CSE 373

NOVEMBER 20TH – TOPOLOGICAL SORT

PROJECT 3

- **500 Internal Error problems**
  - Hopefully all resolved (or close to)
- **P3P1 grades are up (but muted)**
  - Leave canvas comment
  - Emails tomorrow
- **End of quarter**

GRAPHS

- **A graph is composed of two things**
  - A set of vertices
  - A set of edges (which are ordered vertex tuples)
- **Trees are types of graphs**
  - Each of the nodes is a vertex
  - Each pointer from parent to child is an edge
- **Represented as G(V,E) to indicate that V is the set of vertices and E is the set of edges**

GRAPHS

- **Graphs are not an ADT**
  - There is no “functions” that a graph supports
  - Rather, graphs are a theoretical framework for understanding certain types of problems.
  - Travelling salesman, path finding, resource allocating

ANALYZING GRAPHS

- **In graphs, there are two important variables, |V| and |E|**
  - Our analysis can now have two inputs
  - Before, our input size was *n*, now we use |V| and |E|
  - What is the maximum size of |E|? **$O(|V|^2)$**
    - For any vertices a,b, there can exist at most one edge (a,b)
    - A can equal B (this is a self loop)
    - There can be (b,a) -- directed

GRAPHS

- **Paths and Cycles**
  - A path: a set of edges connecting two vertices where all of the edges are connected and neither edges nor vertices are repeated
  - A cycle: a path that starts and ends on the same

GRAPHS

- **Paths and cycles can not have repeated vertices or edges**
  - A path that can repeat vertices or edges is called a walk
  - A path that can repeat vertices but not edges is called a trail
  - A circuit is a trail that starts and ends at the same vertex

GRAPHS

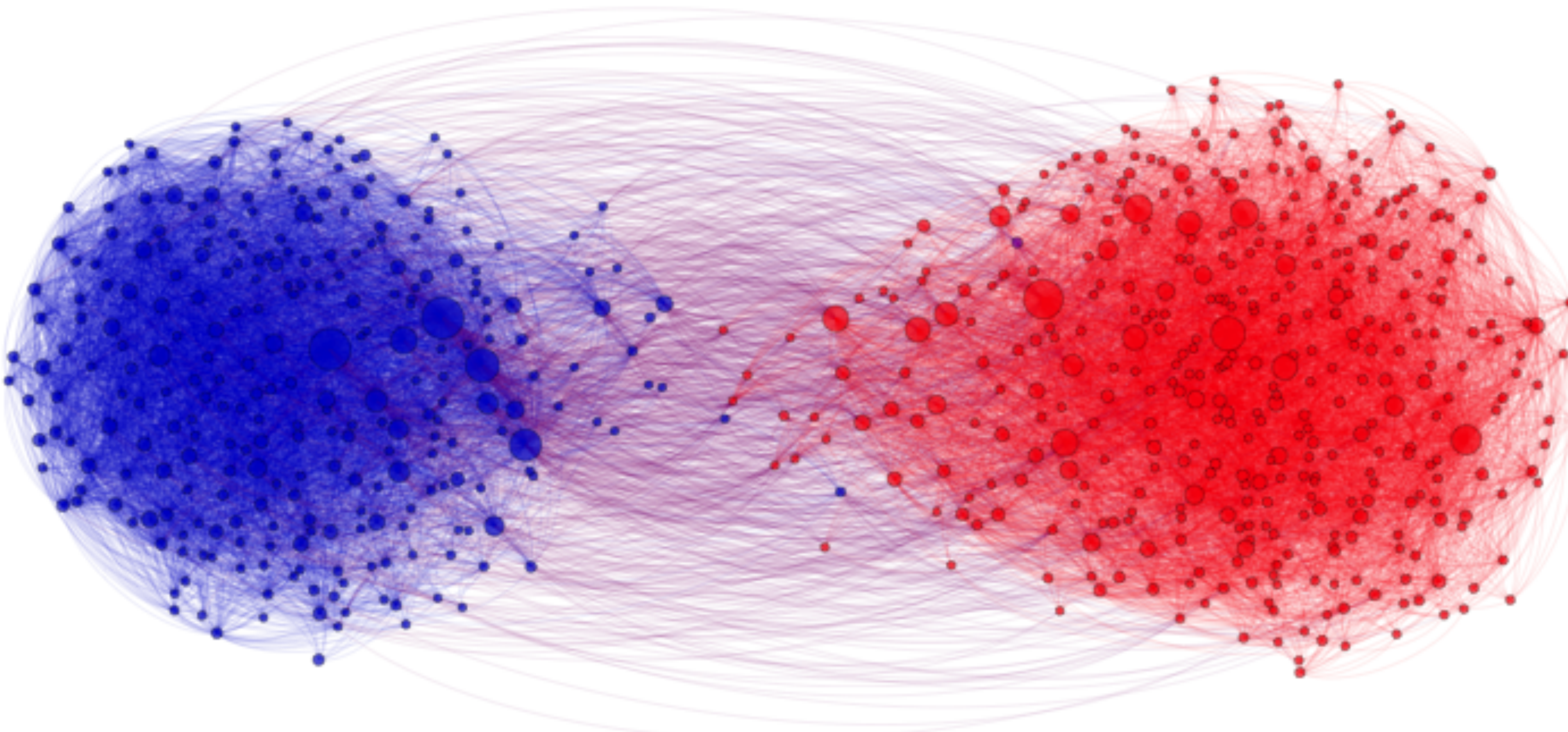
- **Graphs can be either directed or undirected**
  - Undirected graph, if (A,B) is in the set of edges, (B,A) must be in the set of edges
  - Directed graphs, both can be in the set of edges, but those graphs have different connectivity
- **We call a graph *connected* if there is a path between every pair of vertices**

