# CSE 373

## MAY 12TH – MINIMUM SPANNING TREES

# ASSORTED MINUTIAE

- **HW5 is out**
  - Write up has a Minimum spanning tree question, which we're covering today
  - Code due next Wednesday, as usual, Write up will be due on Friday.
- **H2 finally graded**
  - 10 or so students no grade yet, out this weekend. Problems with the script.
- **Feedback on HW4 code out this weekend**

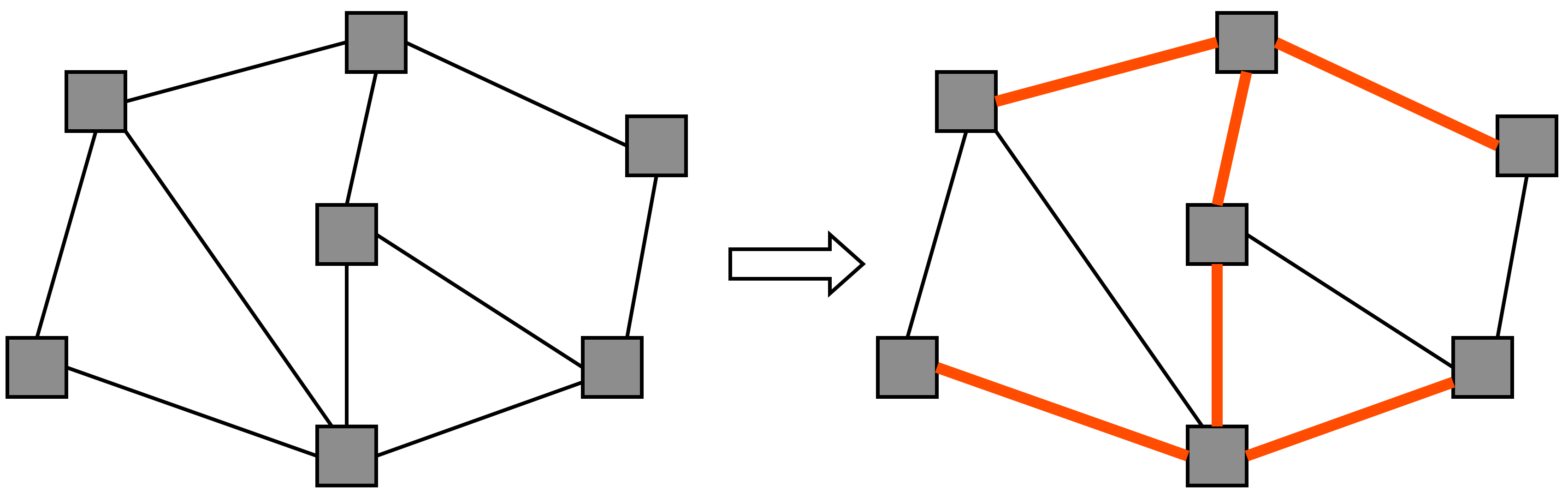
# TODAY'S LECTURE

- **Minimum Spanning Trees**
  - Prim's Algorithm (vertex based solution)
  - Kruskal's Algorithm (edge based solution)

# SPANNING TREES

**Given a *connected* undirected graph G=(V,E), find a subset of edges such that G is still connected**

- A graph **G2**=(**V**,**E2**) such that **G2** is connected and removing any edge from **E2** makes **G2** disconnected



# OBSERVATIONS

1. **Problem not defined if original graph not connected. Therefore, we know |E| >= |V|-1**

2. **Any solution to this problem is a tree**
   - Recall a tree does not need a root; just means acyclic
   - For any cycle, could remove an edge and still be connected

3. **Solution not unique unless original graph was already a tree**

4. **A tree with |V| nodes has |V|-1 edges**
   - So every solution to the spanning tree problem has **|V|-1** edges

# MOTIVATION

A spanning tree connects all the nodes with as few edges as possible

In most compelling uses, we have a *weighted* undirected graph and we want a tree of least total cost

Example: Electrical wiring for a house or clock wires on a chip

Example: A road network if you cared about asphalt cost rather than travel time

This is the minimum spanning tree problem

# LAST CLASS

**Different algorithmic approaches to the spanning-tree problem:**

1. **Do a graph traversal (e.g., depth-first search, but any traversal will do), keeping track of edges that form a tree**

2. **Iterate through edges; add to output any edge that does not create a cycle**

