CSE 373: Data Structures and Algorithms

Lecture 3: Asymptotic Analysis part 2
Math Review, Inductive Proofs, Recursive Functions

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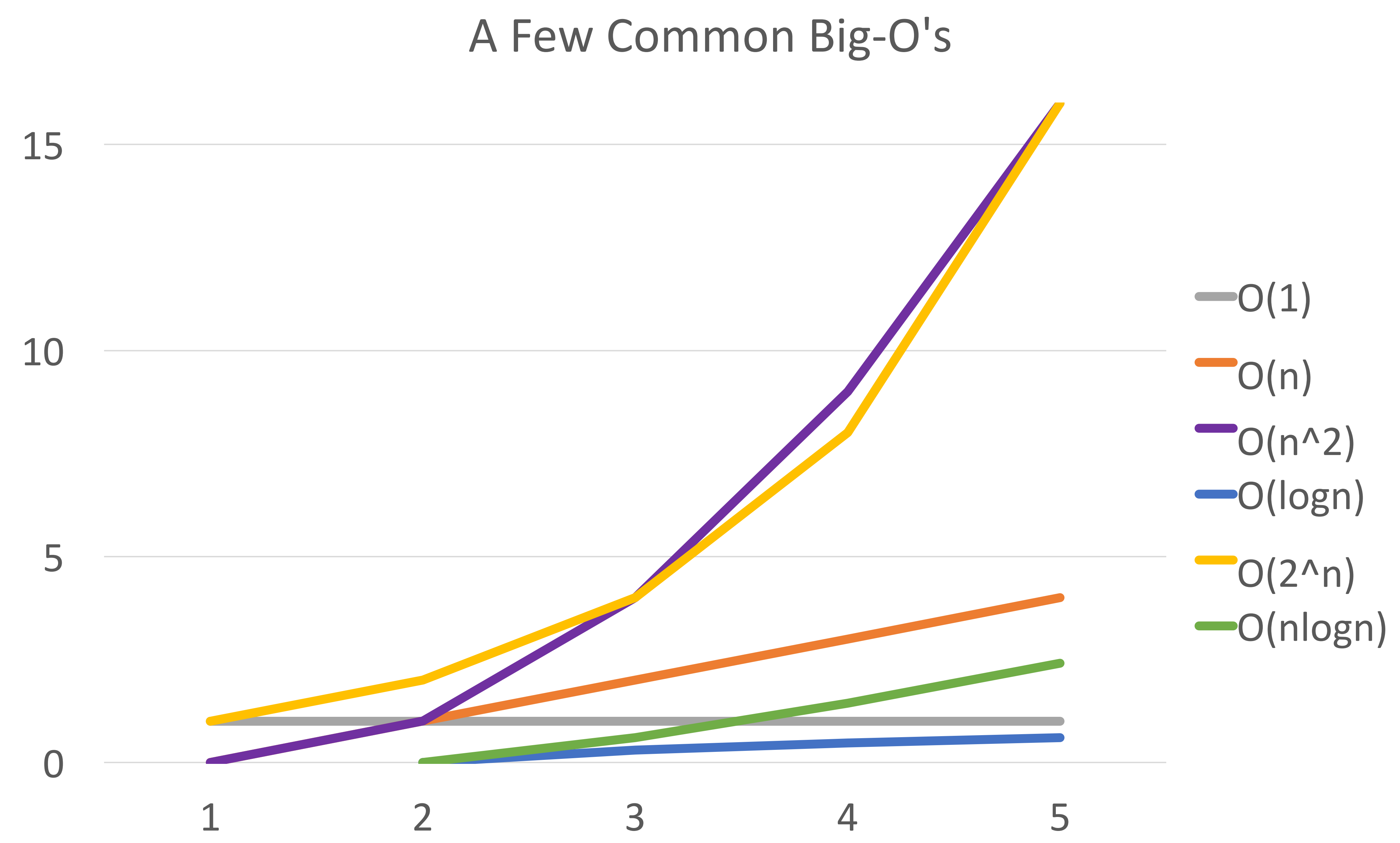
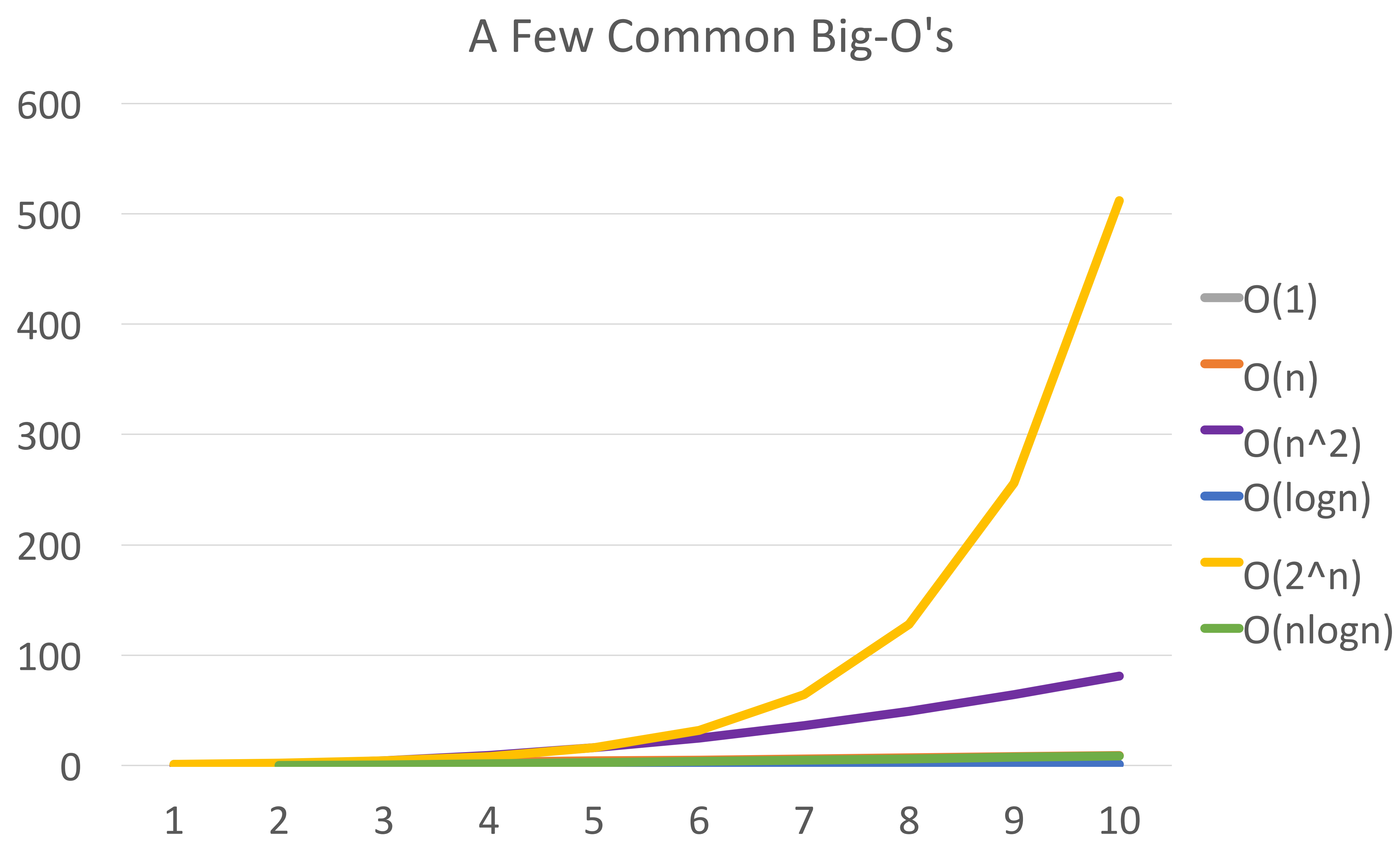
Today:

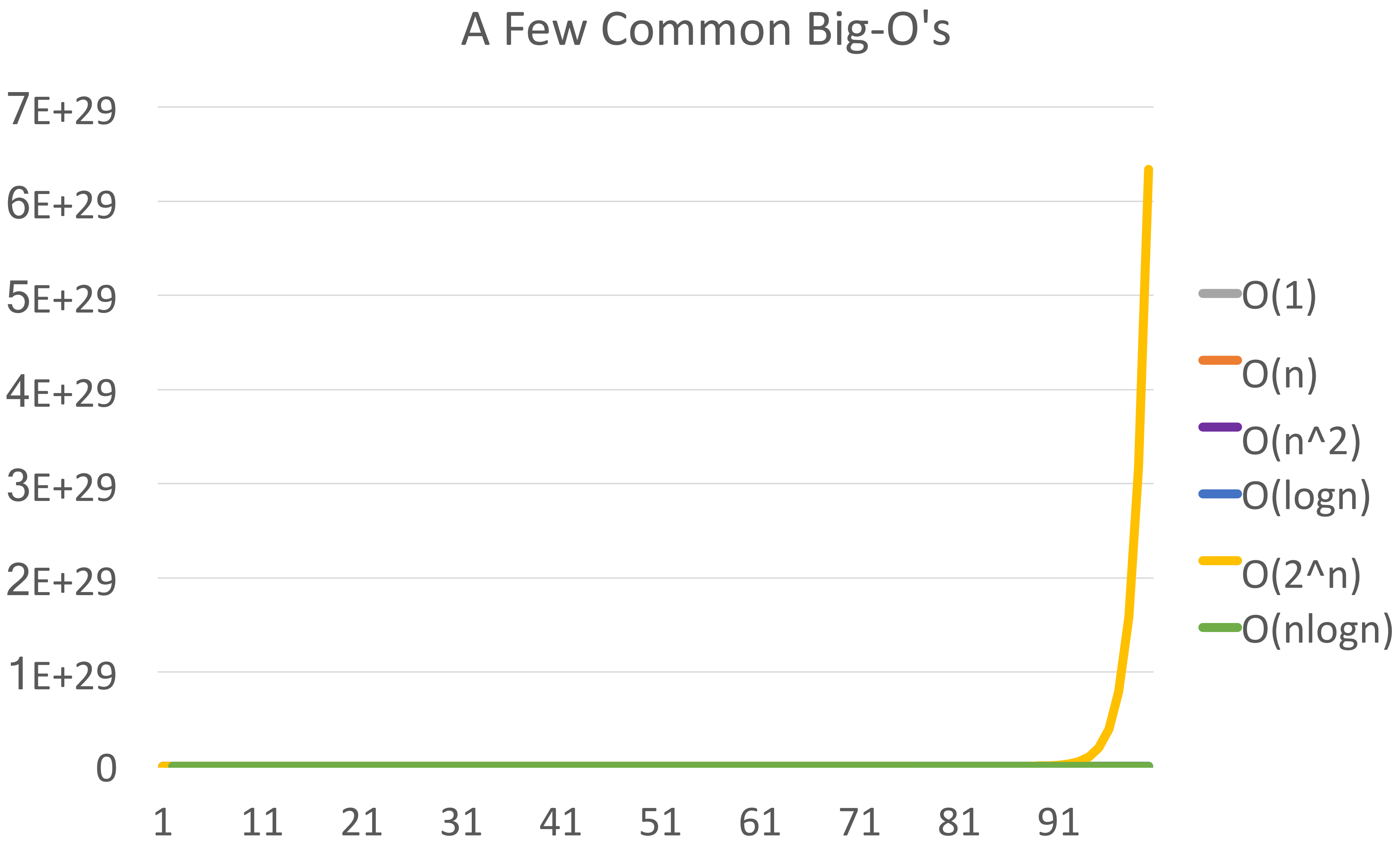
- Brief Math Review (review mostly on your own)
- Continue asymptotic analysis with Big-O
- Proof by Induction
- Recursive Functions

