/2)$ $\qquad$ $T(1) = c$

Base Case: $\boxed{T(1) = c}$

Recurrence Relation: $T(n) = c + T(n/2)$

plug $n/2$ into our recurrence relation

2nd $= c + c + T(n/4)$

plug in $n/4$ into $T(n)$

3rd $= c + c + c + T(n/8)$

$\vdots$

k th $= ck + T\left(\frac{n}{2^k}\right)$

Base case: $n = 1$ $\quad n/2^k = 1 \rightarrow n = 2^k \Rightarrow k = \log_2 n$

$\Rightarrow T(n) = c\log_2 n + c \quad \Rightarrow T(n) = \boxed{O(\log n)}$

(Example #2 continued)

Recap: Solving Recurrence Relations

1. Determine the recurrence relation. What is the base case?
   - $T(n) = 3 + T(n/2)$ $\qquad T(1) = 3$
2. "Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.
   - $$\begin{aligned} T(n) &= 3 + 3 + T(n/4) \\ &= 3 + 3 + 3 + T(n/8) \\ &= \ldots \\ &= 3k + T(n/(2^k)) \end{aligned}$$
3. Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case
   - $n/(2^k) = 1$ means $n = 2^k$ means $k = \mathtt{log}_2\, n$
   - So $T(n) = 10\,\mathtt{log}_2\, n + 8$ (get to base case and do it)
   - So $T(n)$ is $O(\mathtt{log}\, n)$

Common Recurrence Relations

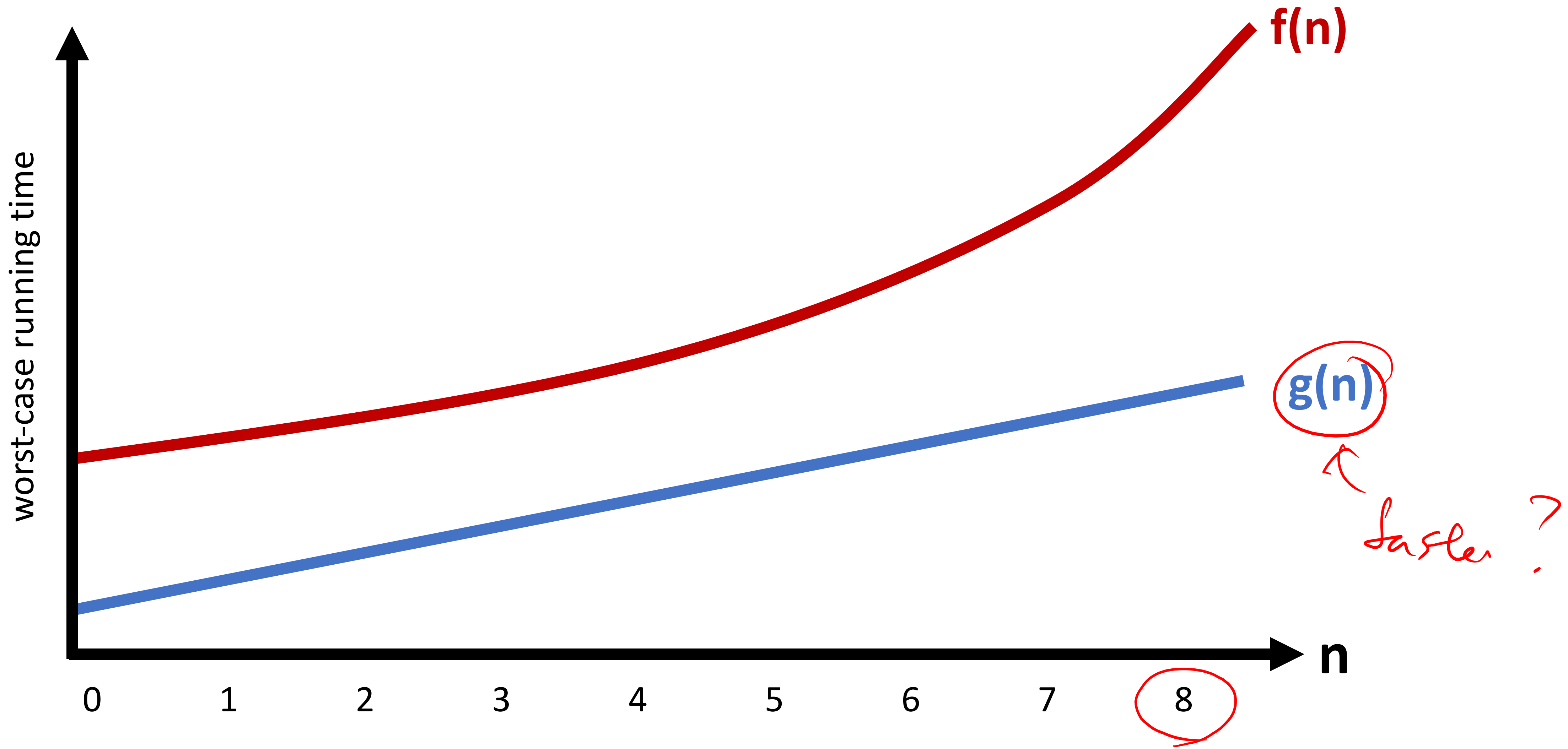
Should know how to solve recurrences but helps to recognize some common ones:

$T(n) = O(1) + T(n-1)$	linear
$T(n) = O(1) + 2T(n/2)$	linear
$T(n) = O(1) + T(n/2)$	logarithmic $O(\log n)$
$T(n) = O(1) + 2T(n-1)$	exponential
$T(n) = O(n) + T(n-1)$	quadratic
$T(n) = O(n) + T(n/2)$	linear (why?)
$T(n) = O(n) + 2T(n/2)$	$O(n \log n)$

Big-O Big Picture

with its formal definition

In terms of Big-O, which function has the faster asymptotic running time?



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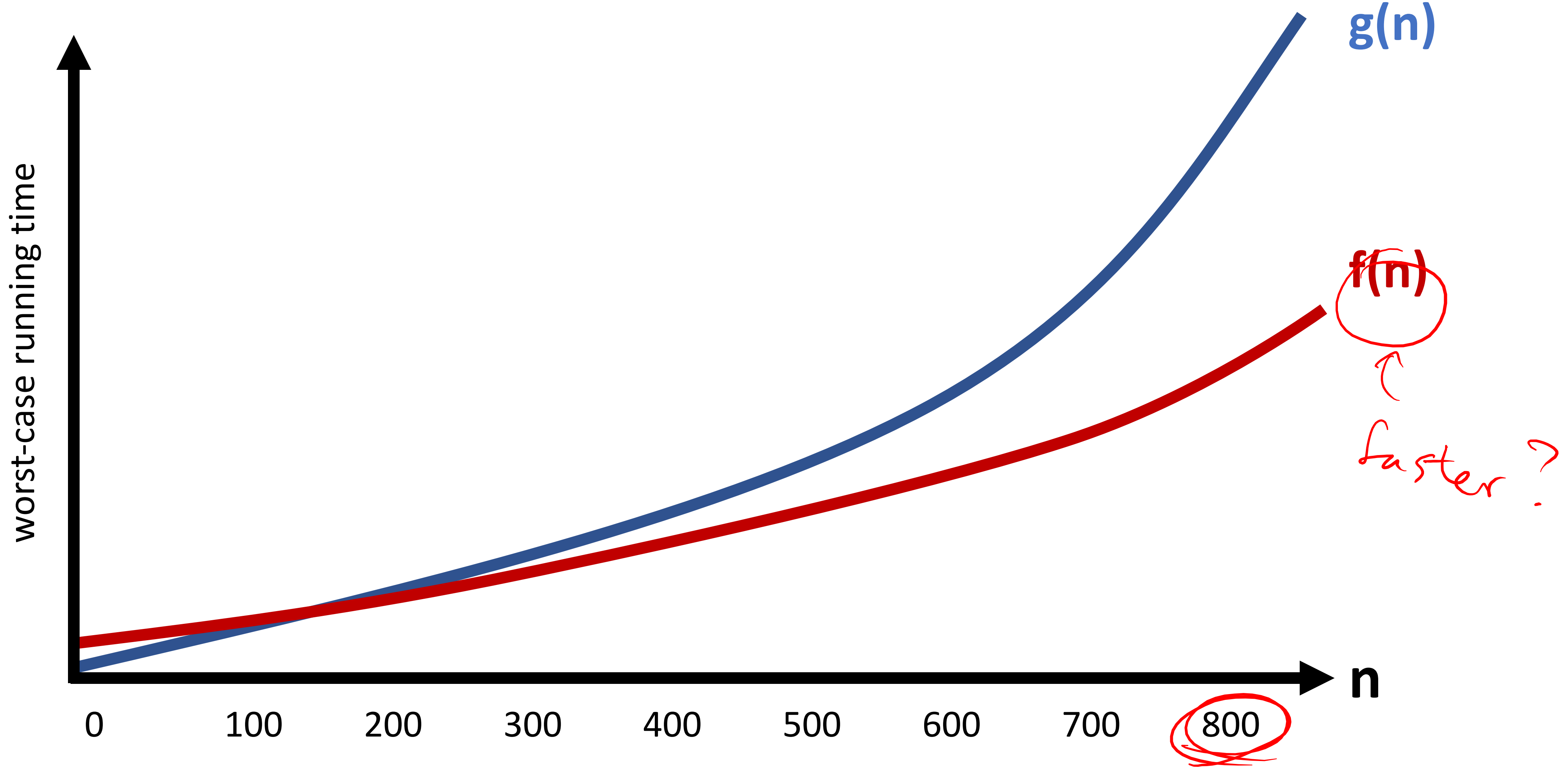