# SPANNING TREE VIA TRAVERSAL

```
spanning_tree(Graph G) {
  for each node v:
      v.marked = false
  dfs(someRandomStartNode)
}
dfs(Vertex a) {  // recursive DFS
  a.marked = true
  for each b adjacent to a:
    if(!b.marked) {
      add(a,b) to output
      dfs(b)
  }
}
```

# MINIMAL SPANNING TREES

- **How do we get a minimal spanning tree from a traversal?**
  - What parts of a traversal can we change?
  - Select which vertex we visit next by which is closest to an old vertex

# PRIM'S ALGORITHM

- **A traversal**
  - Pick a start node
  - Keep track of all of the vertices you can reach
  - Add the vertex that is closest (has the edge with smallest weight) to the current spanning tree.
- **Is this similar to something we've seen before?**

# PRIM'S ALGORITHM

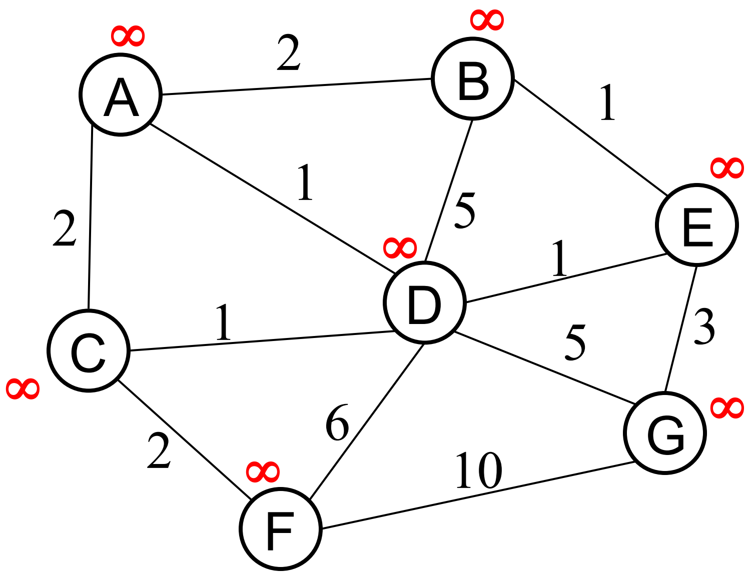
- **Modify Dijkstra's algorithm**
  - Instead of measuring the total length from start to the new vertex, now we only care about the edge from our current spanning tree to new nodes

# THE ALGORITHM

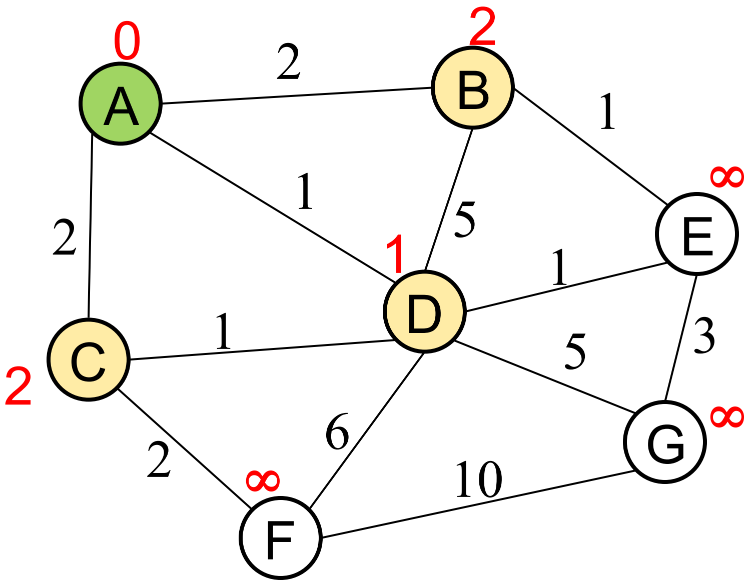
1. **For each node v, set v.cost = $\infty$ and v.known = false**
2. **Choose any node v**
   a) Mark **v** as known
   b) For each edge **(v,u)** with weight **w**, set **u.cost=w** and **u.prev=v**
3. **While there are unknown nodes in the graph**
   a) Select the unknown node **v** with lowest cost
   b) Mark **v** as known and add **(v, v.prev)** to output
   c) For each edge **(v,u)** with weight **w**,

```
if(w < u.cost) {
    u.cost = w;
  u.prev = v;
}
```