Common Big-O Names

$O(1)$	constant (same as $O(k)$ for constant k)
$O(\mathbf{log}\ n)$	logarithmic
$O(n)$	linear
$\mathrm{O}(\mathrm{n}\ \mathbf{log}\ n)$	"n $\mathbf{log}$ n"
$O(n^2)$	quadratic
$O(n^3)$	cubic
$O(n^k)$	polynomial (where is k is any constant)
$O(k^n)$	exponential (where k is any constant > 1)


A Few Common Big-O's
15
10
5
0
1
2
3
4
5
O(1)
O(n)
O(n^2)
O(logn)
O(2^n)
O(nlogn)


A Few Common Big-O's
600
500
400
300
200
100
0
1
2
3
4
5
6
7
8
9
10
O(1)
O(n)
O(n^2)
O(logn)
O(2^n)
O(nlogn)


A Few Common Big-O's
7E+29
6E+29
5E+29
4E+29
3E+29
2E+29
1E+29
0
1
11
21
31
41
51
61
71
81
91
O(1)
O(n)
O(n^2)
O(logn)
O(2^n)
O(nlogn)

Powers of 2: Fun Facts

- A bit is 0 or 1 (just two different “letters” or “symbols”)
- A sequence of *n* bits can represent 2^n distinct things
  (For example, the numbers 1 through 2^n)
- 2^{10} is 1024 (“about a thousand”, kilo in CSE speak)
- 2^{20} is “about a million”, mega in CSE speak
- 2^{30} is “about a billion”, giga in CSE speak

Java: an `int` is 32 bits and signed, so “max int” is “about 2 billion”

a `long` is 64 bits and signed, so “max long” is 2^{63}-1

Which means…

You could give a unique id to…

- Every person in the U.S. with 29 bits
- Every person in the world with 33 bits
- Every person to have ever lived with 38 bits (estimate)
- Every atom in the universe with 250-300 bits

So if a password is 128 bits long and randomly generated,
do you think you could guess it?

Math Review: Logs & Exponents

(Interlude #2 from Big-O)

Logs & Exponents

Definition: $log_a x = y$ if $a^y = x$

- $log_2 32 =$
- $log_{10} 10{,}000 =$

Outside of CSE, $log(x)$ is often short-hand for
In CSE, $log(x)$ is often short-hand for

...but, does it matter?

Can Make a log_2 Out of Any log!

$$log_A x = \frac{log_B(x)}{log_B(A)}$$

so

$$log_2 x = \frac{log_{whatever}(x)}{log_{whatever}(2)}$$

Other Properties of Logarithms

(to review on your own time)

- $log(A * B) = logA + logB$
  - So $log(N^k) = k * logN$
- $log(A/B) = logA - logB$
- $log(logx) = log\ log\ x$
  - Grows as slowly as 2^2 grows quickly
- $log(x)log(x)$ is written $log^2(x)$
  - It is greater than $log(x)$ for all $x > 2$
  - It is not the same as $log\ log\ x$

Floor and Ceiling

(to review on your own time)

$\lfloor X \rfloor$ Floor function: the largest integer $\leq$ X

$\lfloor 2.7 \rfloor = 2$ $\quad$ $\lfloor -2.7 \rfloor = -3$ $\quad$ $\lfloor 2 \rfloor = 2$

$\lceil X \rceil$ Ceiling function: the smallest integer $\geq$ X

$\lceil 2.3 \rceil = 3$ $\quad$ $\lceil -2.3 \rceil = -2$ $\quad$ $\lceil 2 \rceil = 2$

Floor and Ceiling Properties

(to review on your own time)

1. $X - 1 < \lfloor X \rfloor \leq X$
2. $X \leq \lceil X \rceil < X + 1$
3. $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$ if n is an integer

Back to Big-O

What’s the asymptotic runtime of this (semi-)pseudocode?

```
x := 0;
for i=1 to N do
   for j=1 to i do
      x := x + 3;
return x;
```

(Some textbooks format algorithms in this style of semi-pseudocode)

A. O(n)
B. O(n^2)
C. O(n + n/2)
D. None of the above

What’s the asymptotic runtime of this (semi-)pseudocode?

```
x := 0;
for i=1 to N do
  for j=1 to i do
    x := x + 3;
return x;
```

A. O(n)
B. $O(n^2)$
C. O(n + n/2)
D. None of the above

How do we prove the right answer?

Proof by Induction!