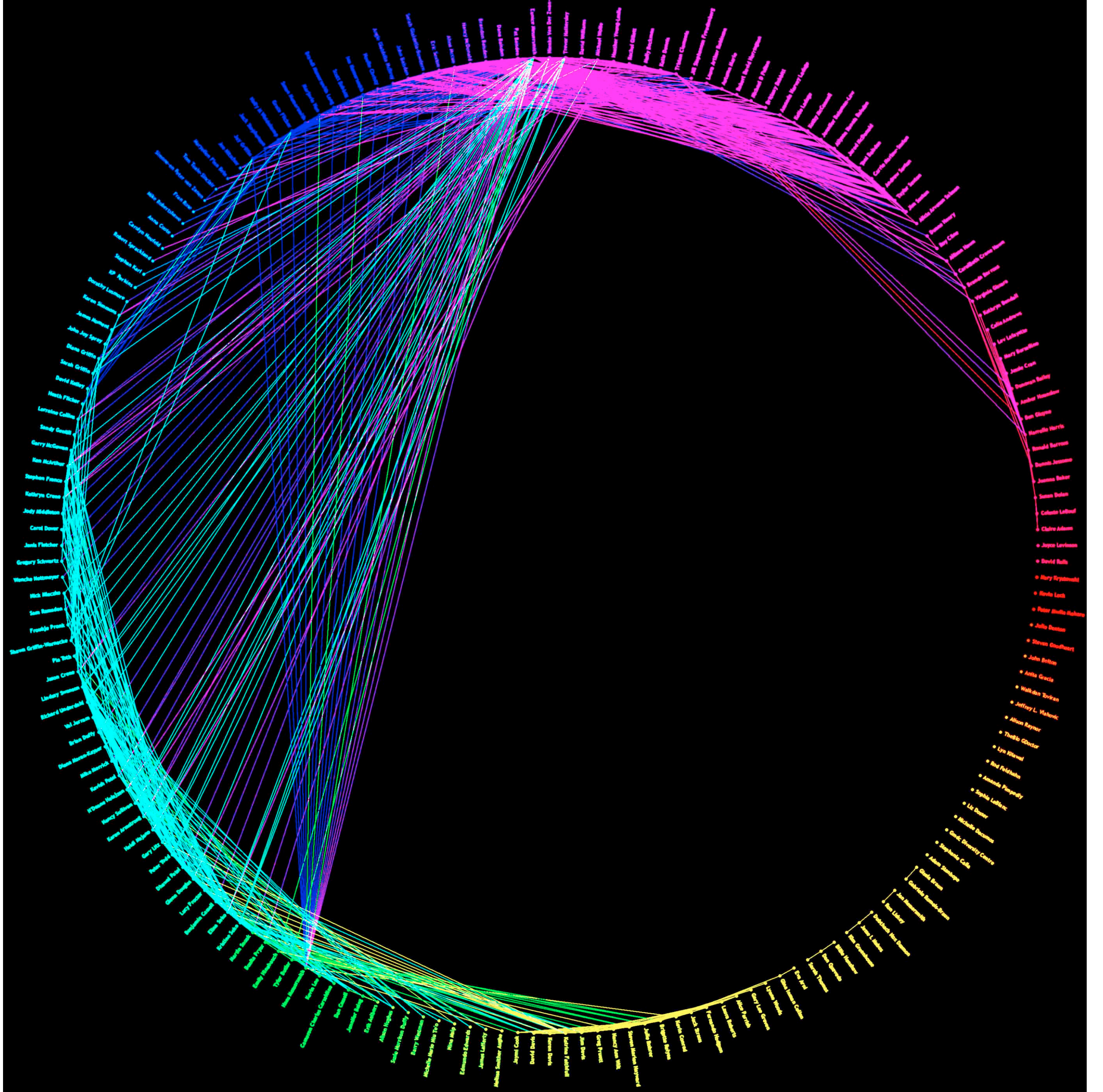
GRAPHS

- **Edges can have weights**
  - This becomes important when we consider path finding algorithms
  - Usually, we consider the weights to be the some attribute pertaining to the edge
  - Each edge has exactly one weight

GRAPHS

- **When we consider graphs, we determine them to be either dense or sparse**
  - Dense graphs are very connected, each vertex is connected to a fraction of the total vertices
  - Sparse graphs are less connected and can be more clustered, each vertex is connected to some constant number of vertices





GRAPHS

- **When graphs are small, it is difficult to distinguish between the two**
  - If we represent Facebook as a graph, where users are vertices and “friendships” are edges, what can we say about the graph?
    - Directed?
    - Connected?
    - Cyclic?
    - Sparse/Dense?

GRAPHS

- **When graphs are small, it is difficult to distinguish between the two**
  - If we represent Facebook as a graph, where users are vertices and “friendships” are edges, what can we say about the graph?
    - Directed? **No, (A,B) means (B,A)**
    - Connected? **Maybe not!**
    - Cyclic? **Yes, mutual friends**
    - Sparse/Dense? **Sparse! 338 average!**

GRAPHS

- **This "value" is called the degree of the vertex**
  - If you have 338 friends, then that vertex has degree 338.
- **In directed graph, we separate this into in-degree and out-degree**
  - Consider Twitter, where friendship isn't symmetric. The number of followers you have is your in-degree and the number of people you follow is your out degree

REPRESENTATION

- **How do we represent graphs on a computer?**
  - Two main approaches

REPRESENTATION

- **How do we represent graphs on a computer?**
  - Two main approaches
    - Adjacency List
    - Adjacency Matrix

ADJACENCY LIST

- If *(u,v)* is an edge, then we say *v* is ***adjacent*** to *u*.
- If we want to store these edges then,
  - For each vertex, we maintain a list of all edges coming *out* of that vertex
- The number of elements coming out of the vertex is called the *out-degree*
- The number of elements coming into the vertex is the *in-degree*

ADJACENCY MATRIX

- Imagine a two dimensional |V| x |V| matrix.
- Let the rows be source vertices, and let the rows be destination vertices
  - If the edge (u,v) is in the graph, then matrix[u][v] is set to true
  - Alternatively, we can set matrix[u][v] to be the weight of the edge

ADJACENCY MATRIX

- Imagine a two dimensional |V| x |V| matrix.
- Let the rows be source vertices, and let the rows be destination vertices
    - If the edge (u,v) is in the graph, then matrix[u][v] is set to true
    - Alternatively, we can set matrix[u][v] to be the weight of the edge
- What is the memory consumption?
    - $O(|V|^2)$, but it implicitly stores in and out vertices
    - If the graph is dense, then this is more efficient

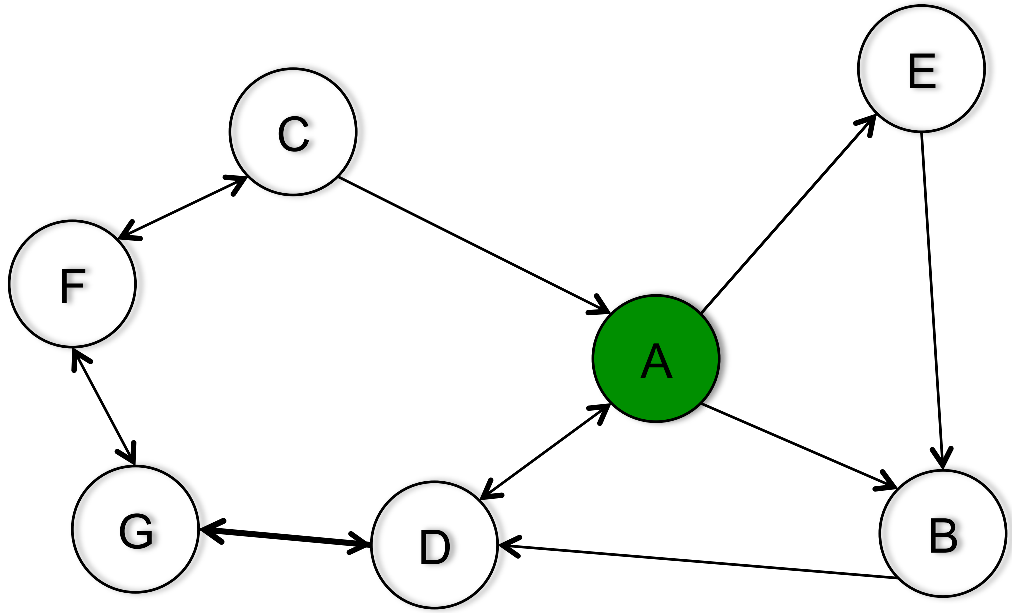
TERMINOLOGY

- **Know the following terms**
  - Vertices and Edges
  - Directed v. Undirected
  - In-degree and out-degree
  - Connected (Strongly connected)
  - Weighted v. unweighted
  - Cyclic v. acyclic
  - DAG: Directed Acyclic Graph