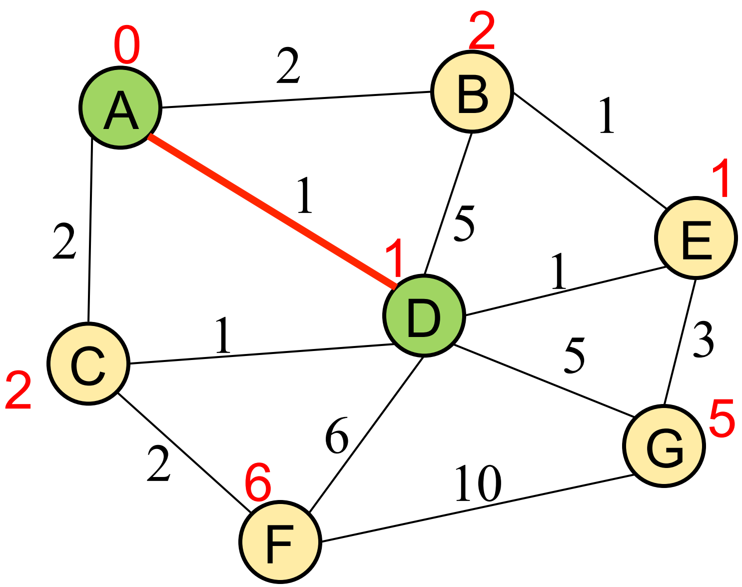
# EXAMPLE



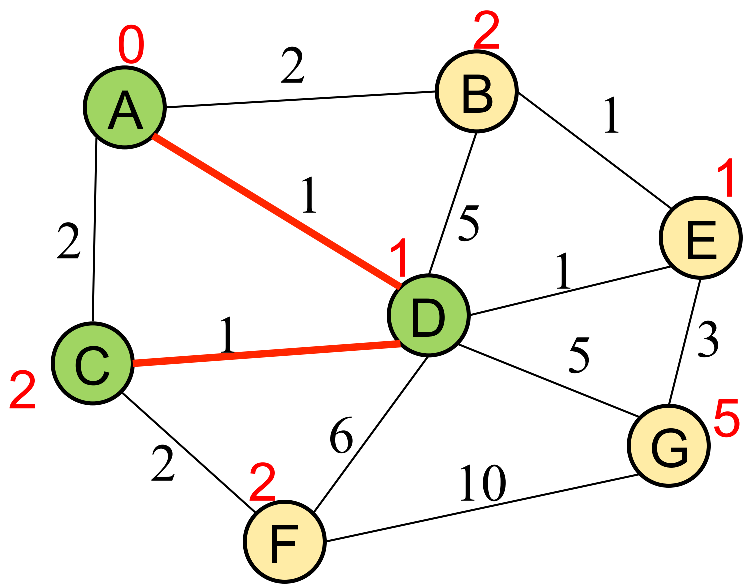

| vertex | known? | cost | prev |
| --- | --- | --- | --- |
| A | | ∞ | |
| B | | ∞ | |
| C | | ∞ | |
| D | | ∞ | |
| E | | ∞ | |
| F | | ∞ | |
| G | | ∞ | |



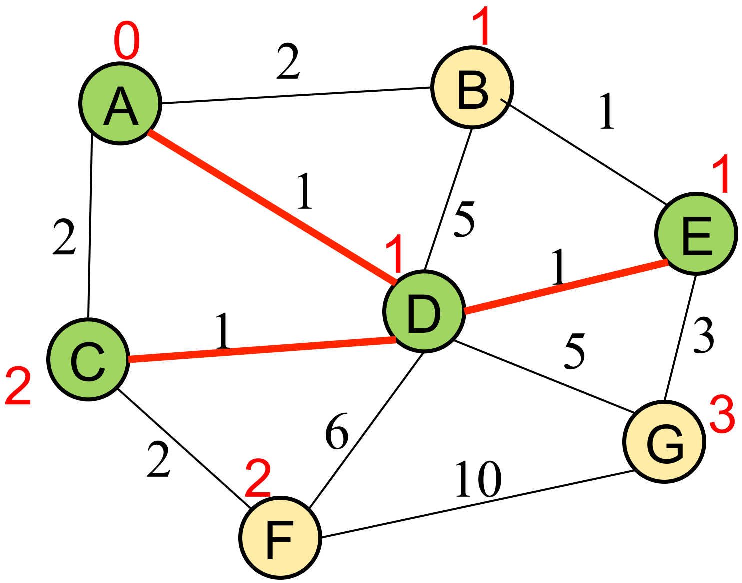

| vertex | known? | cost | prev |
|---|---|---|---|
| A | Y | 0 | |
| B | | 2 | A |
| C | | 2 | A |
| D | | 1 | A |
| E | | ∞ | |
| F | | ∞ | |
| G | | ∞ | |



