CSE 373

MAY 24TH – ANALYSIS AND NON-COMPARISON SORTING

ASSORTED MINUTIAE

- HW6 Out – Due next Wednesday

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  - Only two late days allowed

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  - Regrades also in by that point.

TODAY

- **Merge sort and Quick sort examples**

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- Proving $\Omega(n \log n)$ for comparison sorts

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- Basics of the Recurrence

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- Non-comparison sorting

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- **Proving Ω(n log n) for comparison sorts**
- **Basics of the Recurrence**
- **Non-comparison sorting**
- **JUnit**

JUNIT: TESTING FRAMEWORK

A Java library for unit testing, comes included with Eclipse

- OR can be downloaded for free from the JUnit web site at http://junit.org
- JUnit is distributed as a "JAR" which is a compressed archive containing Java .class files

```
import org.junit.Test;
import static org.junit.Assert.*;

public class name {
  ...

  @Test
  public void name() { // a test case method
    ...
  }
}
```

A method with @Test is flagged as a JUnit test case and run

JUNIT ASSERTS AND EXCEPTIONS

A test will pass if the assert statements all pass and if no exception thrown.
Examples of assert statements:

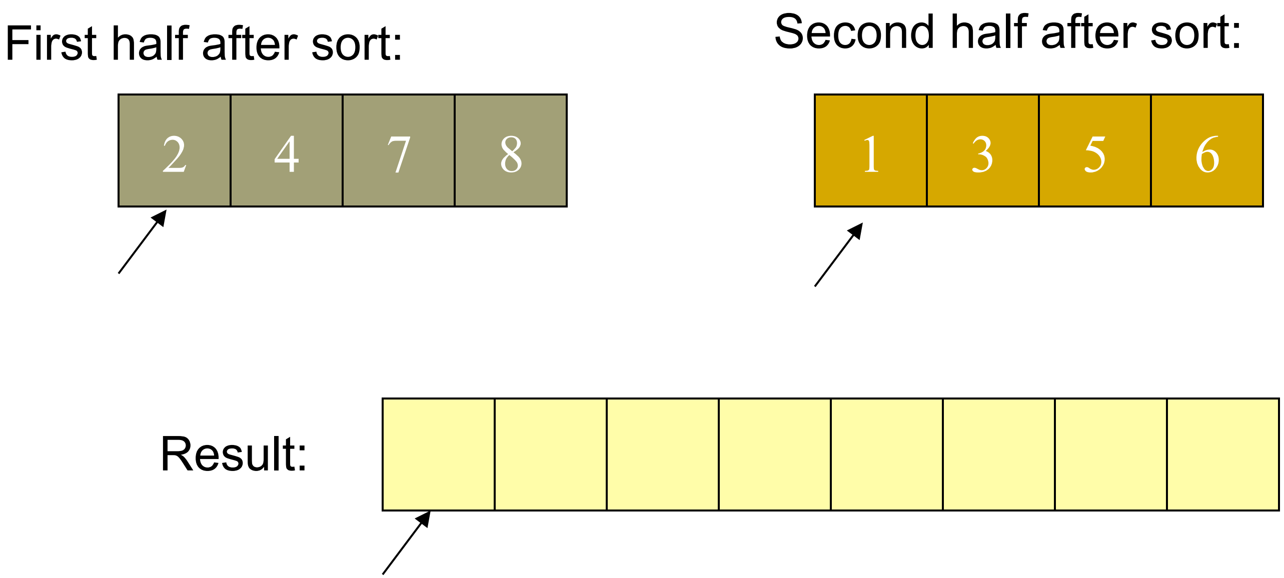
- `assertTrue(value)`
- `assertFalse(value)`
- `assertEquals(expected, actual)`
- `assertNull(value)`
- `assertNotNull(value)`
- `fail()`

Tests can expect exceptions or timeouts

```
@Test(expected = ExceptionType.class)
public void name() {
  ...
}
```

MERGE EXAMPLE

Merge operation: Use 3 pointers and 1 more array



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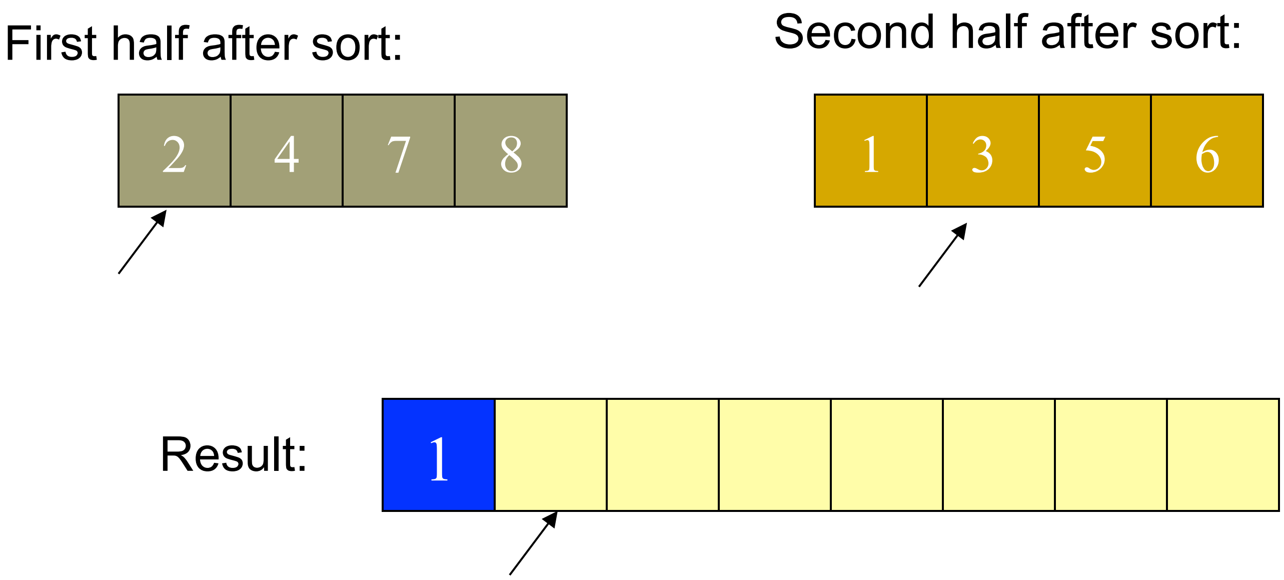

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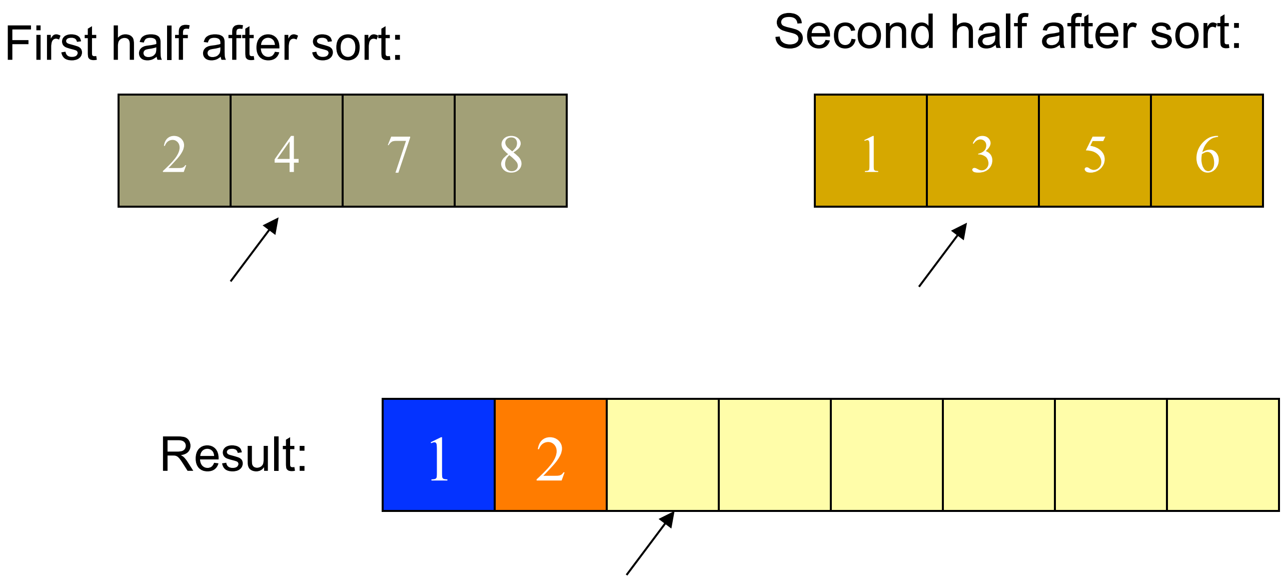