TRAVERSALS

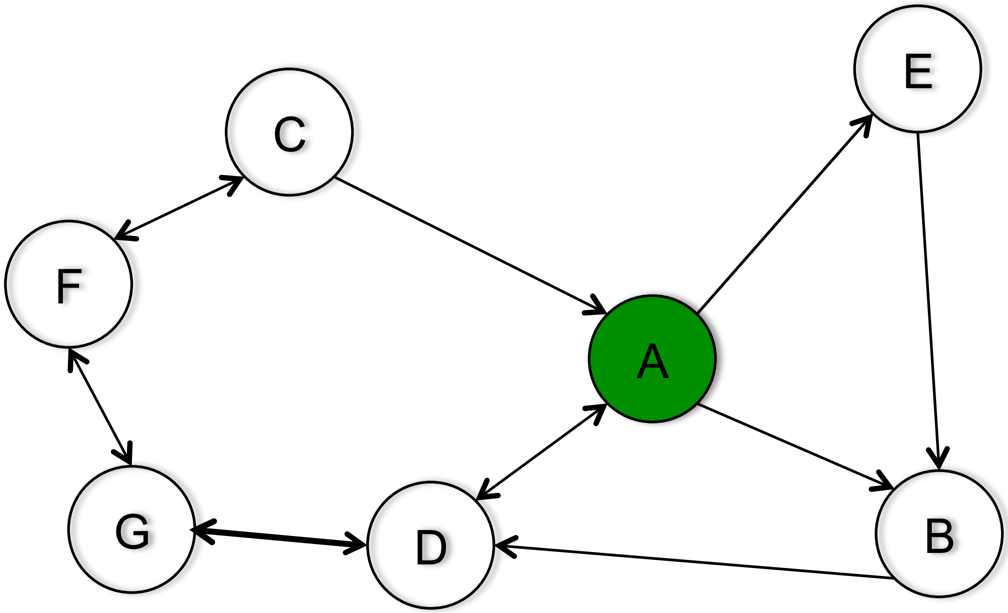
- **Since graphs are abstractions similar to trees, we can also perform traversals.**
  - If a graph is connected, i.e. there is a path between all pairs of vertices, then a traversal can output all nodes if you do it cleverly

TRAVERSAL



- **Depth-first search (prev graph with (D,G) added to make it connected**
  - Traverse the tree with DFS, if there are multiple nodes to choose from, go alphabetically. Start at A.