Inductive Proofs

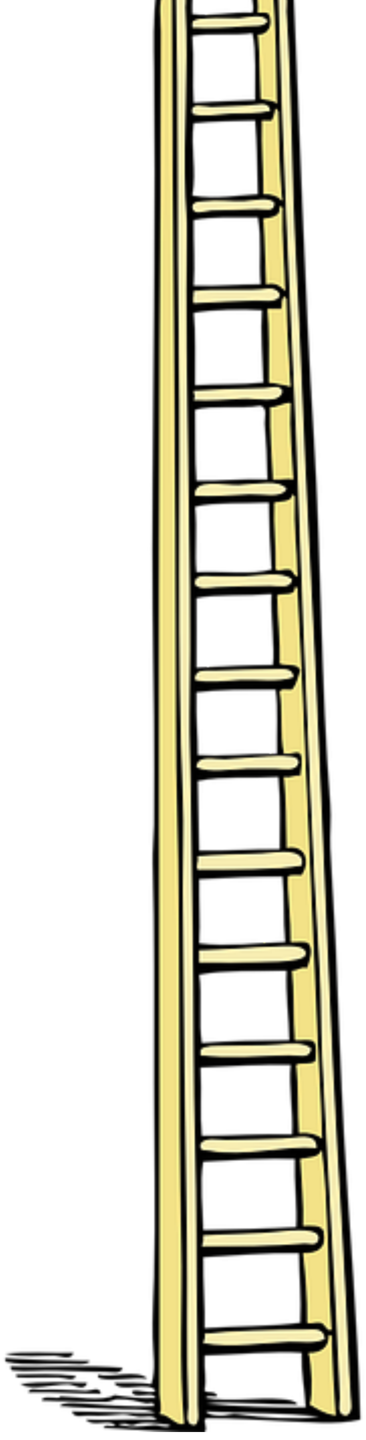
(Interlude from Asymptotic Analysis)

Steps to Inductive Proof

1. If not given, **define n** (or “x” or “t” or whatever letter you use)
2. **Base Case**
3. **Inductive Hypothesis (IHOP):**
   Assume what you want to prove is true for some arbitrary value k
   (or “p” or “d” or whatever letter you choose)
4. **Inductive Step**:
   Use the IHOP (and maybe base case) to prove it's true for n = k+1

Example #0:
Proof that I can climb any length ladder

1. **Let n =** number of rungs on a ladder.
2. **Base Case:** for n = 1
3. **Inductive Hypothesis (IHOP):**
   Assume true for some arbitrary integer n = k.
4. **Inductive Step:** (aiming to prove it's true for n = k+1)
   - By IHOP, I can climb k steps of the ladder.
   - If I’ve climbed that far, I can always climb one more.
   - So I can climb k+1 steps.
   - I can climb forever!



Example #1

Prove that the number of loop iterations is $\frac{n*(n+1)}{2}$

```
x := 0;
for i=1 to N do
  for j=1 to i do
     x := x + 3;
return x;
```

(Extra room for notes)

Example #2:

Prove that $1 + 2 + 4 + 8 + \dots + 2^n = 2^{n+1} - 1$

(Extra room for notes)

Useful Mathematical Property!

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

You'll use it or see it again before the end of CSE 373.

Example #3: (Parody) Reverse Induction!

Proof by Reverse Induction That You Can Always Cage a Lion:

Let n = number of lions

Base Case: There exists some countable, arbitrarily large value of M such that when n = M, the lions are so packed together that it's trivial to cage one.

I**HOP**: Assume this is also true for n = k for some arbitrary value k.

Inductive Step: Then for n = k-1, release a lion to reduce the problem to the case of n = k, which by the IHOP is true.