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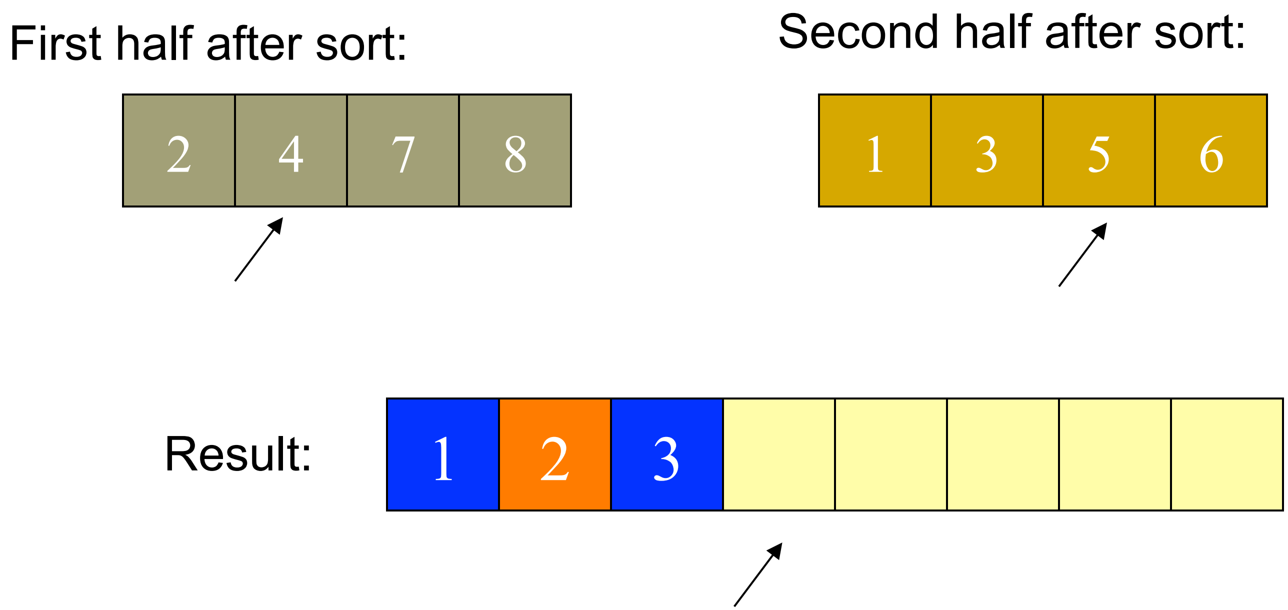

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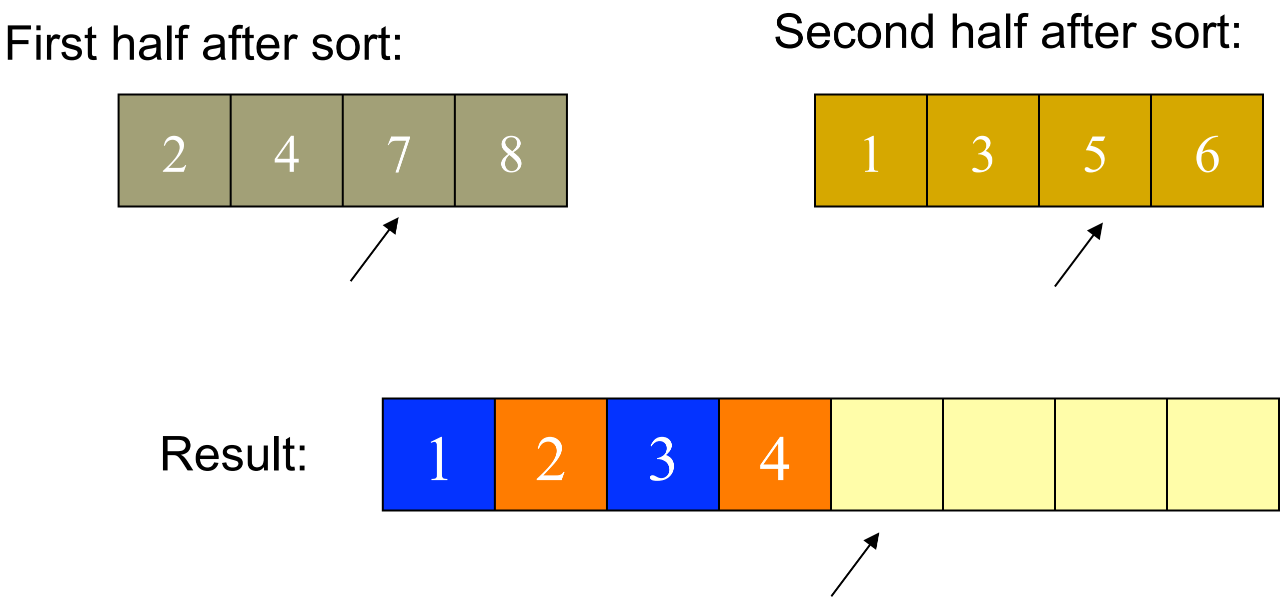

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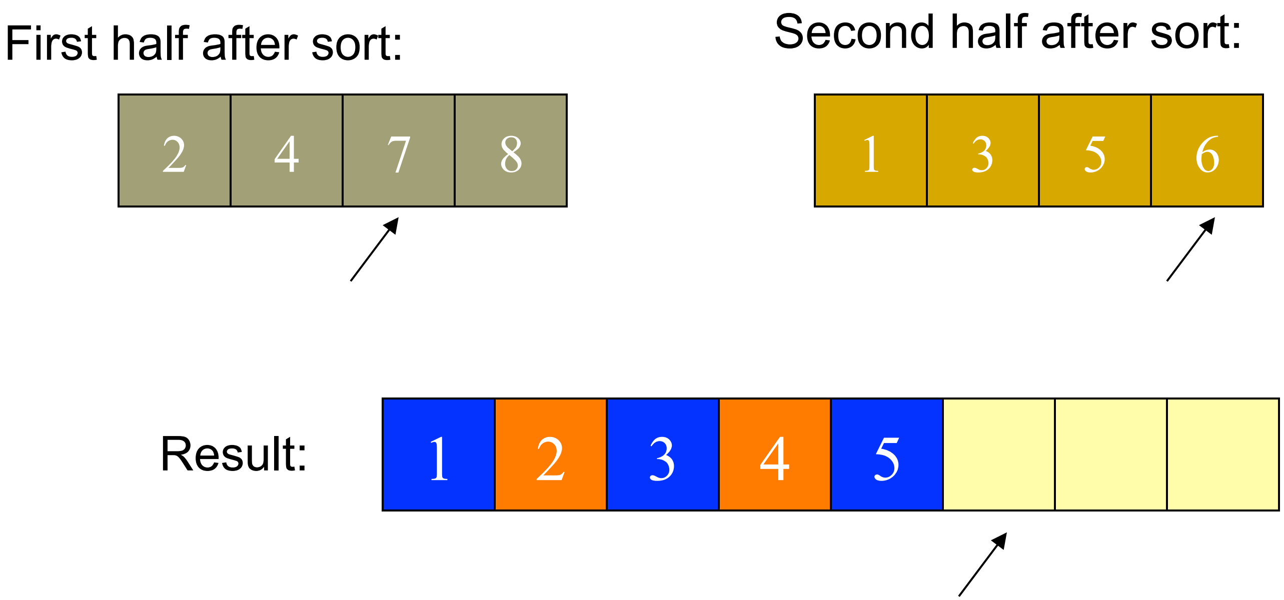