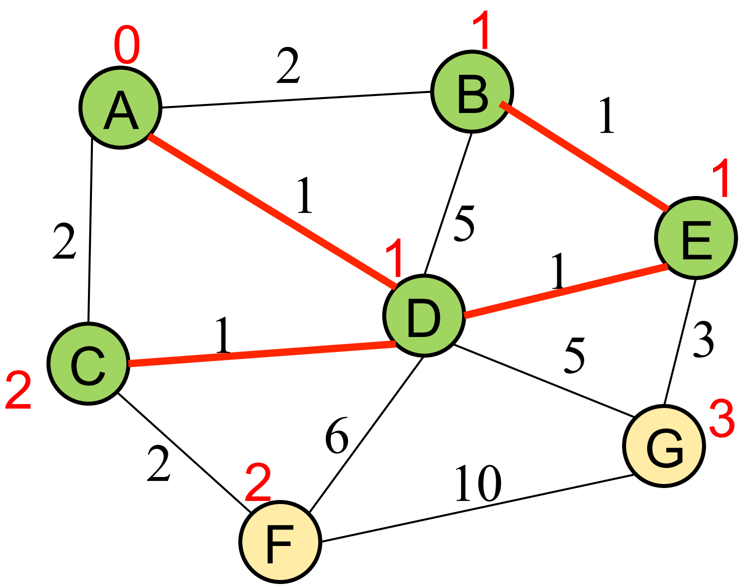

| vertex | known? | cost | prev |
|---|---|---|---|
| A | Y | 0 | |
| B | | 2 | A |
| C | | 1 | D |
| D | Y | 1 | A |
| E | | 1 | D |
| F | | 6 | D |
| G | | 5 | D |




| vertex | known? | cost | prev |
|---|---|---|---|
| A | Y | 0 | |
| B | | 2 | A |
| C | Y | 1 | D |
| D | Y | 1 | A |
| E | | 1 | D |
| F | | 2 | C |
| G | | 5 | D |




| vertex | known? | cost | prev |
|---|---|---|---|
| A | Y | 0 | |
| B | | 1 | E |
| C | Y | 1 | D |
| D | Y | 1 | A |
| E | Y | 1 | D |
| F | | 2 | C |
| G | | 3 | E |




| vertex | known? | cost | prev |
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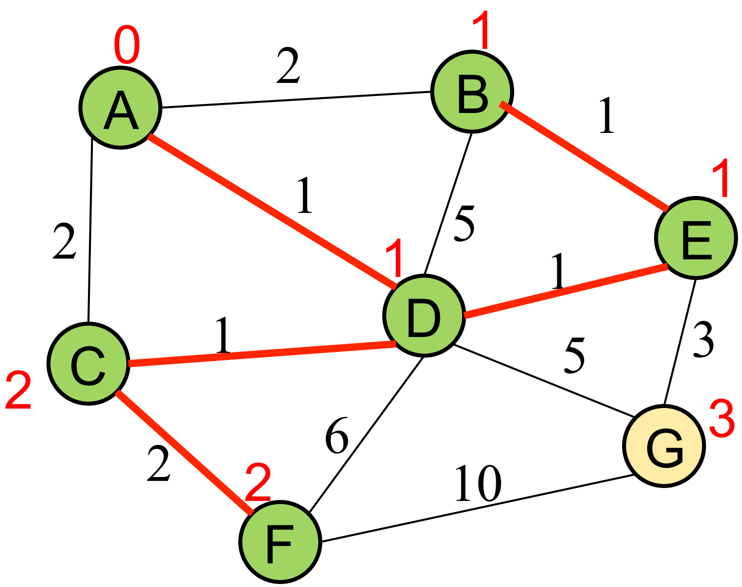


| vertex | known? | cost | prev |
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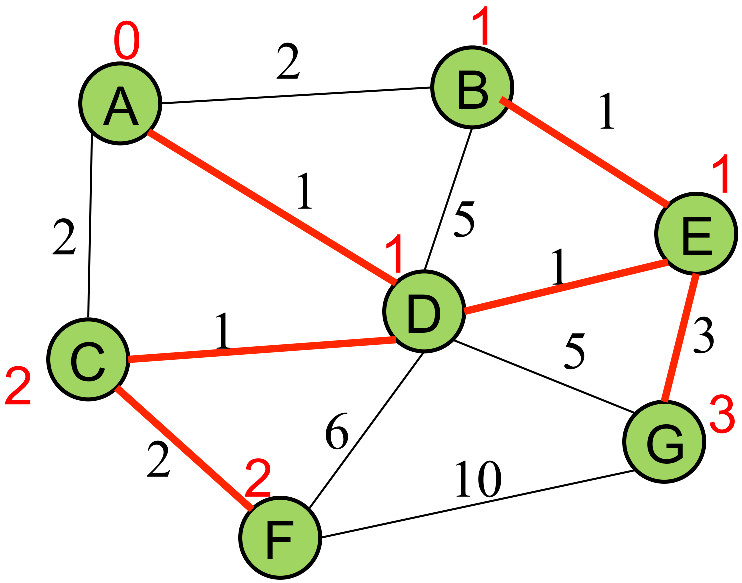