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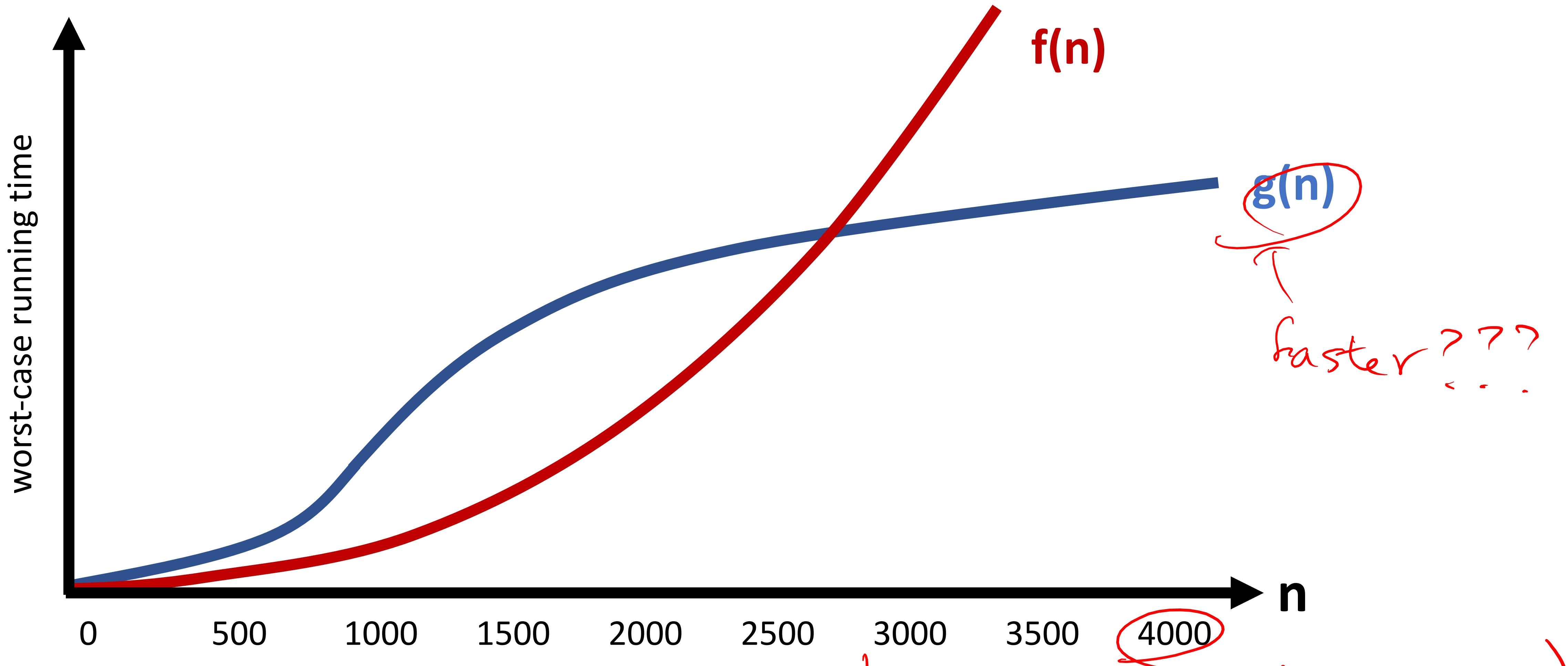


Take-away: Can't base behavior on the first values (even 100000)

That's why asymptotic analysis is about $N \to \infty$

Formal Definition of Big-O

"General Idea" explanation from last week:

Mathematical upper bound describing the behavior of how long a function takes to run in terms of N. (The "shape" as N → ∞)