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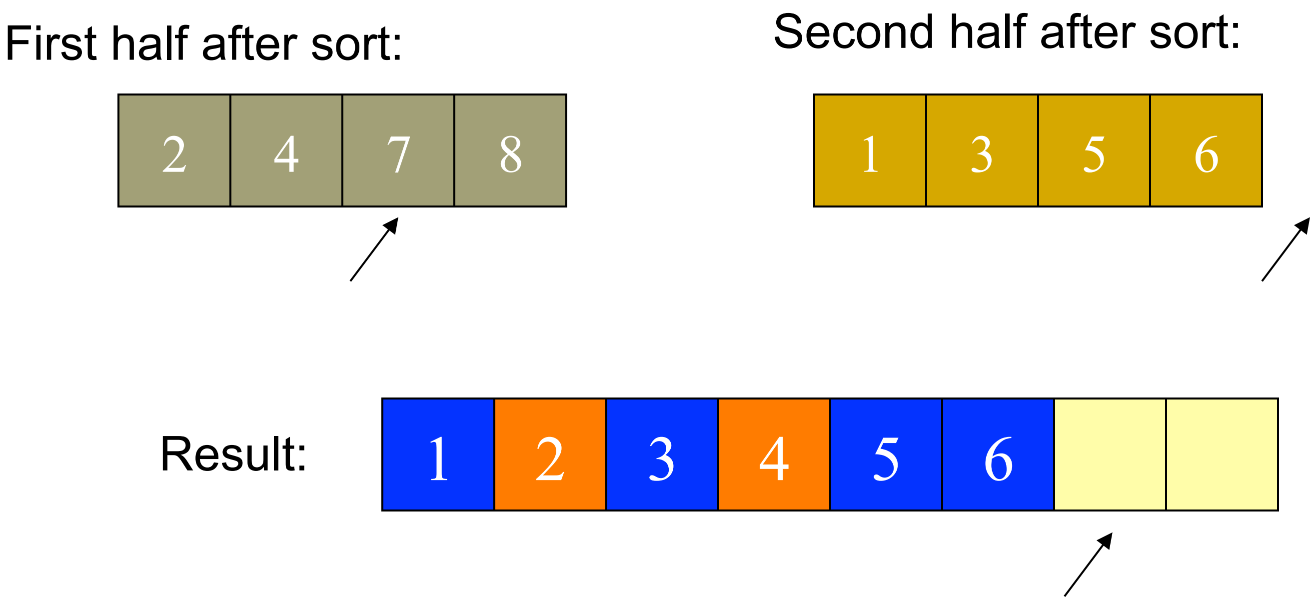

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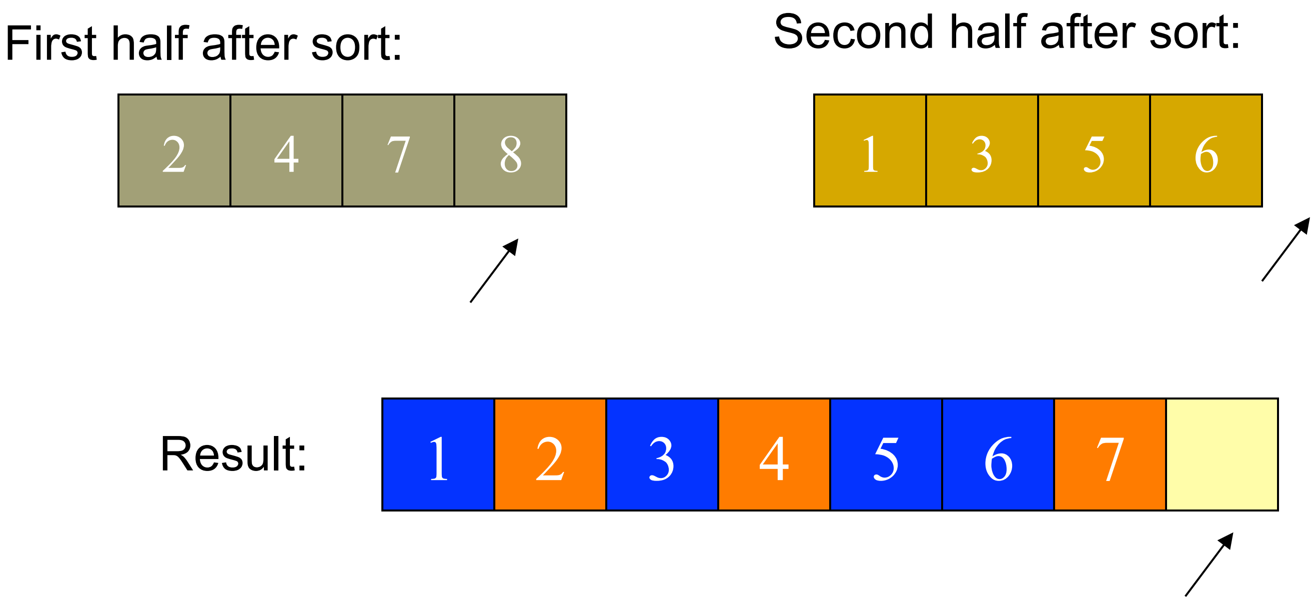