TRAVERSAL

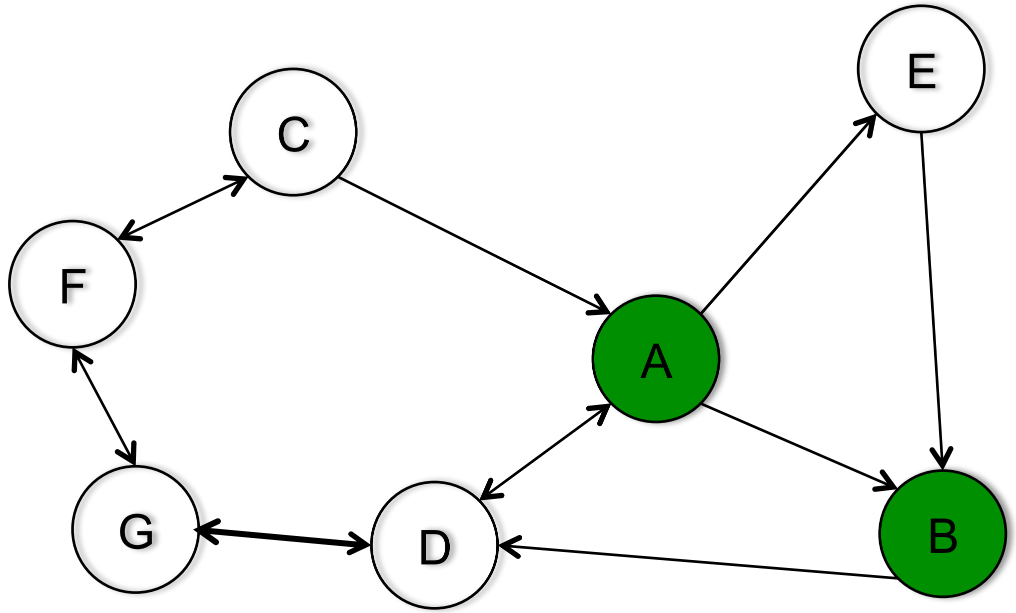


Output: A

Current Node: A

Out-vertices: B, D, E

TRAVERSAL

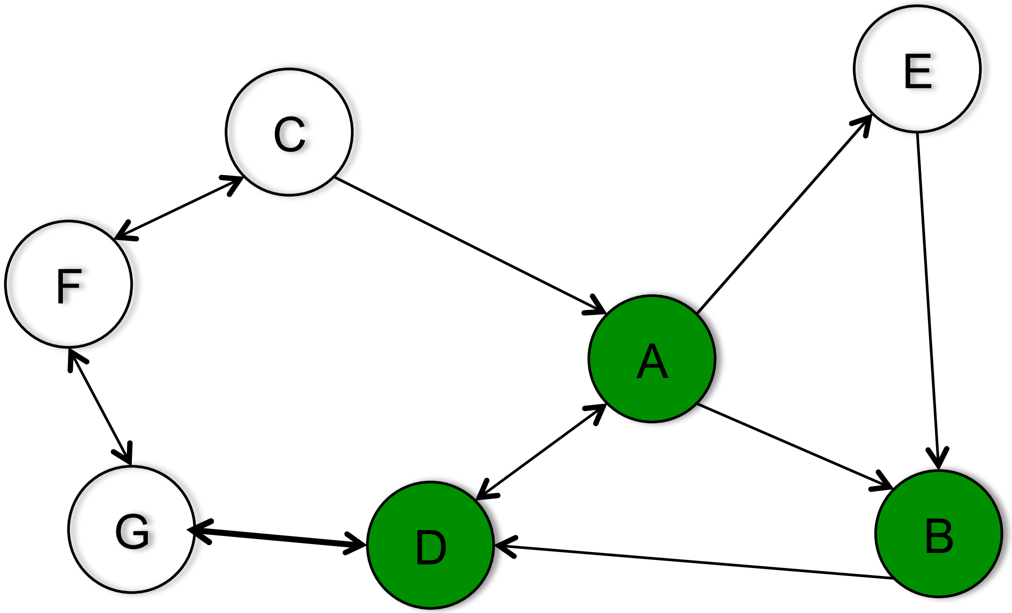


Output: A,B

Current Node: B

Out-vertices: D

TRAVERSAL

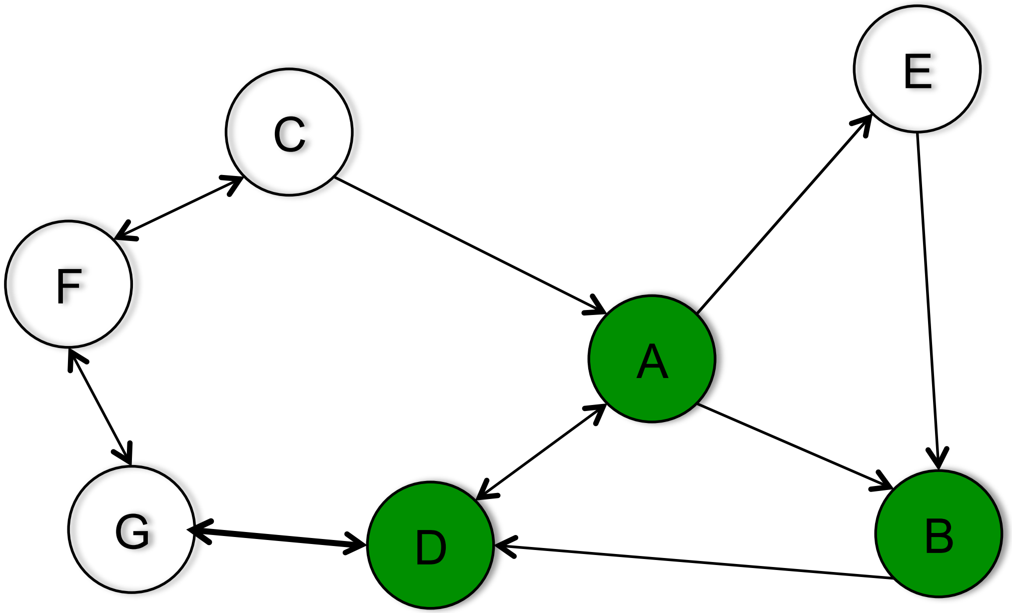


Output: A,B, D

Current Node: D

Out-vertices: A,G

TRAVERSAL

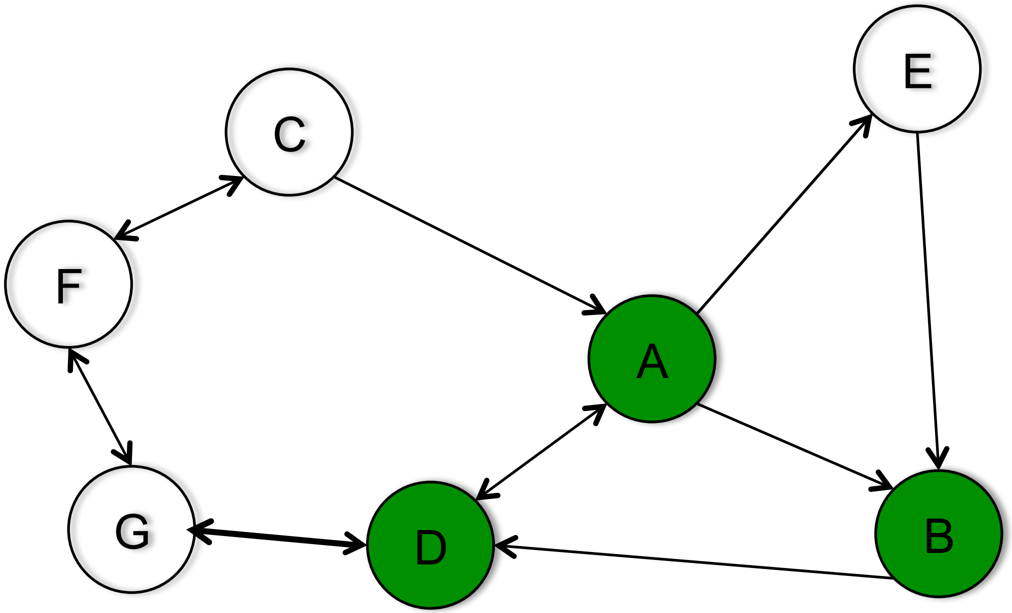


Output: A,B, D, A

Current Node: A

Out-vertices: B,D,E

TRAVERSAL



Output: A,B, D, A

Oh, no! We have repeated output!

Current Node: A

Out-vertices: B,D,E

TRAVERSAL

- **Depth first search needs to check which nodes have been output or else it can get stuck in loops.**
  - This increases the runtime and memory constraints of the traversal
- **In a connected graph, a BFS will print all nodes, but it will repeat if there are cycles and may not terminate**

TRAVERSAL

- **As an aside, in-order, pre-order and post-order traversals only make sense in binary trees, so they aren't important for graphs. However, we do need some way to order our out-vertices (left and right in BST).**

TRAVERSALS

- **For an arbitrary graph and starting node v, find all nodes *reachable* from v.**
  - There exists a path from v
  - Doing something or "processing" each node
  - Determines if an undirected graph is connected? If a traversal goes through all vertices, then it is connected
- **Basic idea**
  - Traverse through the nodes like a tree
  - Mark the nodes as visited to prevent cycles and from processing the same node twice

ABSTRACT IDEA IN PSEUDOCODE

```
void traverseGraph(Node start) {
     Set pending = emptySet()
     pending.add(start)
    mark start as visited
    while(pending is not empty) {
      next = pending.remove()
      for each node u adjacent to next
         if (u is not marked visited) {
            mark u
            pending.add(u)
         }
     }
  }
```

RUNTIME AND OPTIONS

- **Assuming we can add and remove from our "pending" DS in O(1) time, the entire traversal is O(|E|)**
- **Our traversal order depends on what we use for our pending DS.**
  - Stack : DFS
  - Queue: BFS
- **These are the main traversal techniques in CS, but there are others!**

COMPARISON

Breadth-first always finds shortest length paths, i.e., "optimal solutions"

- Better for "what is the shortest path from **x** to **y**"

