CSE 373: Data Structures and Algorithms

Lecture 3: Asymptotic Analysis part 2
Math Review, Inductive Proofs, Recursive Functions

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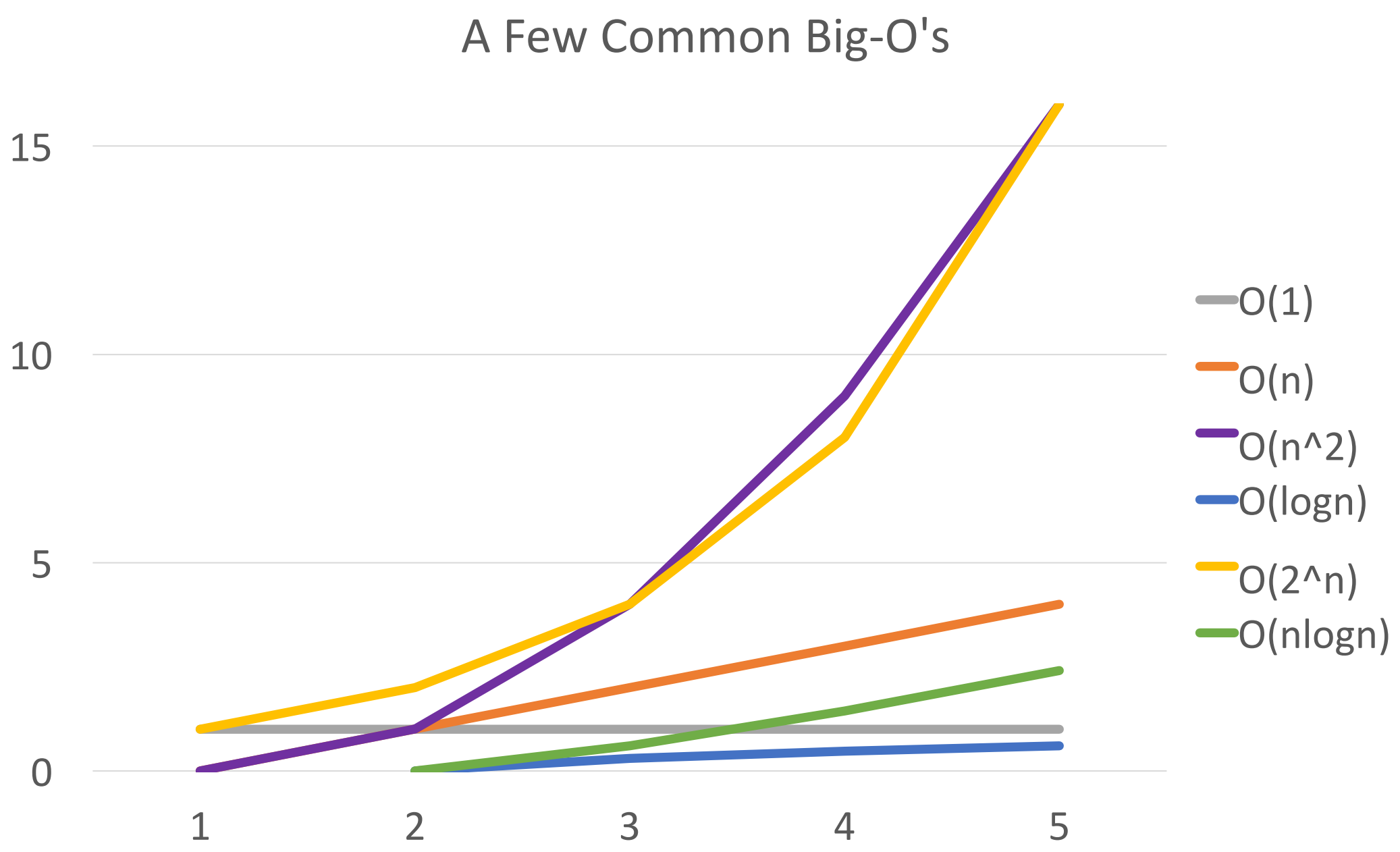
Quarter: Summer 2017