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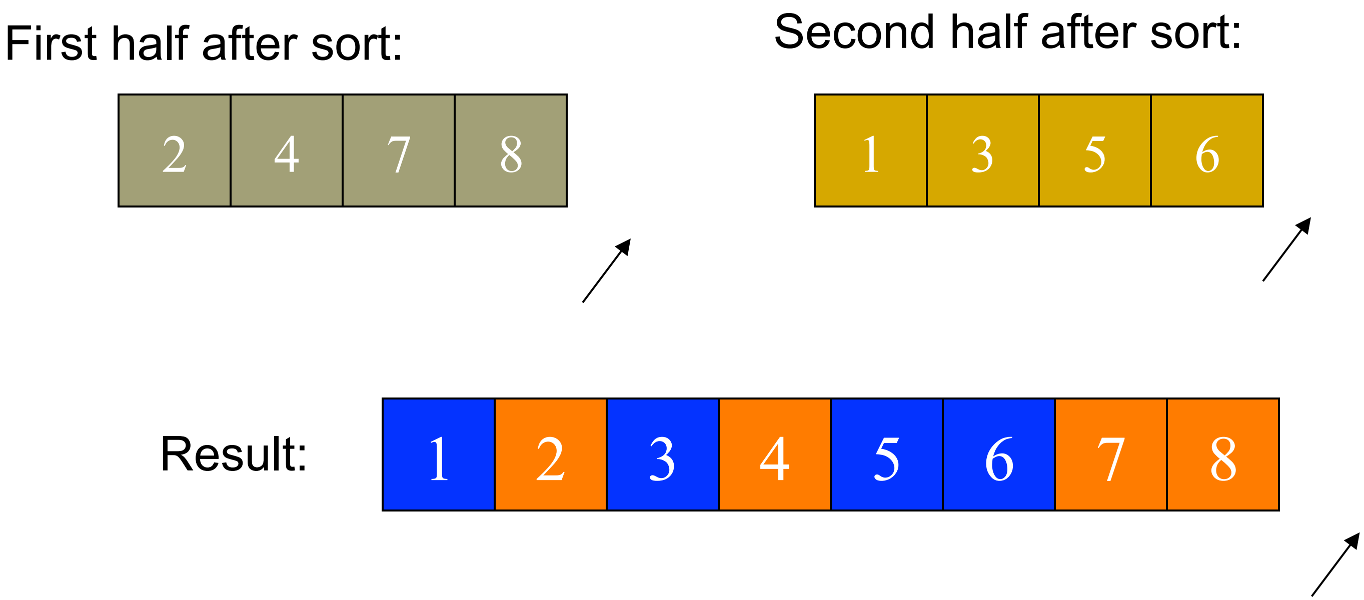
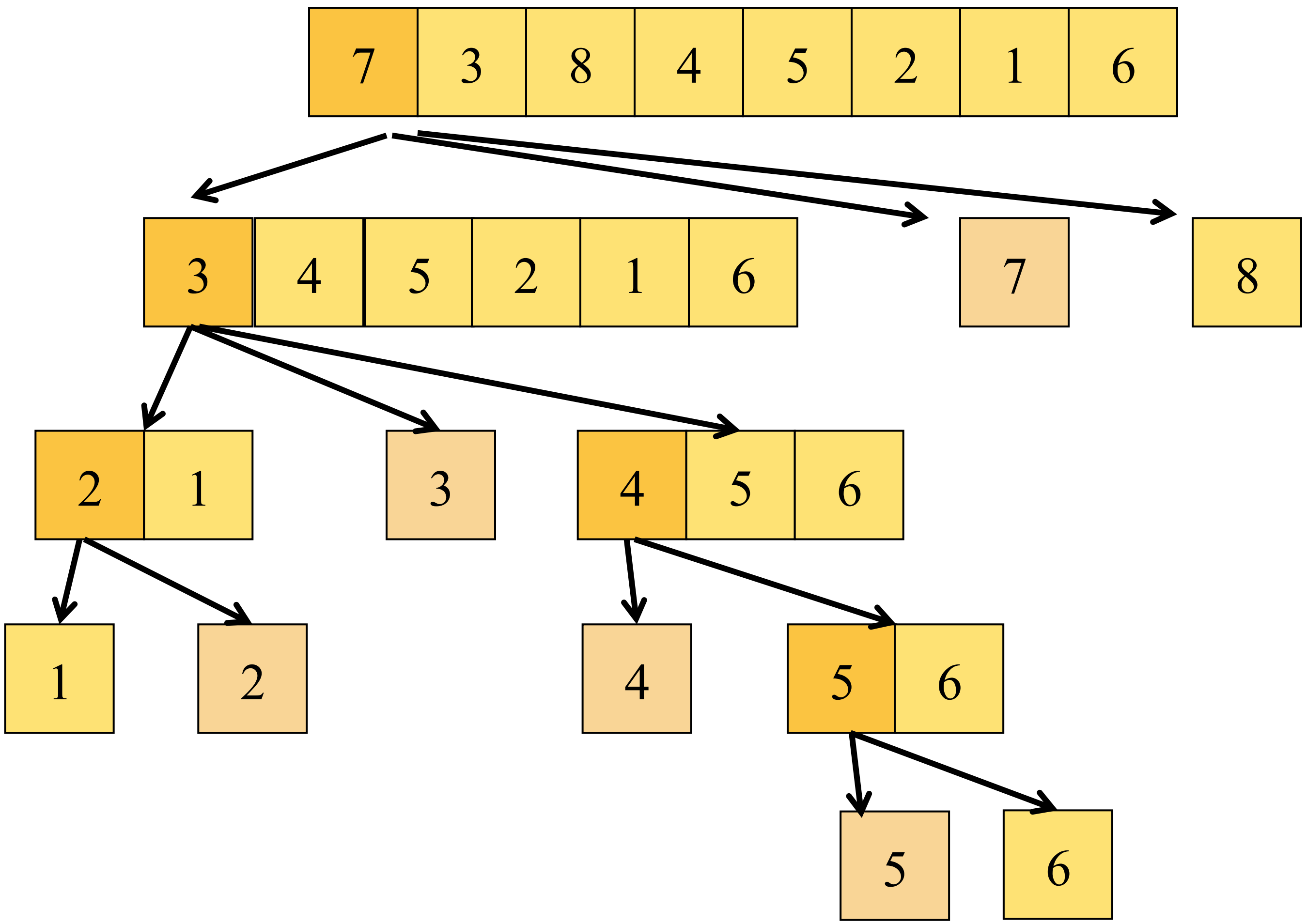


After Merge: copy result into original unsorted array.

Or alternate merging between two size n arrays.