| vertex | known? | cost | prev |
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| B | Y | 1 | E |
| C | Y | 1 | D |
| D | Y | 1 | A |
| E | Y | 1 | D |
| F | Y | 2 | C |
| G | Y | 3 | E |

# PRIM’S ALGORITHM

- **Does this give us the correct solution? Why?**
  - If we consider the “known” cloud as a single vertex, we will never add edges that form a cycle
  - Each time, we take the edge that has minimal weight going out of the vertex.
  - This is the cheapest way of connecting the two subgraphs.

# PRIM'S ALGORITHM

- **What is the runtime?**
  - Traversals go through all of the edges, in the worst case
    - Need to check if an edge forms a cycle or if it has minimal weight.
    - We can check if it forms a cycle by verifying if the other vertex is in the "known cloud" **O(1)**
    - How long to check if it is minimal? **O(log |V|)** if we use a priority queue

# PRIM'S ALGORITHM

- **O(|E| log |V|)**
  - We can use a priority queue to store all of our vertices, and let the edges to them be the priority.
  - Use the decreaseKey() function when the edge to a vertex changes.
  - This also works for Dijkstra's algorithm, but you aren't required to do it for HW5
  - Without the priority queue, both Prim's and Dijkstra's run in **O(|E||V|)**

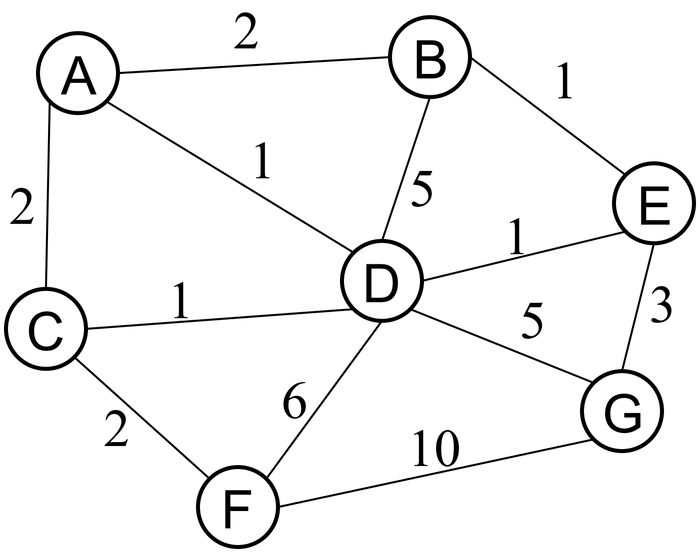
# KRUSKAL’S ALGORITHM

- **Prim’s algorithm works from the vertices, and builds a contiguous spanning tree**
  - The spanning tree grows out from a single vertex
- **Kruskal’s Algorithm adds edges based on their weight**
  - Must check for cycles
  - Use the union-find data structure to speed up this operation

# KRUSKAL'S ALGORITHM

- **Pseudocode:**
  - Sort the edges (or place them into a heap)
  - Create a union-find data structure with all separate vertices
  - For each edge, add it to the minimum spanning tree if the two vertices don't have the same representative in the union find
  - Union the two vertices in the union find
  - Stop after you've added |V|-1 edges