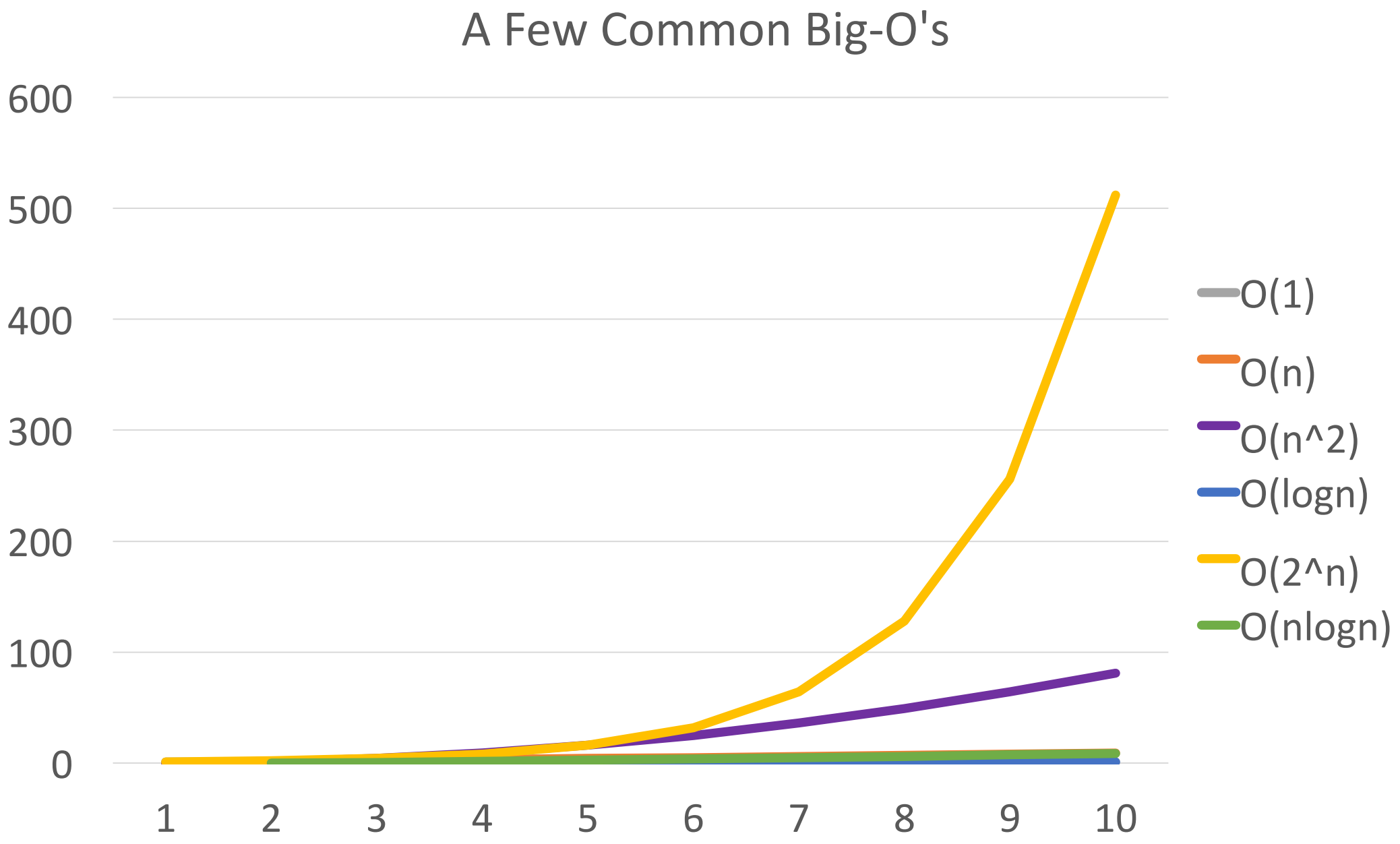
Today:

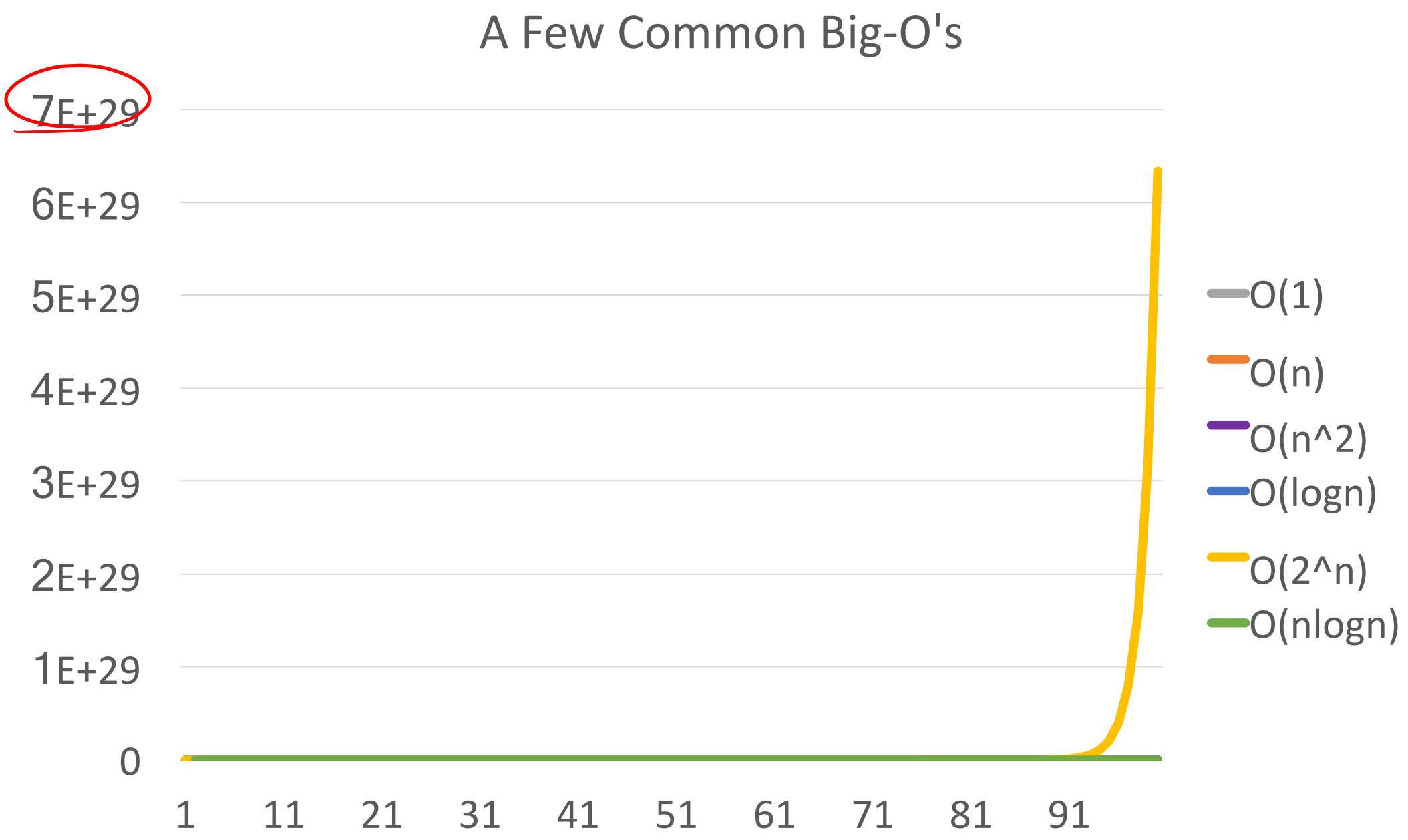
- Brief Math Review (review mostly on your own)
- Continue asymptotic analysis with Big-O
- Proof by Induction
- Recursive Functions

Common Big-O Names

$O(1)$	constant (same as $O(k)$ for constant k)
$O(\mathbf{log}\ n)$	logarithmic
$O(n)$	linear
O(n **log** n)	"n **log** n"
$O(n^2)$	quadratic
$O(n^3)$	cubic
$O(n^k)$	polynomial (where is k is any constant)
$O(k^n)$	exponential (where k is any constant > 1)


A Few Common Big-O's
15
10
5
0
1
2
3
4
5
O(1)
O(n)
O(n^2)
O(logn)
O(2^n)
O(nlogn)


A Few Common Big-O's
600
500
400
300
200
100
0
1
2
3
4
5
6
7
8
9
10
O(1)
O(n)
O(n^2)
O(logn)
O(2^n)
O(nlogn)


A Few Common Big-O's
7E+29
6E+29
5E+29
4E+29
3E+29
2E+29
1E+29
0
1
11
21
31
41
51
61
71
81
91
O(1)
O(n)
O(n^2)
O(logn)
O(2^n)
O(nlogn)

Powers of 2: Fun Facts

0 1 1 0

- A bit is 0 or 1 (just two different "letters" or "symbols")
- A sequence of *n* bits can represent 2^n distinct things
  (For example, the numbers 1 through 2^n)
- 2^{10} is 1024 ("about a thousand", kilo in CSE speak)
- 2^{20} is "about a million", mega in CSE speak
- 2^{30} is "about a billion", giga in CSE speak

Java: an `int` is 32 bits and signed, so "max int" is "about 2 billion"

a `long` is 64 bits and signed, so "max long" is 2^{63}-1

Which means…

You could give a unique id to…

- Every person in the U.S. with 29 bits
- Every person in the world with 33 bits
- Every person to have ever lived with 38 bits (estimate)
- Every atom in the universe with 250-300 bits

So if a password is 128 bits long and randomly generated, do you think you could guess it?