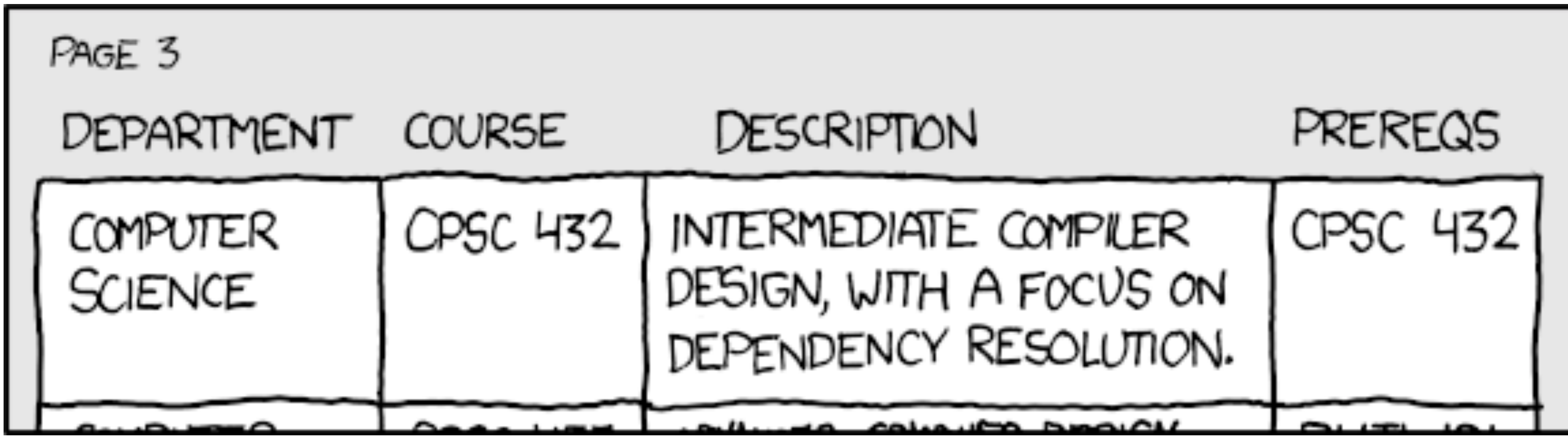
But depth-first can use less space in finding a path

- If *longest path* in the graph is **p** and highest out-degree is **d** then DFS stack never has more than **d*p** elements
- But a queue for BFS may hold *O*(|V|) nodes

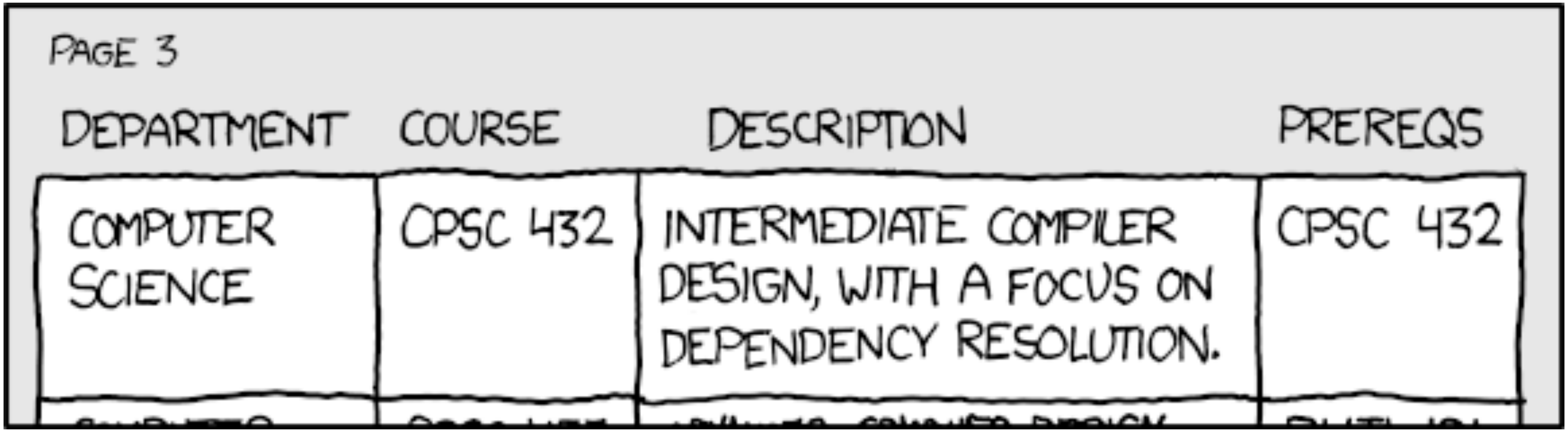
A third approach (useful in Artificial Intelligence)

- *Iterative deepening (IDFS)*:
  - Try DFS but disallow recursion more than **K** levels deep
  - If that fails, increment **K** and start the entire search over
- Like BFS, finds shortest paths. Like DFS, less space.

TOPOLOGICAL SORT



TOPOLOGICAL SORT



- **It’s never too late to start your xkcd addiction**

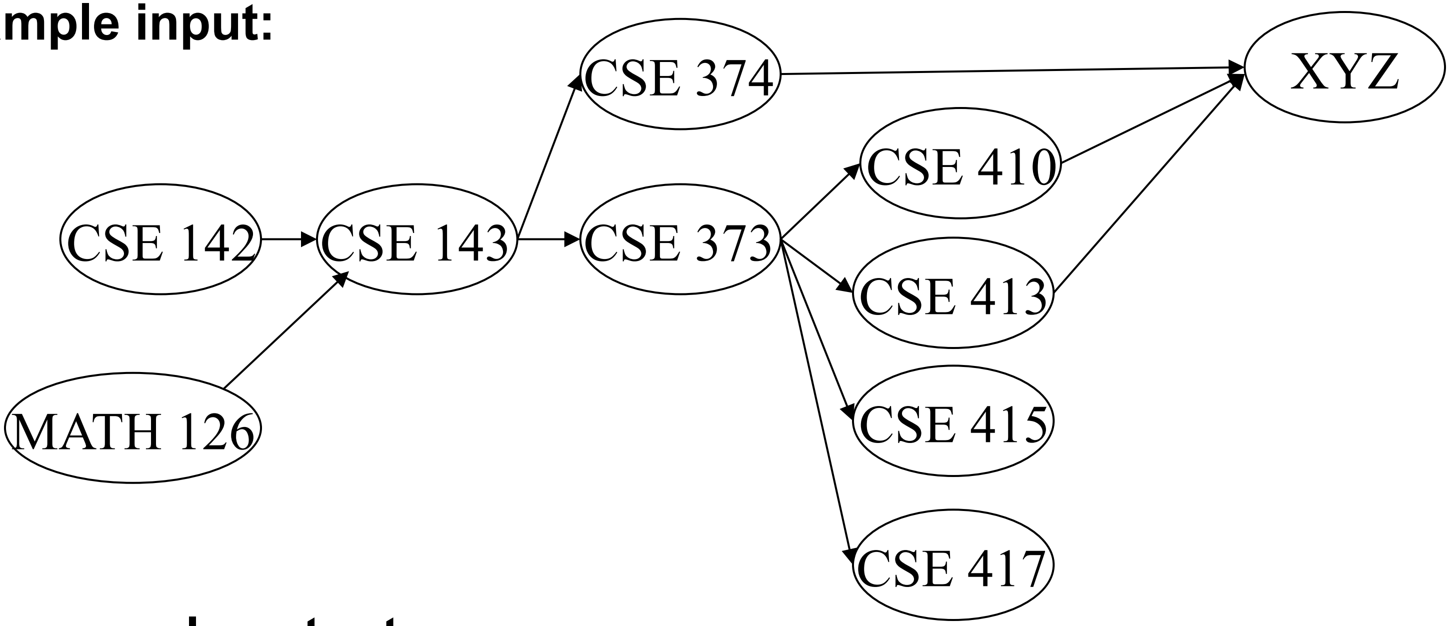
TOPOLOGICAL SORT

- **Topological ordering**
  - One final ordering for graphs
  - Ordering with a focus on dependency resolutions
- **Example, consider a graph where courses are vertices and prerequisites are edges.**
- **A topological ordering is any valid class order**