# EXAMPLE




**Edges in sorted order:**

**1: (A,D), (C,D), (B,E), (D,E)**

**2: (A,B), (C,F), (A,C)**

**3: (E,G)**

**5: (D,G), (B,D)**

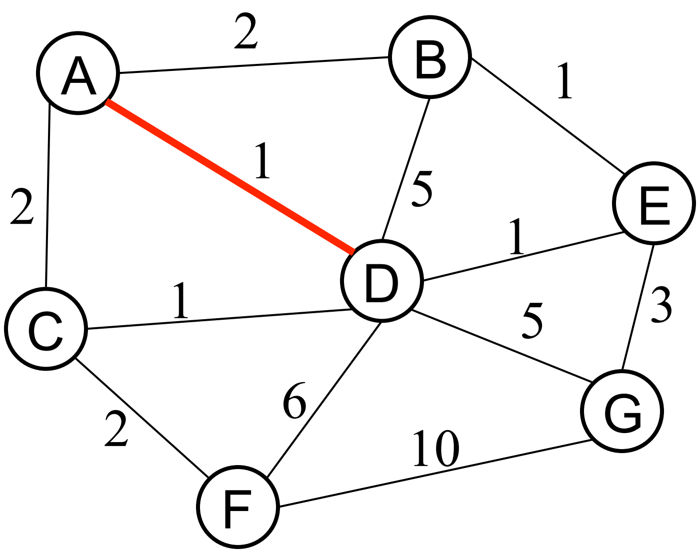
**6: (D,F)**

**10: (F,G)**

Output:

Note: At each step, the union/find sets are the trees in the forest

# EXAMPLE




**Edges in sorted order:**

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**2: (A,B), (C,F), (A,C)**

**3: (E,G)**

**5: (D,G), (B,D)**

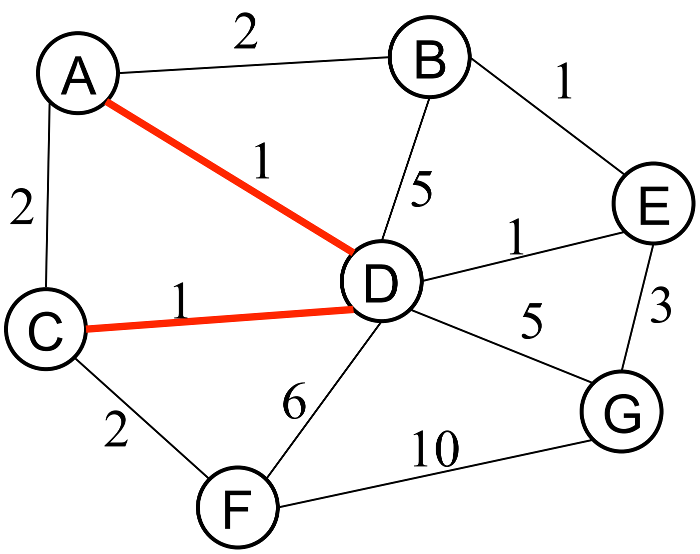
**6: (D,F)**

**10: (F,G)**

Output: (A,D)

Note: At each step, the union/find sets are the trees in the forest

# EXAMPLE




**Edges in sorted order:**

**1: (A,D), (C,D), (B,E), (D,E)**

**2: (A,B), (C,F), (A,C)**

**3: (E,G)**

**5: (D,G), (B,D)**

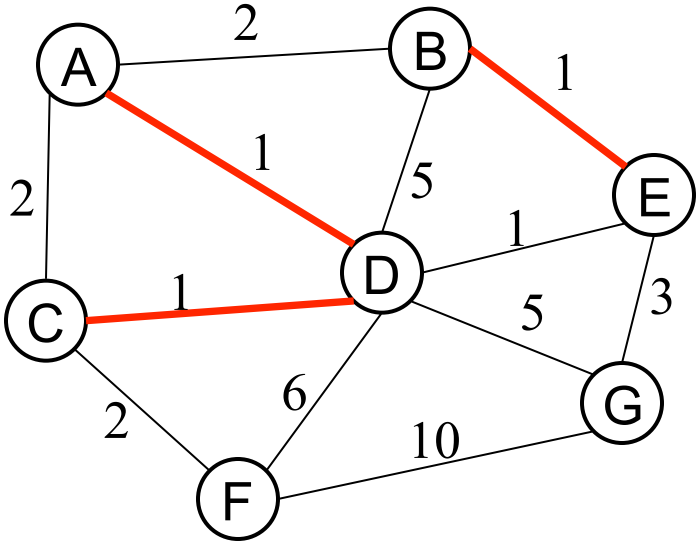
**6: (D,F)**

**10: (F,G)**

Output: (A,D), (C,D)