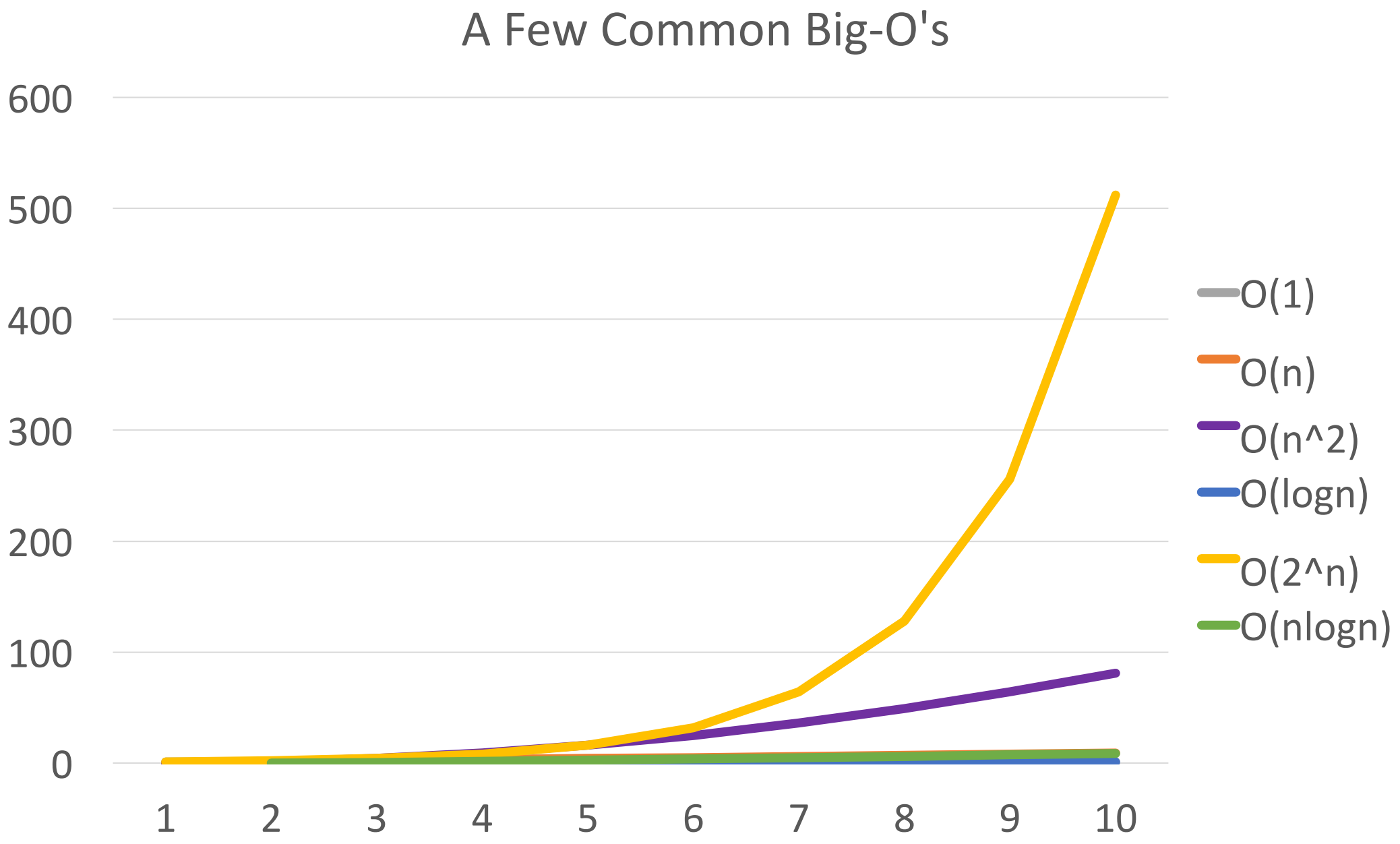
Math Review: Logs & Exponents

(Interlude #2 from Big-O)