TOPOLOGICAL SORT

Problem: Given a DAG `G=(V,E)`, output all vertices in an order such that no vertex appears before another vertex that has an edge to it

Example input:



One example output:

126, 142, 143, 374, 373, 417, 410, 413, XYZ, 415

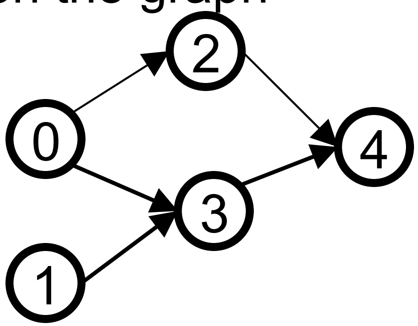
QUESTIONS AND COMMENTS

Why do we perform topological sorts only on DAGs?

- Because a cycle means there is no correct answer

Is there always a unique answer?

- No, there can be 1 or more answers; depends on the graph
- Graph with 5 topological orders:



Do some DAGs have exactly 1 answer?

- Yes, including all lists

Terminology: A DAG represents a partial order and a topological sort produces a total order that is consistent with it

USES OF TOPOLOGICAL SORT

Figuring out how to graduate

Computing an order in which to recompute cells in a spreadsheet

Determining an order to compile files using a Makefile

In general, taking a dependency graph and finding an order of execution

…

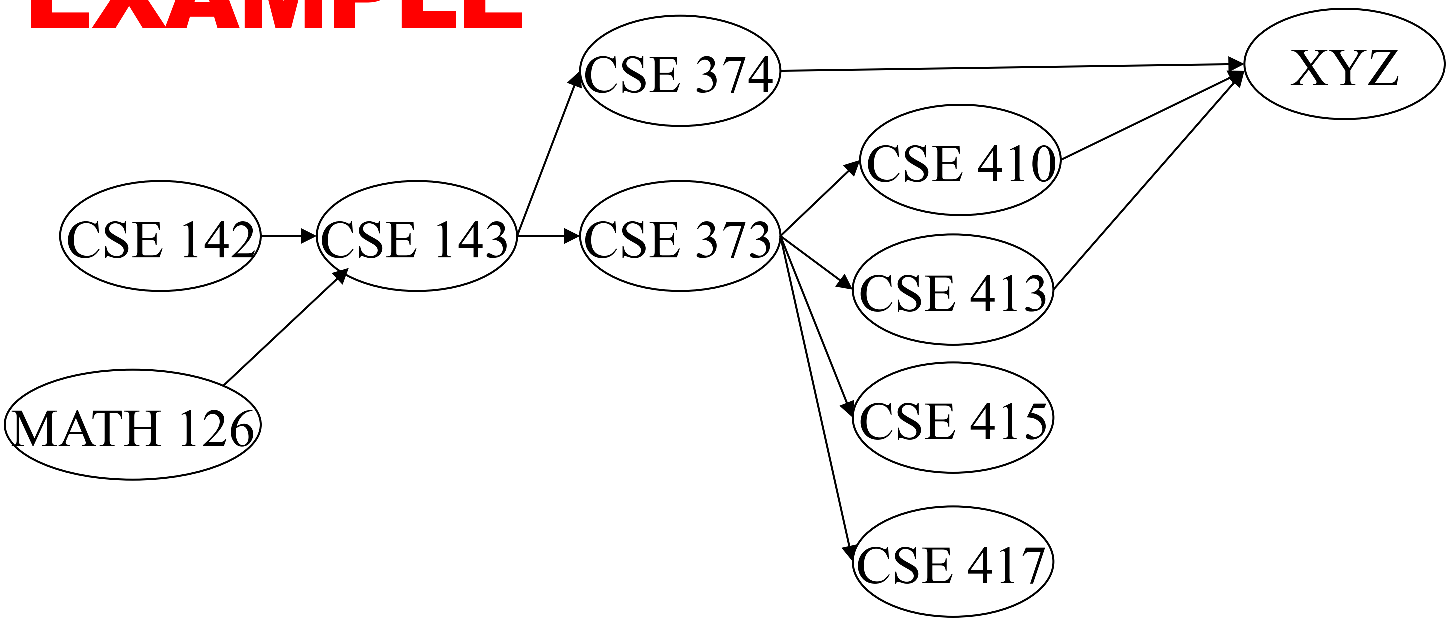
TOPOLOGICAL SORT

1. **Label ("mark") each vertex with its in-degree**
   - Think "write in a field in the vertex"
   - Could also do this via a data structure (e.g., array) on the side

2. **While there are vertices not yet output:**
   a) Choose a vertex **v** with labeled with in-degree of 0
   b) Output **v** and *conceptually* remove it from the graph
   c) For each vertex **u** adjacent to **v** (i.e. **u** such that (**v**,**u**) in **`E`**), decrement the in-degree of **u**