Note: At each step, the union/find sets are the trees in the forest

# EXAMPLE




**Edges in sorted order:**

**1: (A,D), (C,D), (B,E), (D,E)**

**2: (A,B), (C,F), (A,C)**

**3: (E,G)**

**5: (D,G), (B,D)**

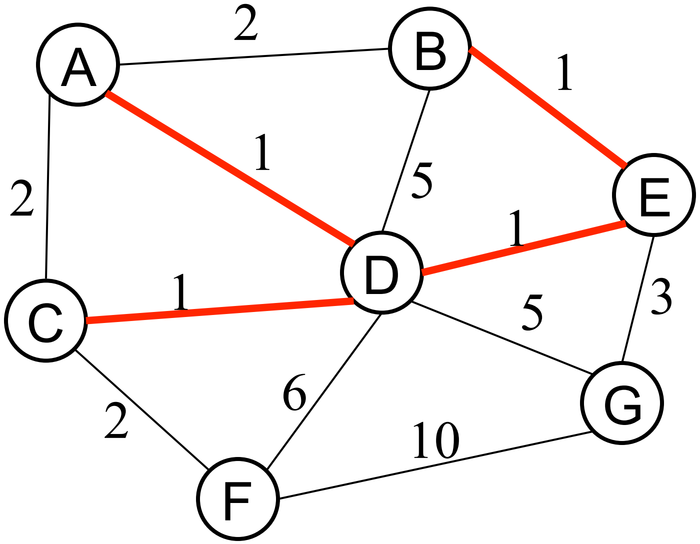
**6: (D,F)**

**10: (F,G)**

Output: (A,D), (C,D), (B,E)

Note: At each step, the union/find sets are the trees in the forest

# EXAMPLE




**Edges in sorted order:**

**1: (A,D), (C,D), (B,E), (D,E)**

**2: (A,B), (C,F), (A,C)**

**3: (E,G)**

**5: (D,G), (B,D)**

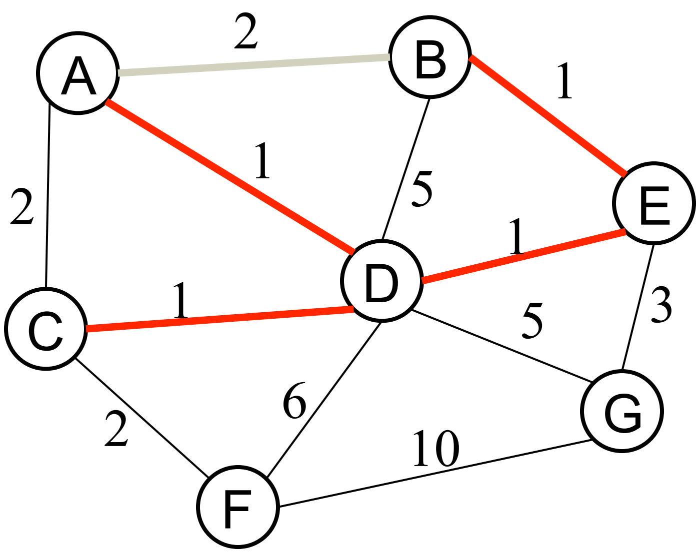
**6: (D,F)**

**10: (F,G)**

Output: (A,D), (C,D), (B,E), (D,E)

Note: At each step, the union/find sets are the trees in the forest

# EXAMPLE




**Edges in sorted order:**

**1: (A,D), (C,D), (B,E), (D,E)**

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Output: (A,D), (C,D), (B,E), (D,E)

Note: At each step, the union/find sets are the trees in the forest

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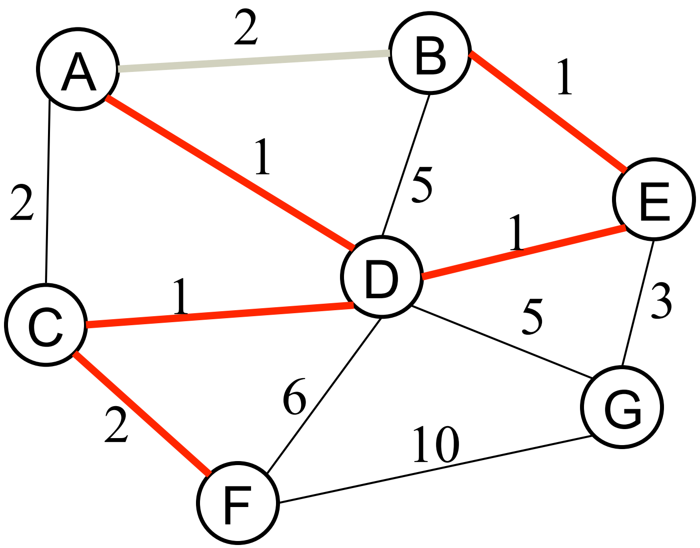