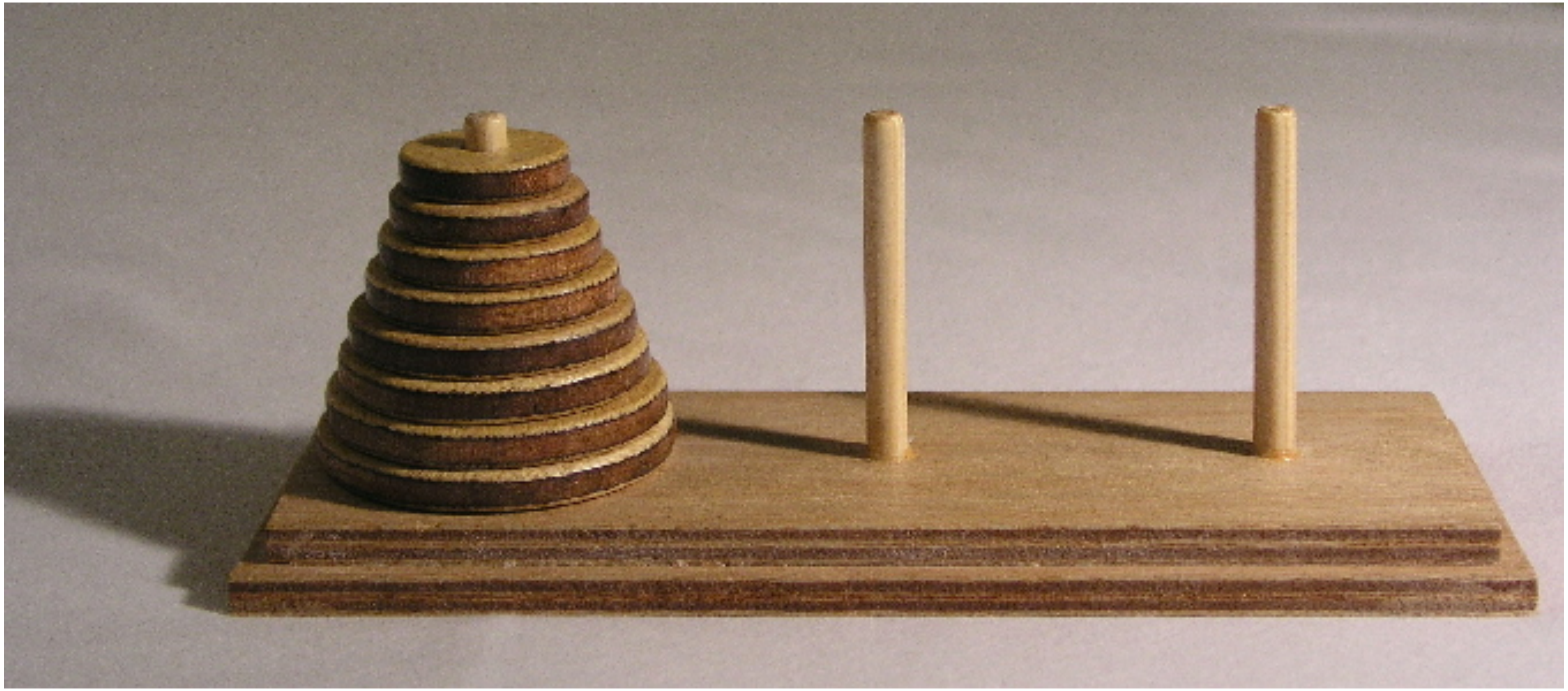
QED :)

Fun fact: Reverse induction is a thing! The math part of the above is actually correct.

Big-O: Recursive Functions

How do we asymptotically analyze recursive functions?

Example #1: Towers of Hanoi



Example #1: Towers of Hanoi

```java
// Prints instructions for moving disks from one
// pole to another, where the three poles are
// labeled with integers "from", "to", and "other".
// Code from rosettacode.org
public void move(int n, int from, int to, int other) {
    if (n == 1) {
        System.out.println("Move disk from pole " + from +
                           " to pole " + to);}
    else {
        move(n - 1, from, other, to);
        move(1, from, to, other);
        move(n - 1, other, to, from);
    }
}
```

Example #1: Towers of Hanoi

```
if (n == 1) {
    System.out.println("Move disk from pole " + from +
                       " to pole " + to);}




else {
    move(n - 1, from, other, to);
    move(1, from, to, other);
    move(n - 1, other, to, from);
}
```

Example #1: Solving the Recurrence Relation

Recurrence Relation:

(continued)

Example #2: Binary Search

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

(Can also be done non-recursively)

```
// Requires the array to be sorted.
// Returns whether k is in array.
public boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
private boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2; // i.e., lo+(hi-lo)/2
   if(lo==hi)          return false;
   if(arr[mid]==k)     return true;
   if(arr[mid]< k)     return help(arr,k,mid+1,hi);
   else                return help(arr,k,lo,mid);
```

What is the recurrence relation?

```
// Requires the array to be sorted.
// Returns whether k is in array.
public boolean find(int[]arr, int k){
    return help(arr,k,0,arr.length);
}
private boolean help(int[]arr, int k, int lo, int hi) {
    int mid = (hi+lo)/2;    // i.e., lo+(hi-lo)/2
    if(lo==hi)          return false;
    if(arr[mid]==k)     return true;
    if(arr[mid]< k)     return help(arr,k,mid+1,hi);
    else                return help(arr,k,lo,mid);
```

A. $2T(n-1) + 3$

B. $T(n-1)*T(n-1) + 3$

C. $T(n/2) + 3$

D. $T(n/2) * T(n/2) + 3$