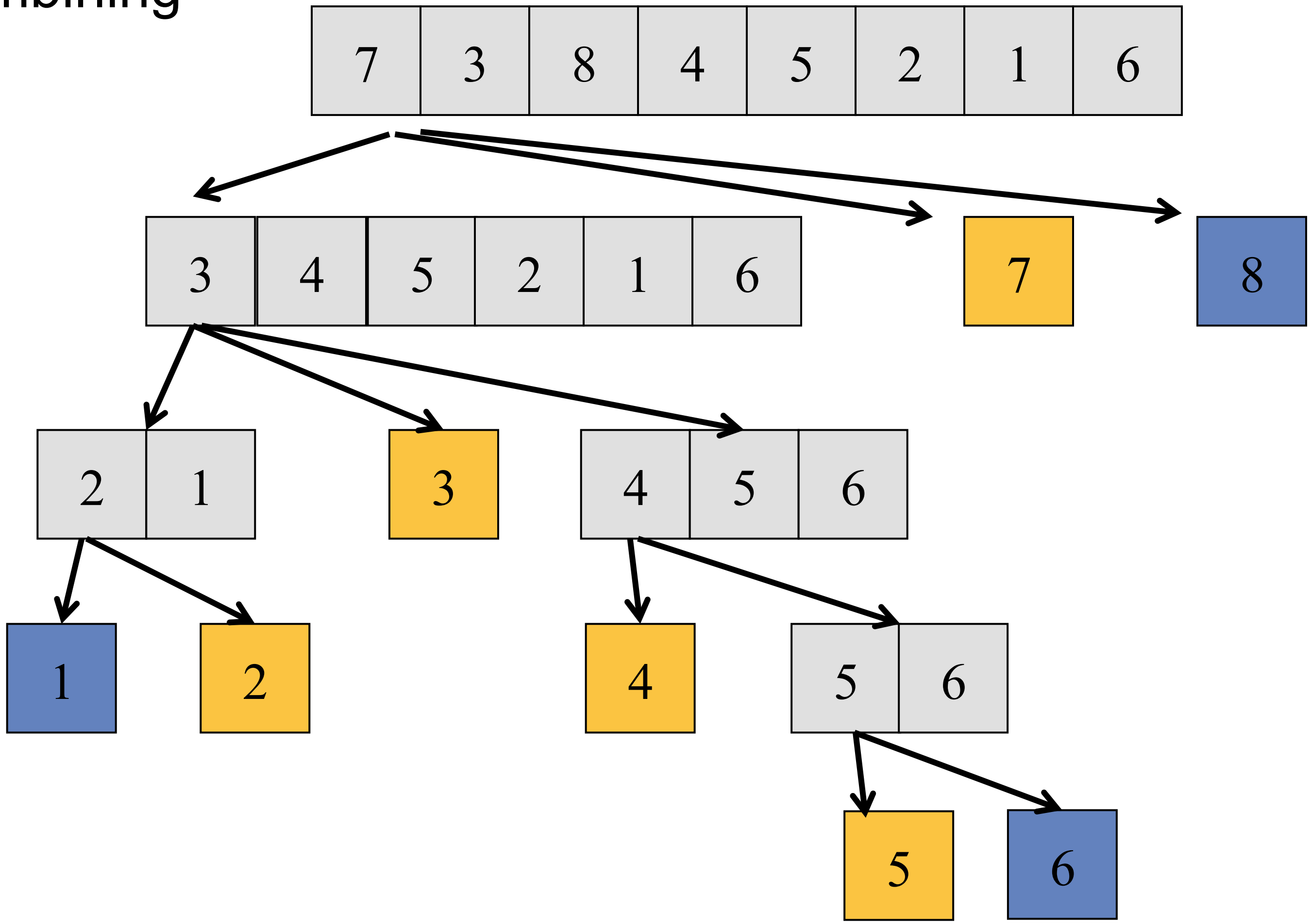
QUICK SORT EXAMPLE: DIVIDE

Pivot rule: pick the element at index 0



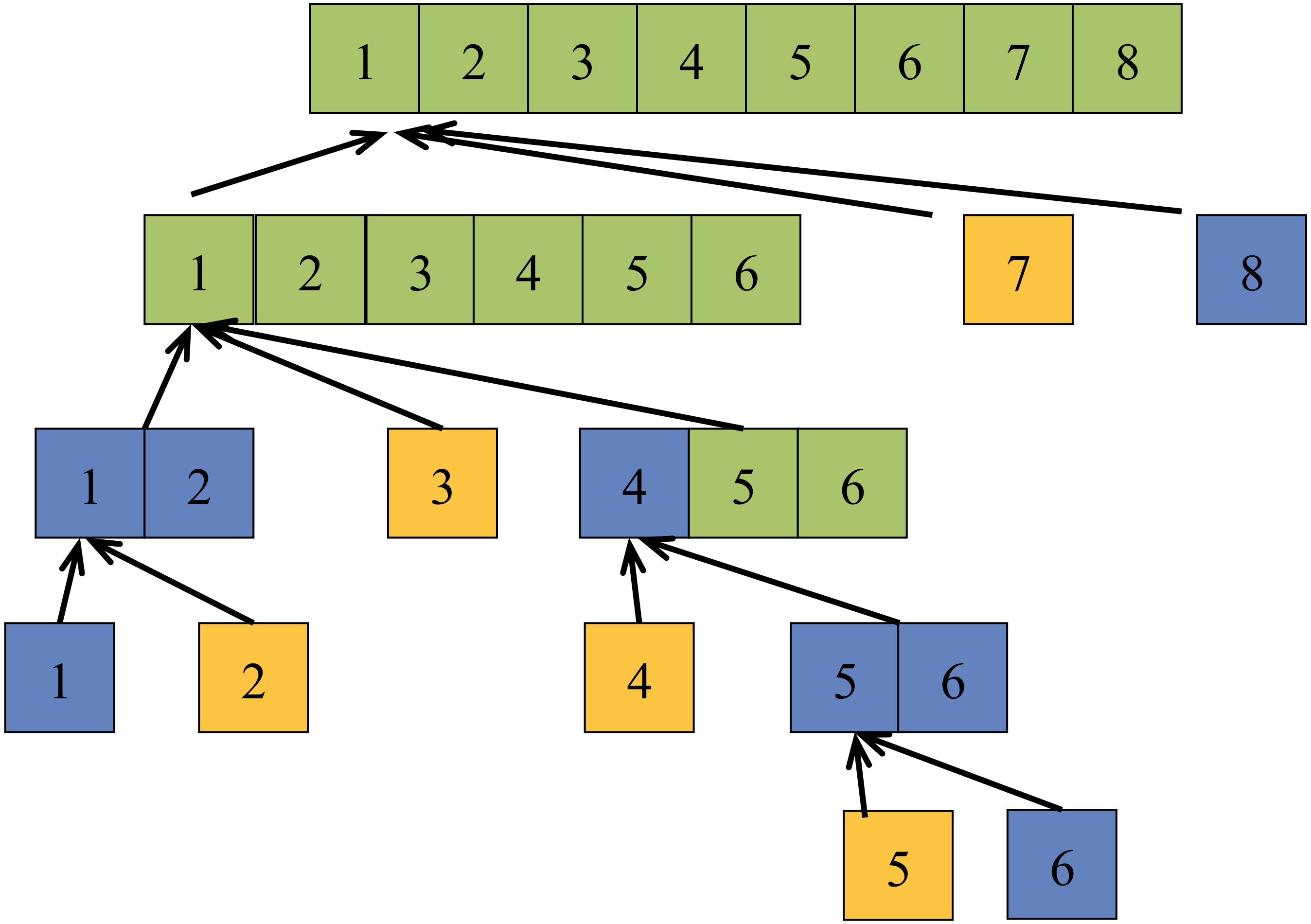
QUICK SORT EXAMPLE: COMBINE

Combine: this is the order of the elements we'll care about when combining



QUICK SORT EXAMPLE: COMBINE

Combine: put left partition < pivot < right partition



MEDIAN PIVOT EXAMPLE

Pick the median of first, middle, and last

7	2	8	4	5	3	1	6

Median = 6

Swap the median with the first value

7	2	8	4	5	3	1	6

Pivot is now at index 0, and we're ready to go

6	2	8	4	5	3	1	7

PARTITIONING

Conceptually simple, but hardest part to code up correctly

- After picking pivot, need to partition in linear time in place

One approach (there are slightly fancier ones):

1. Put pivot in index `lo`
2. Use two pointers `i` and `j`, starting at `lo+1` and `hi-1`
3. `while (i < j)`

```
       if (arr[j] > pivot) j--
       else if (arr[i] < pivot) i++
       else swap arr[i] with arr[j]
```