Logs & Exponents

Definition: $log_a x = y$ if $a^y = x$

- $log_2 32 =$ 5
- $log_{10} 10{,}000 =$ 4


A Few Common Big-O's
600
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O(1)
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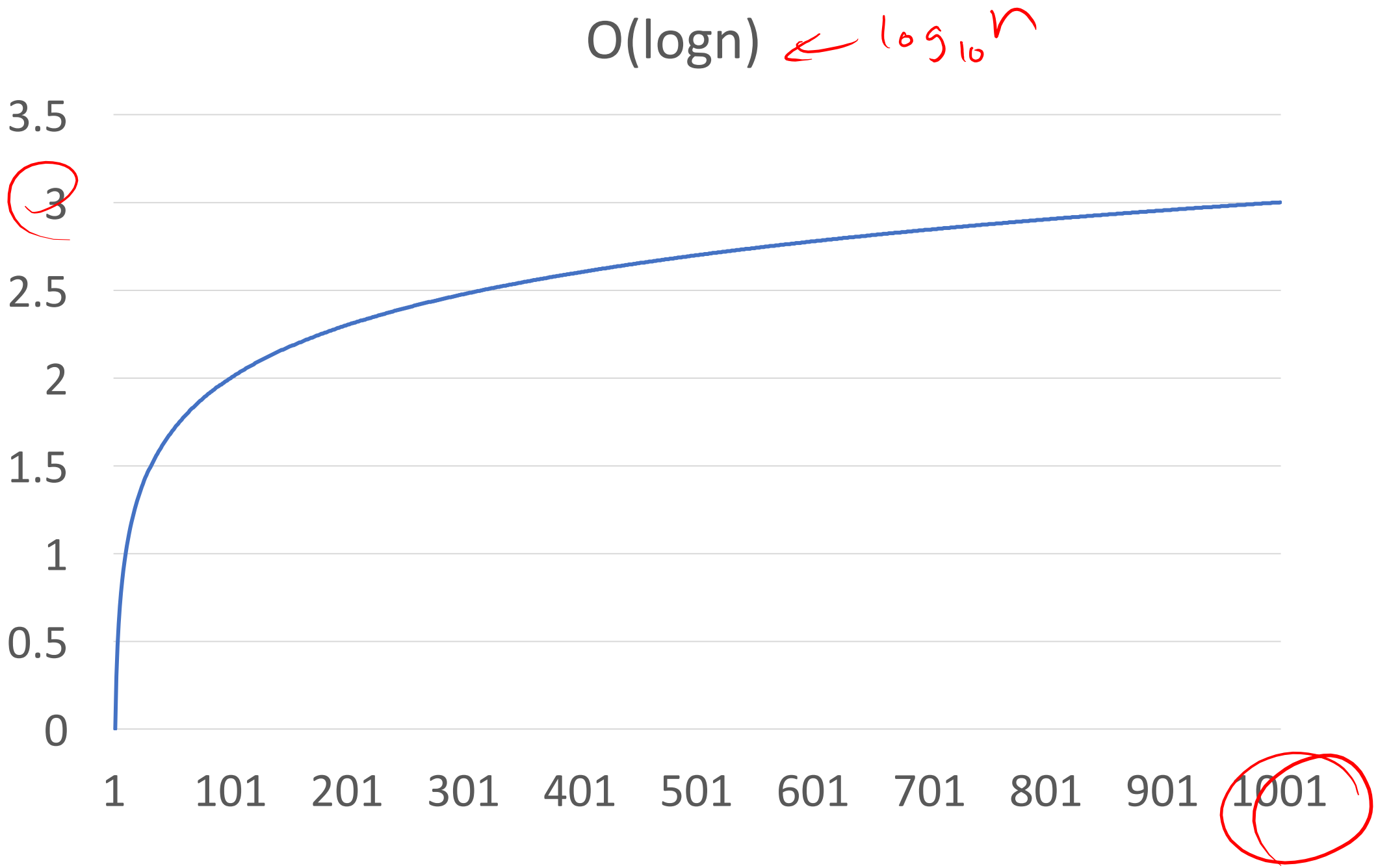

O(logn)
$\log_{10} n$
3.5
3
2.5
2
1.5
1
0.5
0
1
101
201
301
401
501
601
701
801
901
1001

Logs & Exponents

Definition: $log_a x = y$ if $a^y = x$

- $log_2 32 =$
- $log_{10} 10{,}000 =$

Outside of CSE, $log(x)$ is often short-hand for $\log_{10} x$
In CSE, $log(x)$ is often short-hand for $\log_2 x$

...but, does it matter?

Can Make a log_2 Out of Any log!

$$log_A x = \frac{log_B(x)}{log_B(A)}$$

so

$$log_2 x = \frac{log_{whatever}(x)}{log_{whatever}(2)}$$

$$= log_{whatever} x \cdot \left(\frac{1}{log_{whatever}(2)} \right)$$

Other Properties of Logarithms

(to review on your own time)

- $log(A * B) = logA + logB$
  - So $log(N^k) = k * logN$
- $log(A/B) = logA - logB$
- $log(logx) = log\ log\ x$
  - Grows as slowly as 2^2 grows quickly
- $log(x)log(x)$ is written $log^2(x)$
  - It is greater than $log(x)$ for all $x > 2$
  - It is not the same as $log\ log\ x$

Floor and Ceiling

(to review on your own time)

$\lfloor X \rfloor$ Floor function: the largest integer $\leq$ X

$$\lfloor 2.7 \rfloor = 2 \qquad \lfloor -2.7 \rfloor = -3 \qquad \lfloor 2 \rfloor = 2$$

$\lceil X \rceil$ Ceiling function: the smallest integer $\geq$ X

$$\lceil 2.3 \rceil = 3 \qquad \lceil -2.3 \rceil = -2 \qquad \lceil 2 \rceil = 2$$