**Edges in sorted order:**

**1: (A,D), (C,D), (B,E), (D,E)**

**2: (A,B), (C,F), (A,C)**

**3: (E,G)**

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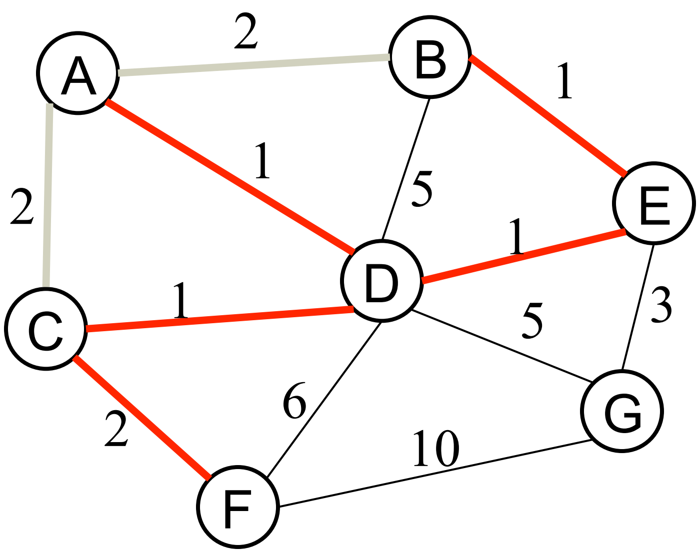
Output: (A,D), (C,D), (B,E), (D,E), (C,F)

Note: At each step, the union/find sets are the trees in the forest

# EXAMPLE




**Edges in sorted order:**

**1: (A,D), (C,D), (B,E), (D,E)**

**2: (A,B), (C,F), (A,C)**

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Output: (A,D), (C,D), (B,E), (D,E), (C,F)

Note: At each step, the union/find sets are the trees in the forest

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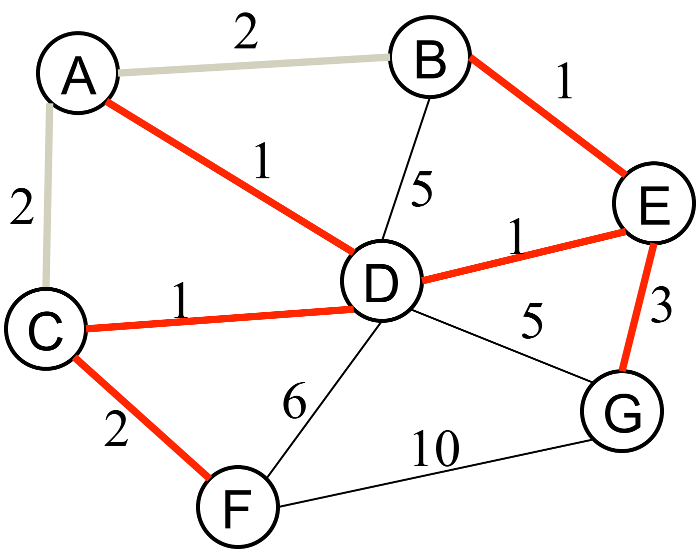


**Edges in sorted order:**

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**6: (D,F)**

**10: (F,G)**

Output: (A,D), (C,D), (B,E), (D,E), (C,F), (E,G)

Note: At each step, the union/find sets are the trees in the forest

# KRUSKAL'S ALGORITHM

- **Runtime**
  - Put edges into a heap **O(|E|)** Floyd's method!
  - Until the MST is complete:
    - Pull the minimum edge out of the heap **O(log |E|)**
    - Check if it forms a cycle **O(log |V|)**
  - **How many times does the loop run? O(E)**
  - **O(|E| log |E|)**

# COMPARISONS

- Prim's
  - O(|E| log |V|)
- Kruskal's
  - O(|E| log |E|)
- Since |E| must be at least |V|-1 for the graph to be connected, which do we prefer?

# COMPARISONS

- **Prim's**
  - **O(|E| log |V|)**
- **Kruskal's**
  - **O(|E| log |E|)**
- **Since |E| must be at least |V|-1 for the graph to be connected, which do we prefer?**
  - Since |E| is at most $|V|^2$, log|E| is at most log($|V|^2$) which is 2log|V|.
  - So log|E| is O(log|V|)

# CONCLUSIONS

- **Prim's and Kruskal's both run in O(|E| log |V|)**
- **An undirected graph has a unique minimum spanning tree if all of its edge weights are unique.**
- **If graphs have multiple edges of the same weight, it is possible (but not necessary) that there are many spanning trees of the same weight**

# NEXT WEEK

- **Graph algorithm runtimes**
- **Conclude Graphs**
- **New Algorithm Analysis technique**
  - Recurrences
- **Start sorting**