4. Swap pivot with `arr[i]` *

***skip step 4 if pivot ends up being least element**

EXAMPLE

Step one: pick pivot as median of 3

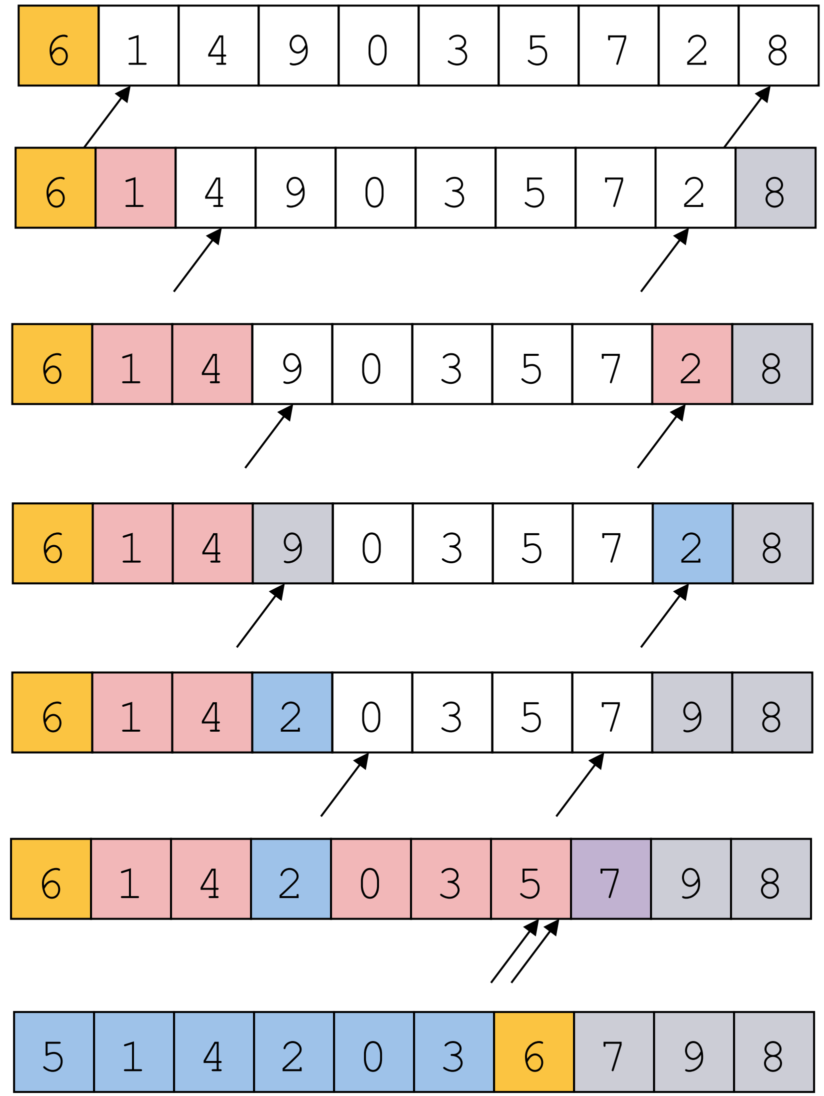
- **`lo`** = 0, **`hi`** = 10

0	1	2	3	4	5	6	7	8	9
8	1	4	9	**0**	3	5	2	7	**6**

- Step two: move pivot to the `lo` position

0	1	2	3	4	5	6	7	8	9
6	1	4	9	**0**	3	5	2	7	**8**

QUICK SORT PARTITION EXAMPLE



ASYMPTOTIC RUNTIME OF RECURSION

Recurrence Definition:

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Example: Fibonacci numbers.

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To analyze the runtime of recursive code, we use a recurrence by splitting the work into two pieces:

- Non-Recursive Work
- Recursive Work

RECURSIVE VERSION OF SUM:

```
int sum(int[] arr){
   return help(arr,0,arr.length);
}
int help(int[] arr, int lo, int hi) {
   if(lo==hi)   return 0;
   if(lo==hi-1) return arr[lo];
   int mid = (hi+lo)/2;
   return help(arr,lo,mid) + help(arr,mid,hi);
}
```

What's the recurrence T(n)?

- Non-Recursive Work:
- Recursive Work:

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}
```

What's the recurrence T(n)?

- Non-Recursive Work: O(1)
- Recursive Work: T(n/2) * 2 halves

T(n) = O(1) + 2*T(n/2)

SOLVING THAT RECURRENCE RELATION

1. **Determine the recurrence relation. What is the base case?**
   - *If* $T(1) = 1$, then $T(n) = 1 + 2^*T(n/2)$

2. **"Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.**
   - $T(n) = 1 + 2 * T(n / 2)$
     $= 1 + 2 + 2 * T(n / 4)$
     $= 1 + 2 + 4 + \ldots$ for log(n) times
     $= \ldots$
     $= 2^{(\log n)} – 1$

3. **Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case**
   - So $T(n)$ is $O(n)$

Explanation: it adds each number once while doing little else

SOLVING RECURRENCE RELATIONS EXAMPLE 2

1. **Determine the recurrence relation. What is the base case?**
   - *If* $T(n) = 10 + T(n/2)$ and $T(1) = 10$

2. **"Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.**
   - $T(n) = 10 + 10 + T(n/4)$
     $= 10 + 10 + 10 + T(n/8)$
     $= \ldots$
     $= 10k + T(n/(2^k))$

3. **Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case**
   - $n/(2^k) = 1$ means $n = 2^k$ means $k = \log_2 n$
   - So $T(n) = 10 \log_2 n + 8$ (get to base case and do it)
   - So $T(n)$ is $O(\log n)$

REALLY COMMON RECURRENCES

You can recognize some really common recurrences:

$T(n) = O(1) + T(n\text{-}1)$	linear
$T(n) = O(1) + 2T(n/2)$	linear
$T(n) = O(1) + T(n/2)$	logarithmic $O($`log` n)
$T(n) = O(1) + 2T(n\text{-}1)$	exponential
$T(n) = O(n) + T(n\text{-}1)$	quadratic
$T(n) = O(n) + T(n/2)$	linear
$T(n) = O(n) + 2T(n/2)$	$O($n `log` n) (divide and conquer sort)

Note big-Oh can also use more than one variable

Example: can sum all elements of an *n*-by-*m* matrix in $O(nm)$

QUICK SORT ANALYSIS

Best-case: Pivot is always the median

T(0)=T(1)=1

$T(n)=2T(n/2) + n$ -- linear-time partition

Same recurrence as mergesort: $O(n$ `log` $n)$

Worst-case: Pivot is always smallest or largest element

T(0)=T(1)=1

$T(n) = 1T(n\text{-}1) + n$

Basically same recurrence as selection sort: $O(n^2)$

Average-case (e.g., with random pivot)

- O(n `log` n), not responsible for proof

HOW FAST CAN WE SORT?

Heapsort & mergesort have $O(n \log n)$ worst-case running time

Quicksort has $O(n \log n)$ average-case running time

- ***Assuming* our comparison *model***: The only operation an algorithm can perform on data items is a 2-element comparison. There is no lower asymptotic complexity, such as $O(n)$ or $O(n \log \log n)$

COUNTING COMPARISONS

No matter what the algorithm is, it cannot make progress without doing comparisons

- **Intuition**: Each comparison can *at best* eliminate *half* the remaining possibilities of possible orderings

Can represent this process as a *decision tree*

- Nodes contain “set of remaining possibilities”
- Edges are “answers from a comparison”
- The algorithm does not actually build the tree; it’s what our *proof* uses to represent “the most the algorithm could know so far” as the algorithm progresses

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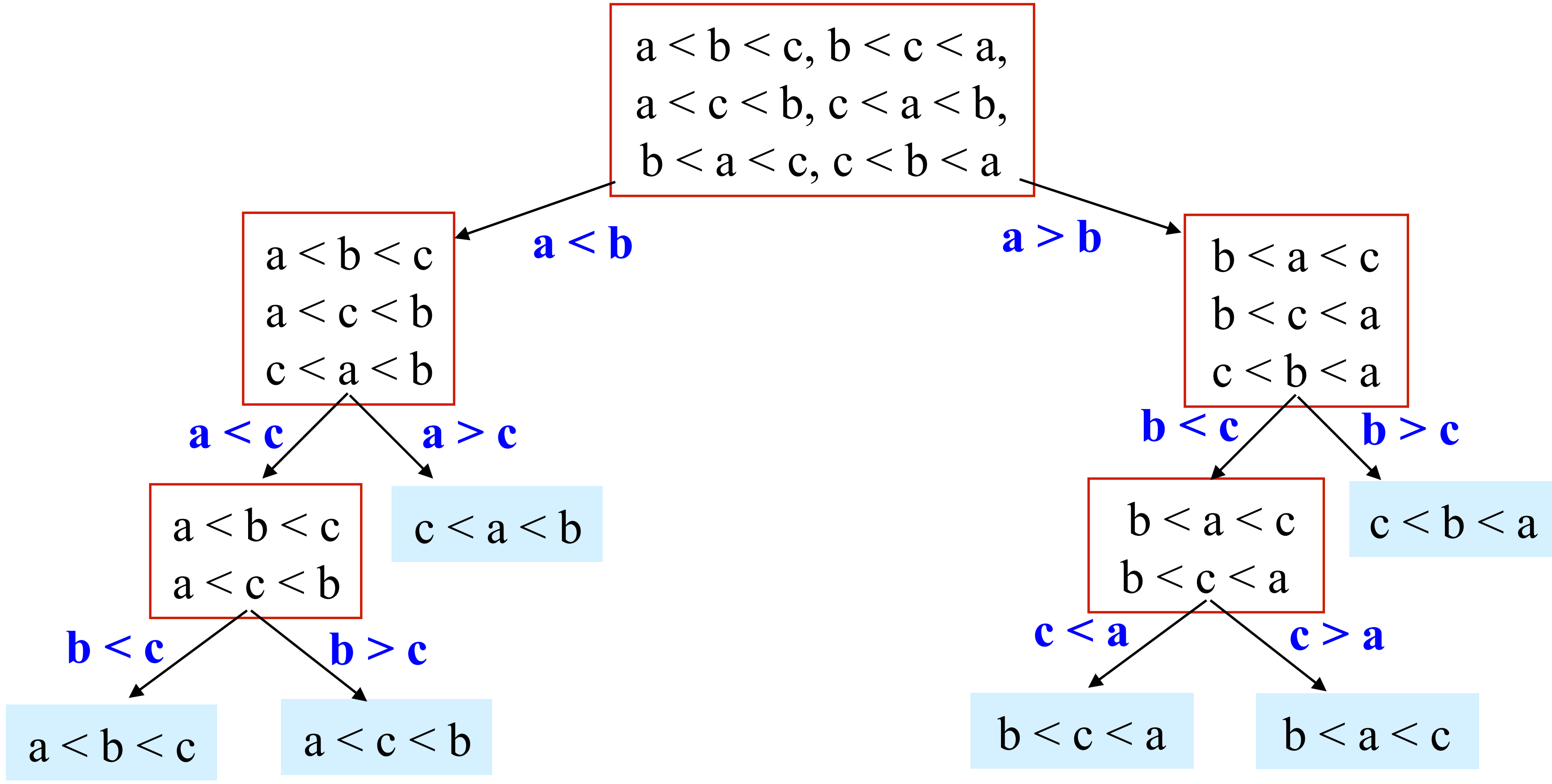
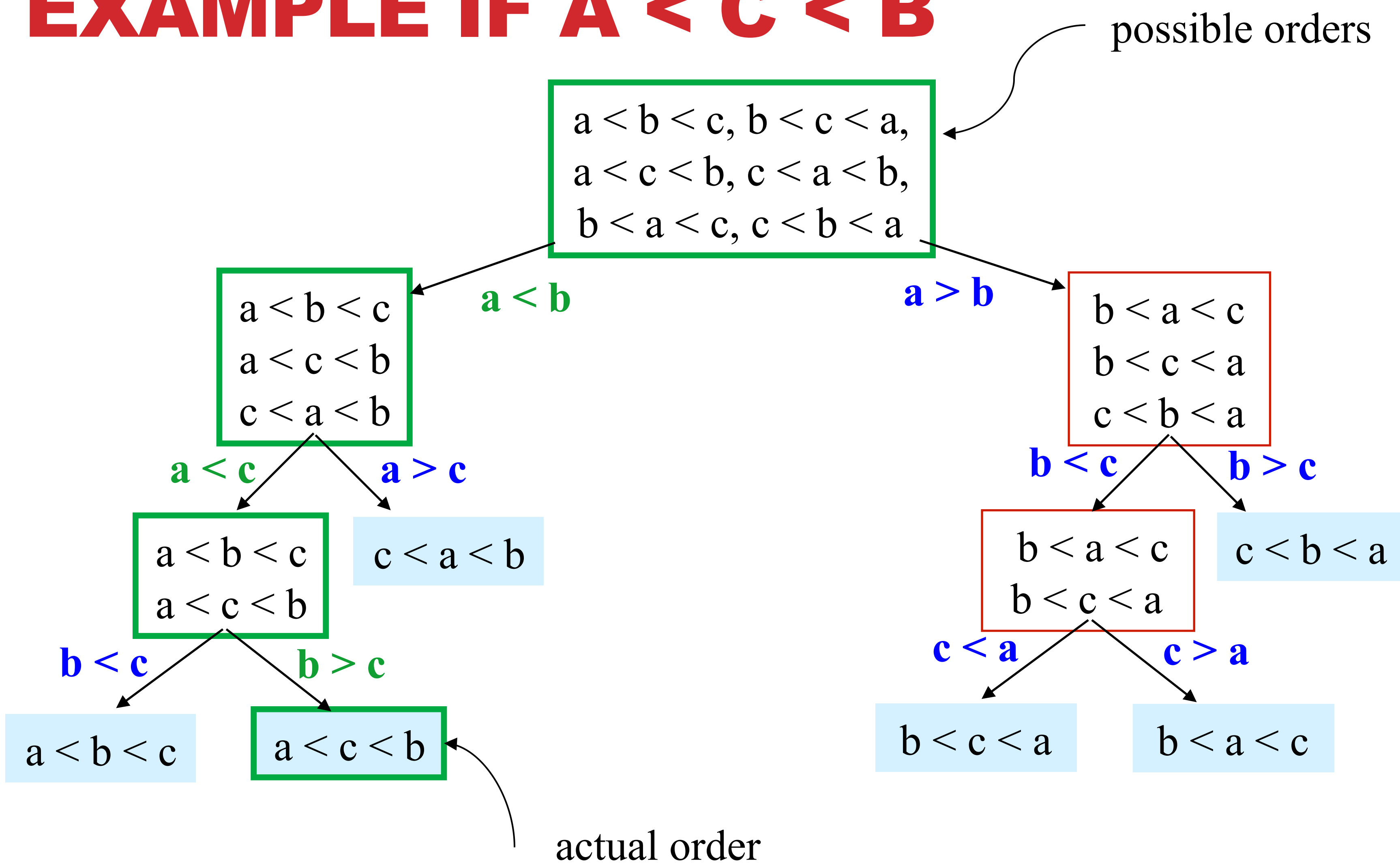
DECISION TREE FOR N = 3



- The leaves contain all the possible orderings of a, b, c

EXAMPLE IF A < C < B



DECISION TREE

A binary tree because each comparison has 2 outcomes (we're comparing 2 elements at a time)

Because any data is possible, any algorithm needs to ask enough questions to produce all orderings.

The facts we can get from that:

1. Each ordering is a different leaf (only one is correct)
2. Running *any* algorithm on *any* input will *at best* correspond to a root-to-leaf path in *some* decision tree. Worst number of comparisons is the longest path from root-to-leaf in the decision tree for input size n
3. There is no worst-case running time better than the height of a tree with *<num possible orderings>* leaves

POSSIBLE ORDERINGS

Assume we have *n* elements to sort. How many *permutations* of the elements (possible orderings)?

- For simplicity, assume none are equal (no duplicates)

Example, *n*=3

$a[0]<a[1]<a[2]$ $a[0]<a[2]<a[1]$

$a[1]<a[0]<a[2]$

$a[1]<a[2]<a[0]$ $a[2]<a[0]<a[1]$

$a[2]<a[1]<a[0]$

In general, *n* choices for least element, *n*-1 for next, *n*-2 for next, …

- *n*(*n*-1)(*n*-2)…(2)(1) = ***n*!** possible orderings