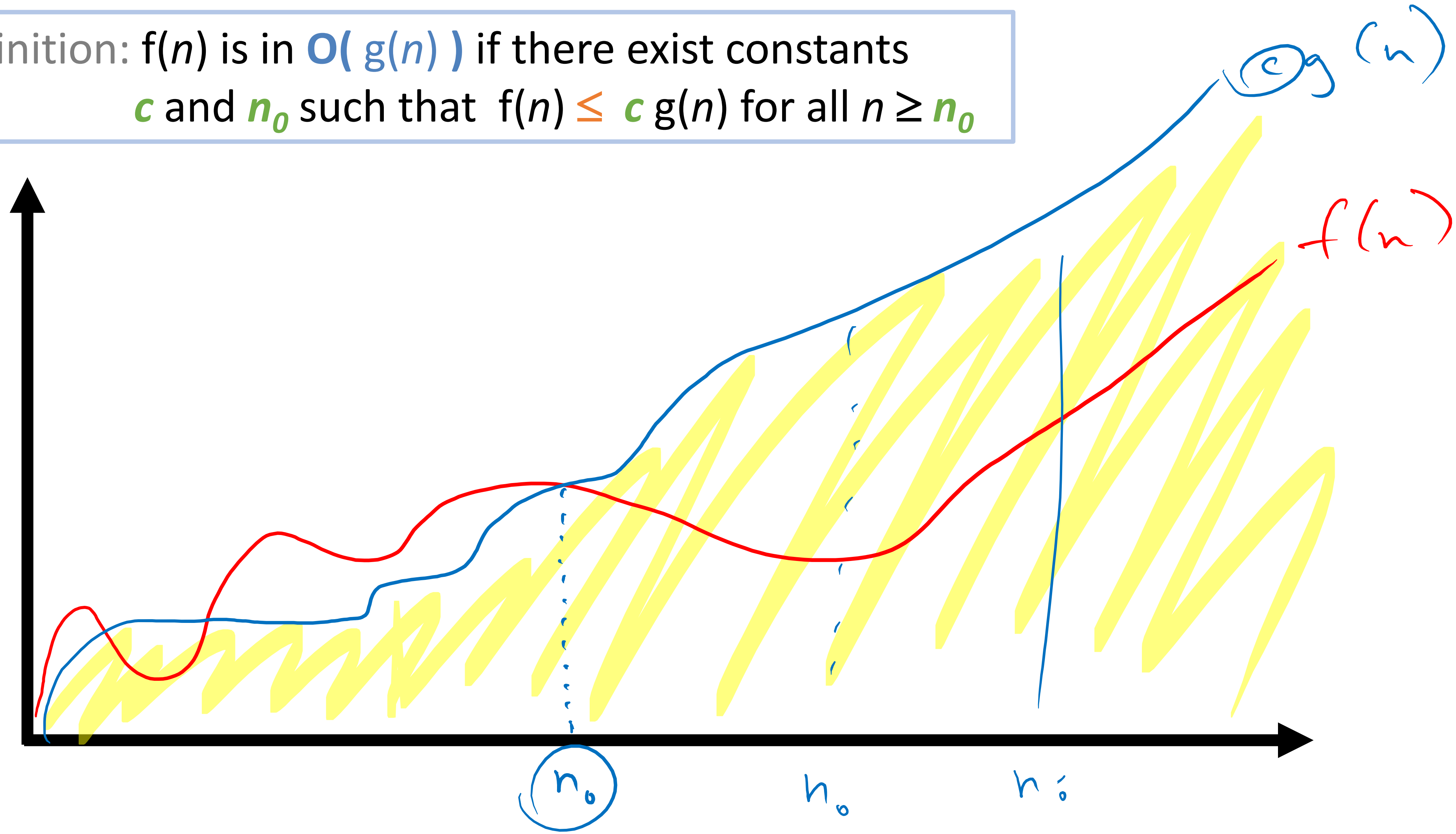
Formal definition of Big-O:

$f(n)$ is in $O(g(n))$ if there exists constants

c and n_0 s.t. $f(n) \leq c\,g(n)$ for all $n \geq n_0$

Formal Definition of Big-O

Definition: $f(n)$ is in $\mathbf{O(}\ g(n)\ \mathbf{)}$ if there exist constants c and n_0 such that $f(n) \leq c\, g(n)$ for all $n \geq n_0$



Using the Formal Definition of Big-O

Definition: f(n) is in **O(g(n))** if there exist constants c and n_0 such that f(n) $\leq$ c g(n) for all $n \geq n_0$

To show f(n) is in **O(g(n))**, pick a c large enough to "cover the constant factors" and n_0 large enough to "cover the lower-order terms"

$g(n) \leftarrow O(f(n))$

Example:

Let f(n) = $3n^2+18$ and g(n) = n^2

$c = 5 \qquad n_0 = 10$

$3n^2+18 \leq 5n^2 \qquad \forall n \geq n_0 ?$

$18 \leq 2n^2 \qquad \forall n \geq 10 ?$

$9 \leq n^2$

Example:

Let f(n) = $3n^2+18$ and g(n) = n^5 $\leftarrow O(n^2)$

$O(n^5)$

$O(2^n)$

$c = 3 \qquad n_0 = 10$

$3n^2+18 \leq 3n^5 \qquad \forall n \geq 10$

$18 \leq 3(n^5 - n^2)$

$6 \leq n^5 - n^2$