Floor and Ceiling Properties

(to review on your own time)

1. $X - 1 < \lfloor X \rfloor \leq X$
2. $X \leq \lceil X \rceil < X + 1$
3. $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$ if n is an integer

Back to Big-O

What's the asymptotic runtime of this (semi-)pseudocode?

```
x := 0;
for i=1 to N do
  for j=1 to i do
     x := x + 3;
return x;
```

worst case = N iterations

N=1 1

=2 1+2

=3 1+2+3

$\sum_{i=1}^{n} i = \frac{n+1}{2}$

$\frac{n(n+1)}{2}$

$O(n^2)$

A. O(n)

B. $O(n^2)$

C. O(n + n/2)

D. None of the above

How do we prove the right answer?

Proof by Induction!

Inductive Proofs

(Interlude from Asymptotic Analysis)

Steps to Inductive Proof

☺ ✯ ♡ = ∑ ♣

1. If not given, **define n** (or “x” or “t” or whatever letter you use)
2. **Base Case** P(#) is true
3. **Inductive Hypothesis (IHOP):** assume true P(k)
   Assume what you want to prove is true for some arbitrary value k (or “p” or “d” or whatever letter you choose)
4. **Inductive Step**: show true P(k+1)
   Use the IHOP (and maybe base case) to prove it's true for n = k+1

☺ ✯ ♡ = ∑ ♣ for k

plug in k+1 in ☺ ✯ ♡

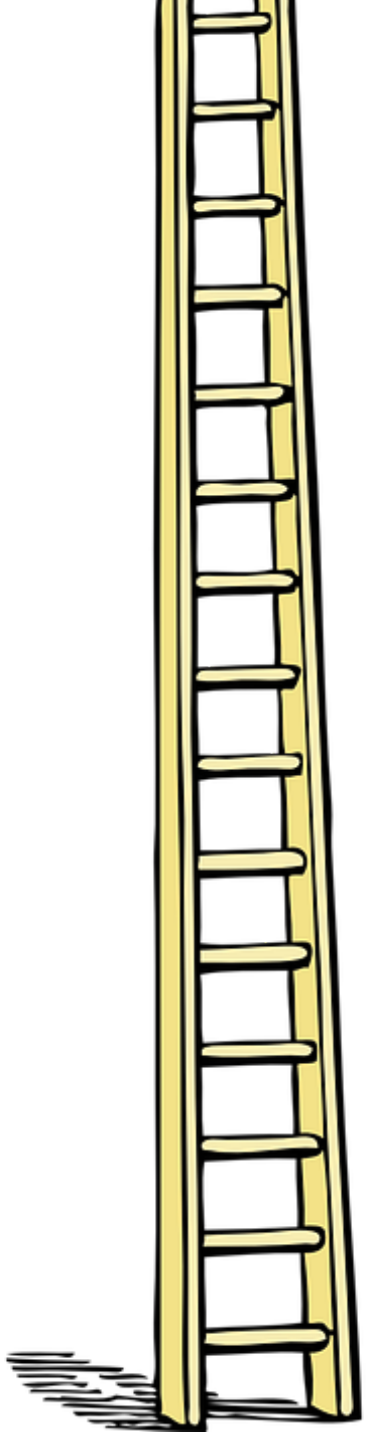
IHOP

⋮

= ∑ ♣ for k+1

Example #0:
Proof that I can climb any length ladder

1. **Let n =** number of rungs on a ladder.
2. **Base Case:** for n = 1 ✓
3. **Inductive Hypothesis (IHOP):**
   Assume true for some arbitrary integer n = k.
4. **Inductive Step:** (aiming to prove it's true for n = k+1)
   - By IHOP, I can climb k steps of the ladder.
   - If I've climbed that far, I can always climb one more.
   - So I can climb k+1 steps.
   - I can climb forever!



Example #1

Prove that the number of loop iterations is $\frac{n * (n+1)}{2}$

```
x := 0;
for i=1 to N do
  for j=1 to i do
     x := x + 3;
return x;
```

① $n = N$

② Base Case: true for $n=1$

$n=1$ $\quad \frac{1(1+1)}{2} = \frac{2}{2} = 1$ ✓

③ IHOP: assume true $n=k$ where k is an arbitrary integer >1

to $i=k$: same as $n=k$

add last step $i=k+1$

↳ adds $k+1$

③ Inductive step

(Extra room for notes)

(Prove $\frac{n(n+1)}{2}$ iterations)

Inductive step: (Goal: show it's true for $n = k+1$)

$$\frac{(k+1)((k+1)+)}{2}$$

For $N = k+1$ = (iterations for $N = k$) + $(k+1)$

by IHOP $= \frac{k(k+1)}{2} + k+1$

$$= \frac{k^2+k}{2} + \frac{2(k+1)}{2}$$

$$= \frac{k^2+k+2k+2}{2} = \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)((k+1)+1)}{2}$$