That means with n! possible leaves, best height for tree is log(n!), given that best case tree splits leaves in half at each branch

RUNTIME

That proves runtime is at least $\Omega(\log(n!))$. Can we write that more clearly?

$$
\begin{aligned}
\lg(n!) &= \lg(n(n-1)(n-2)\ldots 1) && \text{[Def. of } n!\text{]} \\
&= \lg(n) + \lg(n-1) + \ldots \lg\left(\frac{n}{2}\right) + \lg\left(\frac{n}{2} - 1\right) + \ldots \lg(1) && \text{[Prop. of Logs]} \\
&\geq \lg(n) + \lg(n-1) + \ldots + \lg\left(\frac{n}{2}\right) \\
&\geq \left(\frac{n}{2}\right) \lg\left(\frac{n}{2}\right) \\
&= \left(\frac{n}{2}\right)(\lg n - \lg 2) \\
&= \frac{n \lg n}{2} - \frac{n}{2} \\
&\in \Omega(n \lg(n))
\end{aligned}
$$

Nice! Any sorting algorithm must do *at best* (1/2)*(*n*`log` *n* – *n*) comparisons: Ω(*n*`log` *n*)

SORTING

- This is the lower bound for comparison sorts

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- **How can non-comparison sorts work better?**
  - They need to know something about the data
- **Strings and Ints are very well ordered**
  - If I told you to put “Apple” into a list of words, where would you put it?

BUCKETSORT

If all values to be sorted are known to be integers between 1 and *K* (or any small range):

- Create an array of size *K*
- Put each element in its proper bucket (a.k.a. bin)
- *If* data is only integers, no need to store more than a *count* of how times that bucket has been used

Output result via linear pass through array of buckets

`count` array	
1	3
2	1
3	2
4	2
5	3

- Example:
  K=5
  input (5,1,3,4,3,2,1,1,5,4,5)
  output: 1,1,1,2,3,3,4,4,5,5,5

ANALYZING BUCKET SORT

Overall: *O*(*n*+*K*)

- Linear in *n*, but also linear in *K*

Good when *K* is smaller (or not much larger) than *n*

- We don't spend time doing comparisons of duplicates

Bad when *K* is much larger than *n*

- Wasted space; wasted time during linear *O*(*K*) pass

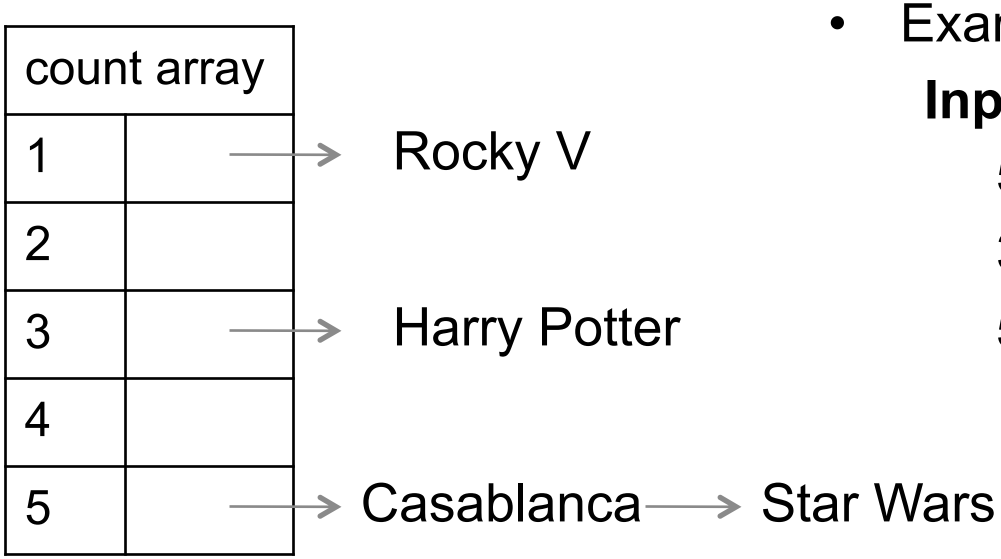
For data in addition to integer keys, use list at each bucket

BUCKET SORT

Most real lists aren't just keys; we have data

Each bucket is a list (say, linked list)

To add to a bucket, insert in *O*(1) (at beginning, or keep pointer to last element)



- Example: Movie ratings; scale 1-5

  Input:

  5: Casablanca

  3: Harry Potter movies

  5: Star Wars Original Trilogy

  1: Rocky V

- Result: 1: Rocky V, 3: Harry Potter, 5: Casablanca, 5: Star Wars
- Easy to keep 'stable'; Casablanca still before Star Wars

RADIX SORT

Radix = "the base of a number system"

- Examples will use base 10 because we are used to that
- In implementations use larger numbers
  - For example, for ASCII strings, might use 128

Idea:

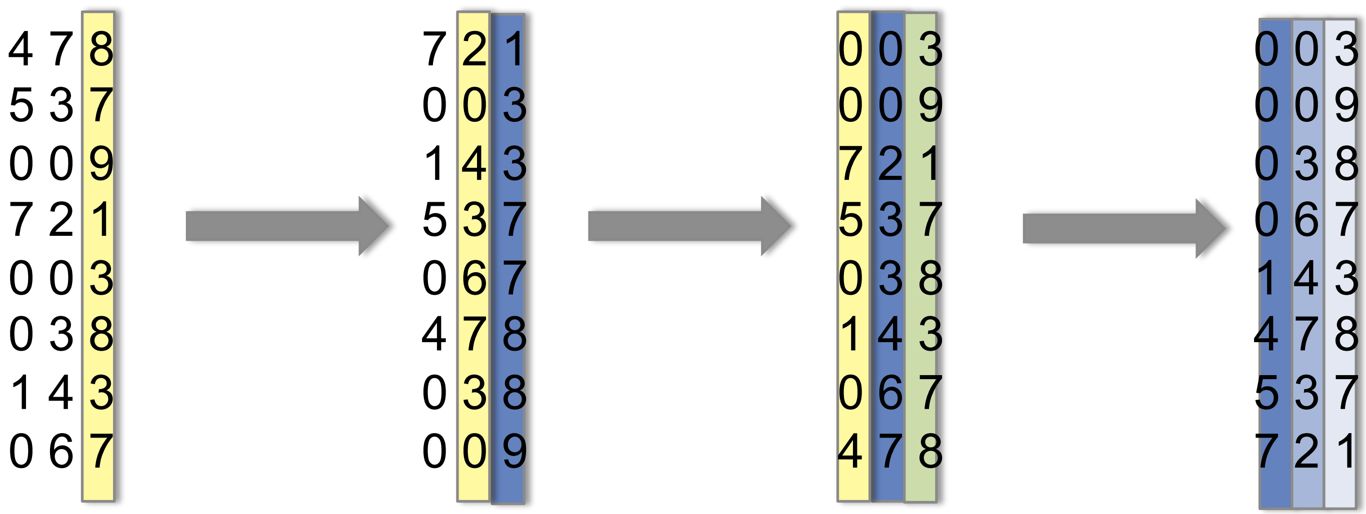
- Bucket sort on one digit at a time
  - Number of buckets = radix
  - Starting with *least* significant digit
  - Keeping sort *stable*
- Do one pass per digit
- **Invariant**: After *k* passes (digits), the last *k* digits are sorted

RADIX SORT EXAMPLE

Radix = 10

Input: 478, 537, 9, 721, 3, 38, 143, 67

3 passes (input is 3 digits at max), on each pass, stable sort the input highlighted in yellow



ANALYSIS

Input size: n

Number of buckets = Radix: B

Number of passes = "Digits": P

Work per pass is 1 bucket sort: $O(B+n)$

Total work is $O(P(B+n))$

Compared to comparison sorts, sometimes a win, but often not

- Example: Strings of English letters up to length 15
  - Run-time proportional to: $15*(52 + n)$
  - This is less than n log n only if $n > 33{,}000$
  - Of course, cross-over point depends on constant factors of the implementations

SORTING TAKEAWAYS

Simple $O(n^2)$ sorts can be fastest for small n

- Selection sort, Insertion sort (latter linear for mostly-sorted)
- Good for “below a cut-off” to help divide-and-conquer sorts

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$O(n \texttt{ log } n)$ sorts

- Heap sort, in-place but not stable nor parallelizable
- Merge sort, not in place but stable and works as external sort
- Quick sort, in place but not stable and $O(n^2)$ in worst-case
    - Often fastest, but depends on costs of comparisons/copies

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Non-comparison sorts

- Bucket sort good for small number of possible key values
- Radix sort uses fewer buckets and more phases

Best way to sort? It depends!

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