QED

($p(n+1) = p(n) + n+1$ true for <u>this</u> problem)

Example #2:

n = variable
k = number

Prove that $1 + 2 + 4 + 8 + \ldots + 2^n = 2^{n+1} - 1$

1. $(n = n)$

2. Base case: $n = 0$ $\quad 2^0 = 1 = 2^{0+1} - 1 = 2 - 1 = 1$

true for $n = 0$ ✓

true $k = 0$

3. IHOP: assume true for $n = k$ $\quad \sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

4. Inductive step: (Goal: show it's true for $n = k+1$

wts: $\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$)

IHOP = $n = k+1$
LS = $n = k+2$

(Extra room for notes)

(wts $\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$)

(IHOP: $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$)

$$\sum_{i=0}^{k+1} 2^i = \left(\sum_{i=0}^{k} 2^i\right) + 2^{k+1}$$

split summation

$$= 2^{k+1} - 1 + 2^{k+1}$$

by IHOP

$2(2^x) = 2^{x+1}$?

$$2(2^x) = \underbrace{(2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2)}_{x \text{ times}} 2 = 2^{k+1} + 2^{k+1} - 1$$

$$2^{x+1} = \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdots 2}_{x+1 \text{ times}} = 2(2^{k+1}) - 1 = 2^{k+2} - 1$$

QED

Useful Mathematical Property!

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

You'll use it or see it again before the end of CSE 373.

Example #3: (Parody) Reverse Induction!

Proof by Reverse Induction That You Can Always Cage a Lion:

Let n = number of lions

Base Case: There exists some countable, arbitrarily large value of M such that when n = M, the lions are so packed together that it's trivial to cage one.

I**HOP**: Assume this is also true for n = k for some arbitrary value k.

Inductive Step: Then for n = k-1, release a lion to reduce the problem to the case of n = k, which by the IHOP is true.

QED :)

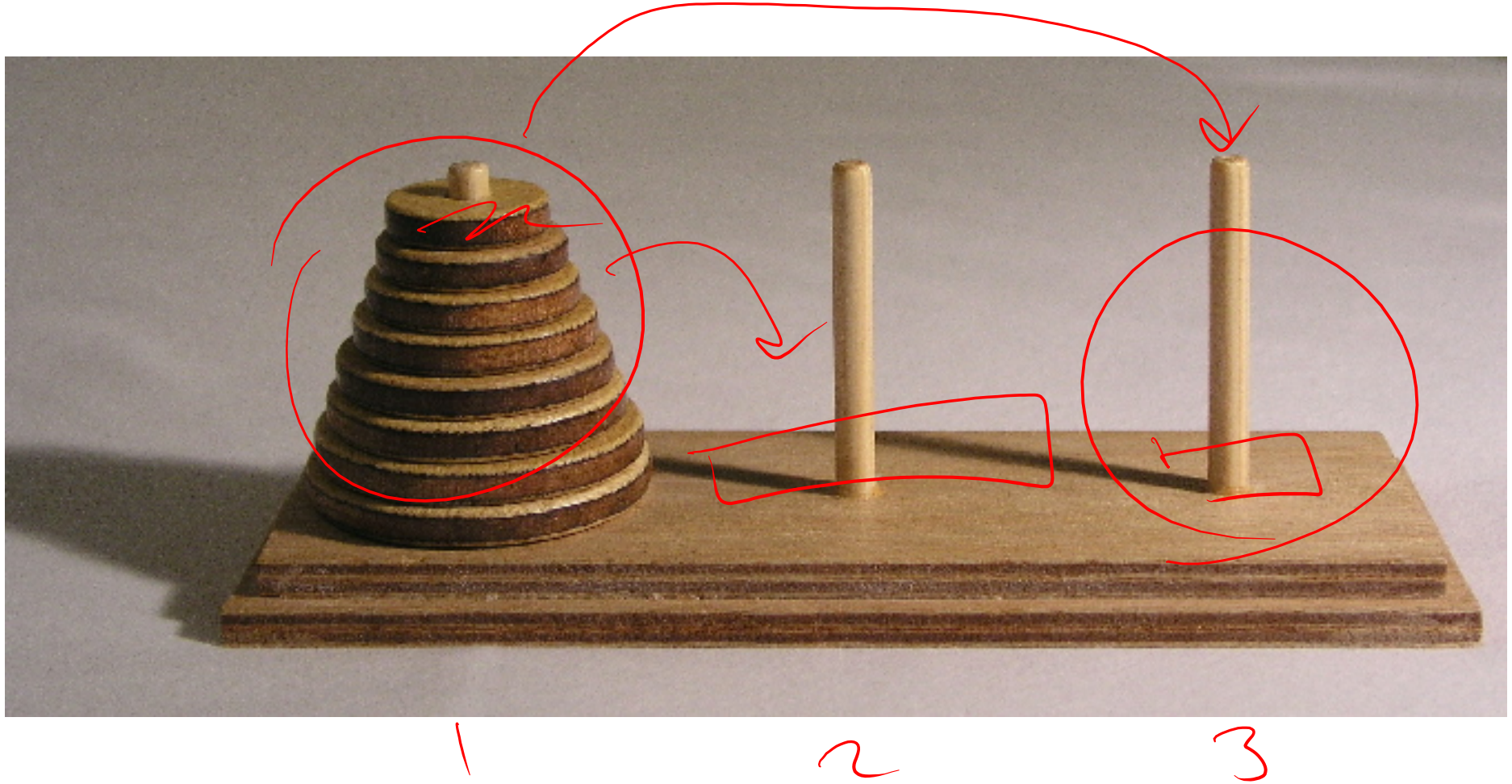
Fun fact: Reverse induction is a thing! The math part of the above is actually correct.

Recursive fn: fn that calls itself

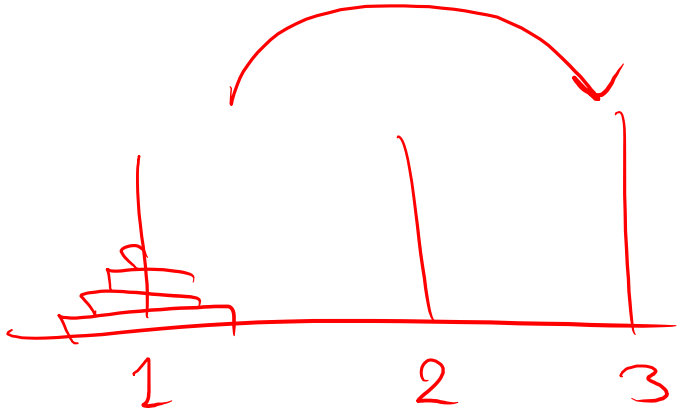
Big-O: Recursive Functions

How do we asymptotically analyze recursive functions?

Example #1: Towers of Hanoi



Example #1: Towers of Hanoi



from = 1 to = 3

other = 2

```
// Prints instructions for moving disks from one
// pole to another, where the three poles are
// labeled with integers "from", "to", and "other".
// Code from rosettacode.org
public void move(int n, int from, int to, int other) {
    if (n == 1) {
        System.out.println("Move disk from pole " + from +
                           " to pole " + to);}
    else {
        move(n - 1, from, other, to);
        move(1, from, to, other);
        move(n - 1, other, to, from);
    }
}
```

Base case

Recursion

Example #1: Towers of Hanoi

move

$H(n)$

Base Case: $H(1) = 1$

```
if (n == 1) {
    System.out.println("Move disk from pole " + from +
                       " to pole " + to);}
```

$O(1)$

"1 unit of time"

Recursive Step:

```
else {
    move(n - 1, from, other, to);
    move(1, from, to, other);
    move(n - 1, other, to, from);
}
```

$H(n-1)$

$H(1) = 1$

$H(n-1)$

Recurrence Relation

All together: $H(n) = H(n-1) + 1 + H(n-1)$

$$\boxed{H(n) = 1 + 2H(n-1)}$$

Example #1: Solving the Recurrence Relation

Recurrence Relation: $H(n) = 1 + 2H(n-1)$

$n-k$

Expanding (plug in for $H(n)$).

1st $\quad H(n) = 1 + 2H(n-1)$

2nd $\quad = 1 + 2(1 + 2H(n-1))$

$H(n) = 1 + 2 + 4H(n-2)$

3rd $\quad H(n) = 1 + 2 + 4 + 8H(n-3)$

Pattern

$\vdots$

kth $\quad = 2^0 + 2^1 + 2^2 + \cdots 2^{k-1} + 2^k H(n-k)$

(continued)

$$= 2^k - 1 + 2^k H(n-k)$$

Base case is at $H(1)$, so

let's solve $n-k=1$

$\Rightarrow k = n-1$

$H(1) = 1$

Plug it in:

$$H(n) = 2^k - 1 + 2^k H(n-k)$$

$$= 2^{n-1} - 1 + 2^{n-1} \underbrace{H(n-[n-1])}_{H(1)}$$

$$= 2^{n-1} - 1 + 2^{n-1}(1)$$

$$= 2(2^{n-1}) - 1 = 2^n - 1$$

$H(n) = 2^n - 1$

$\boxed{O(2^n)}$