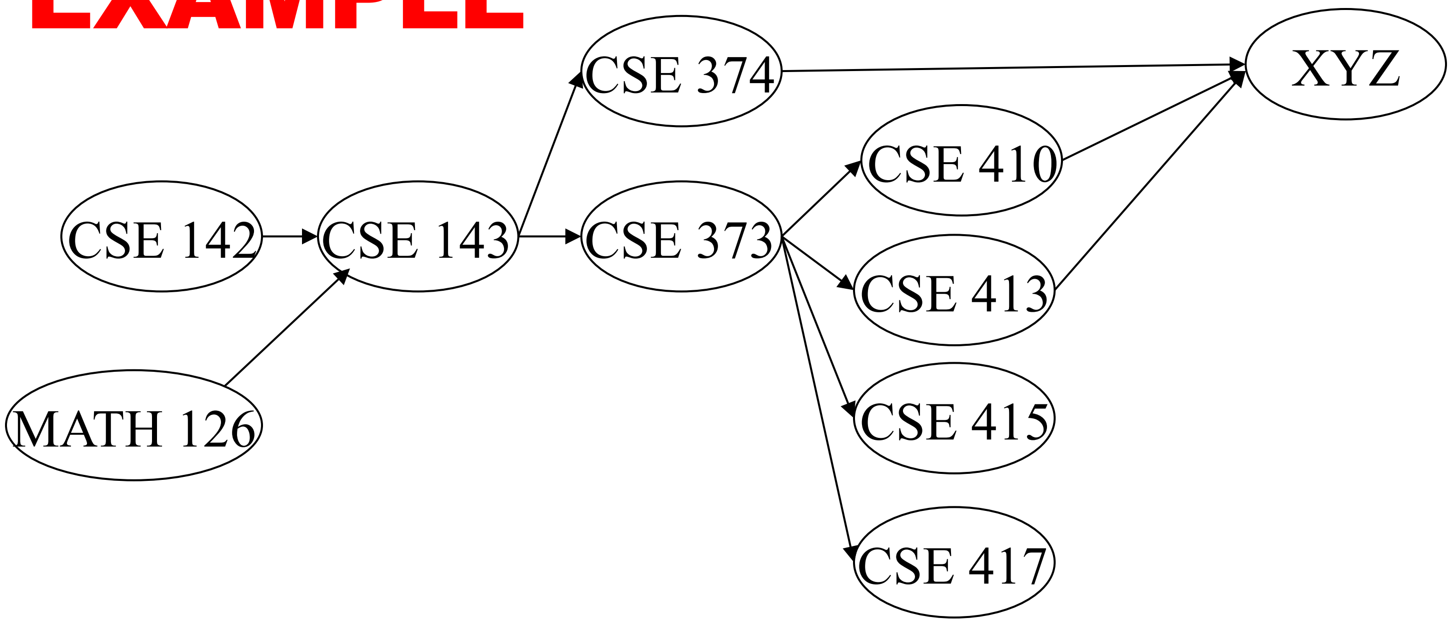
EXAMPLE

Output:



Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?										
In-degree:	0	0	2	1	1	1	1	1	1	3

EXAMPLE

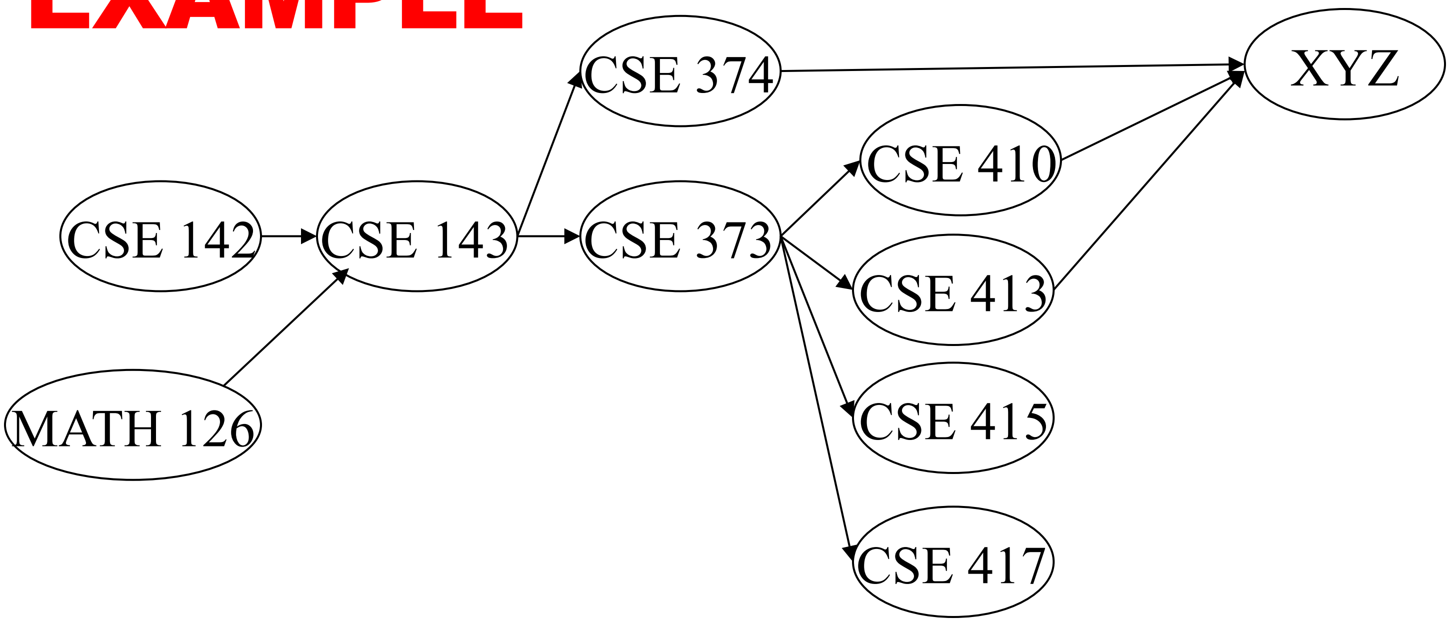


Output:

126

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x									
In-degree:	0	0	~~2~~ 1	1	1	1	1	1	1	3

EXAMPLE



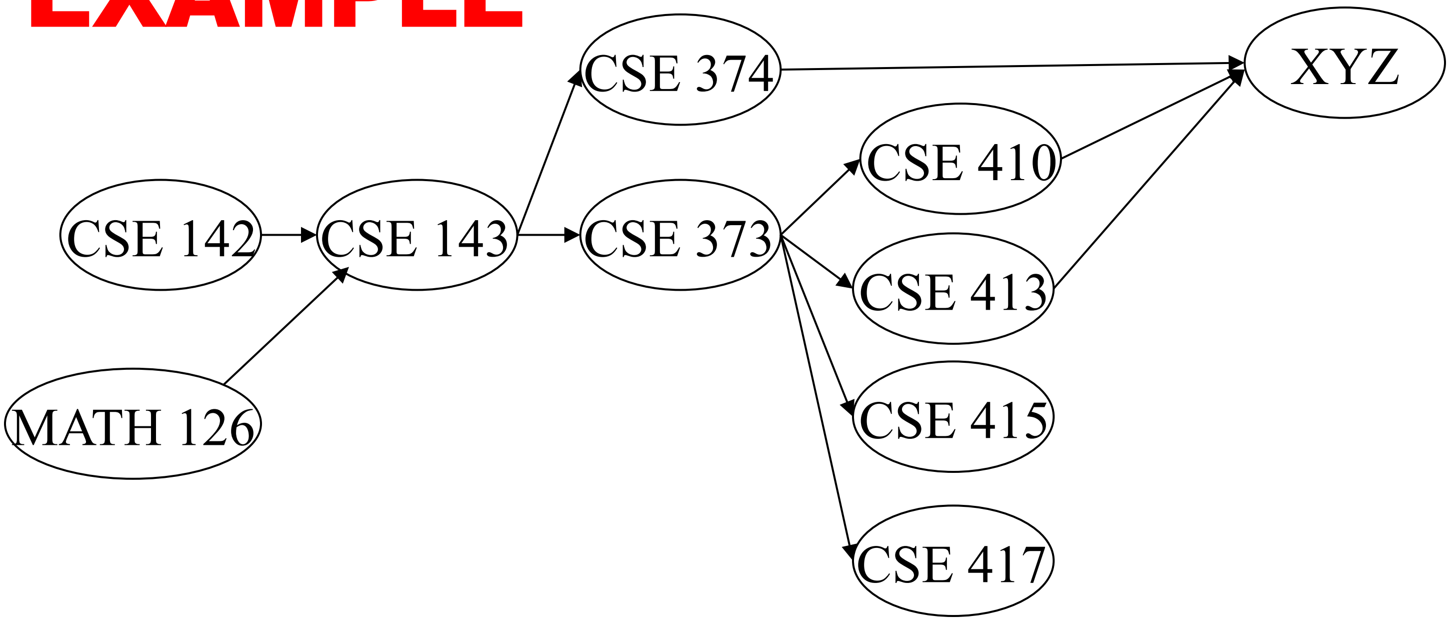
Output:

126

142

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x								
In-degree:	0	0	~~2~~ ~~1~~ 0	1	1	1	1	1	1	3

EXAMPLE



Output:

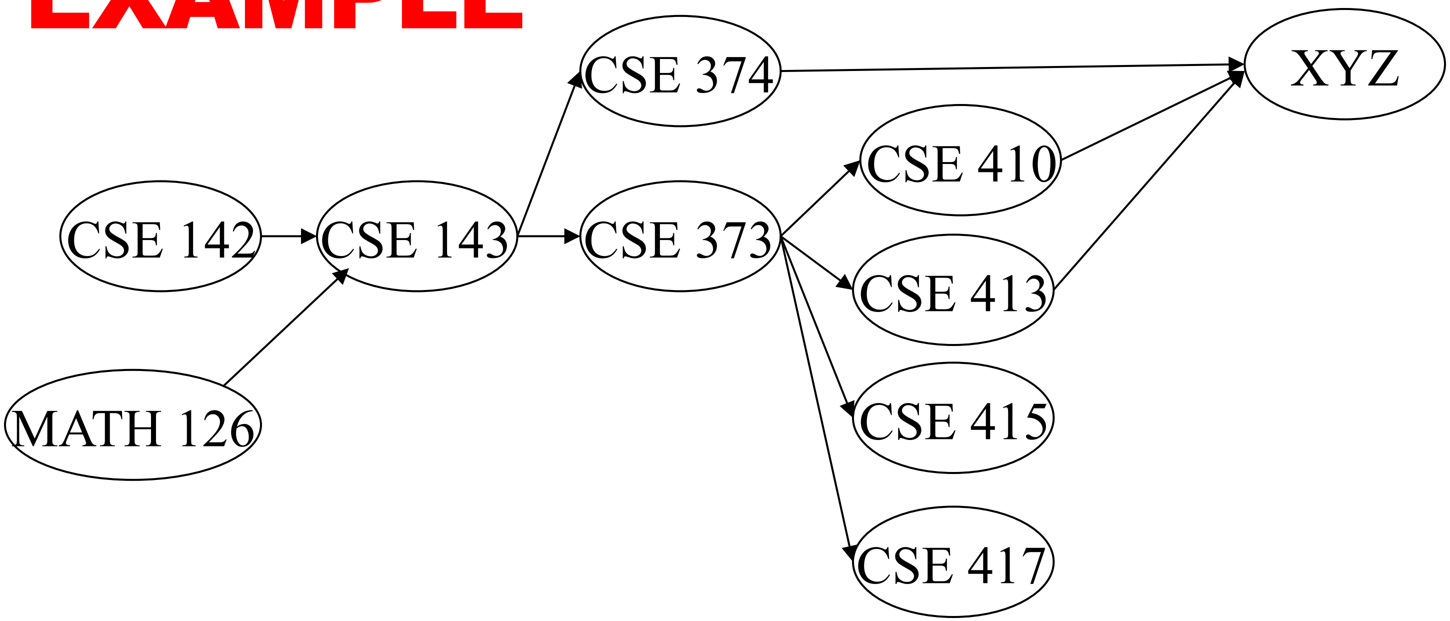
126

142

143

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x							
In-degree:	0	0	~~2~~ ~~1~~ 0	~~1~~ 0	~~1~~ 0	1	1	1	1	3

EXAMPLE



Output:

126

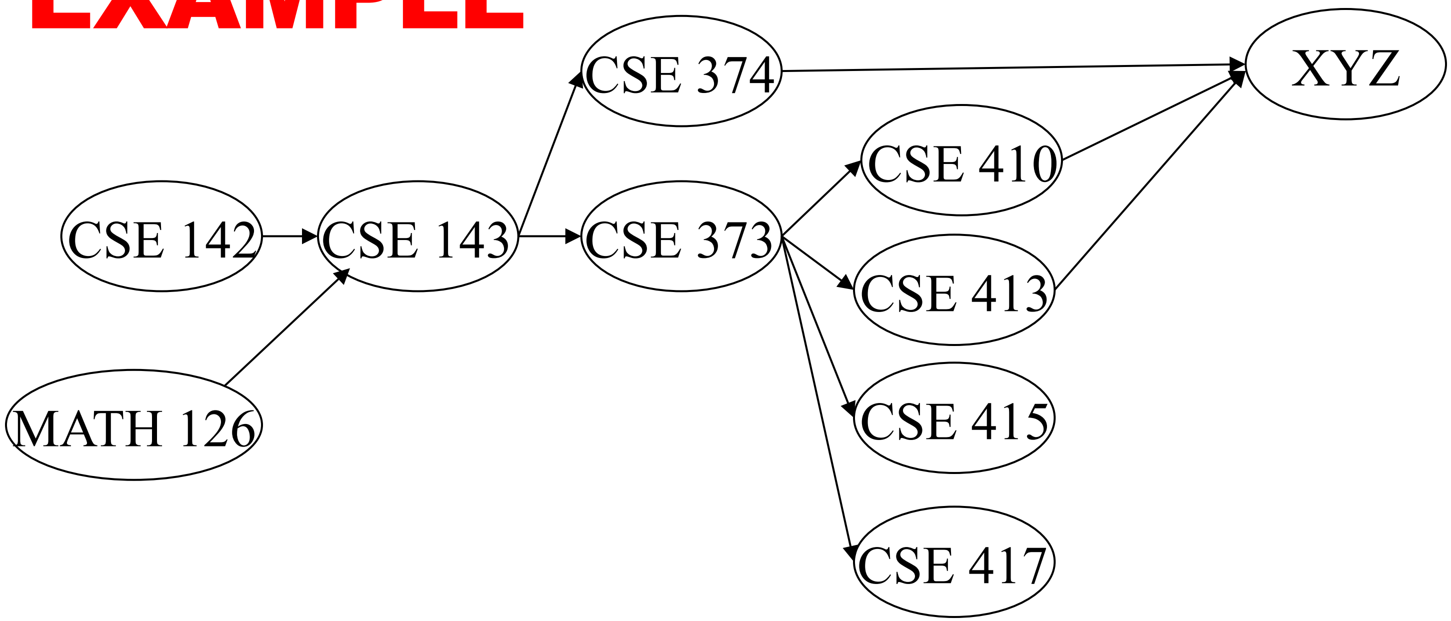
142

143

374

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x						
In-degree:	0	0	~~2~~ ~~1~~ 0	~~1~~ 0	~~1~~ 0	1	1	1	1	~~3~~ 2

EXAMPLE



Output:

126

142

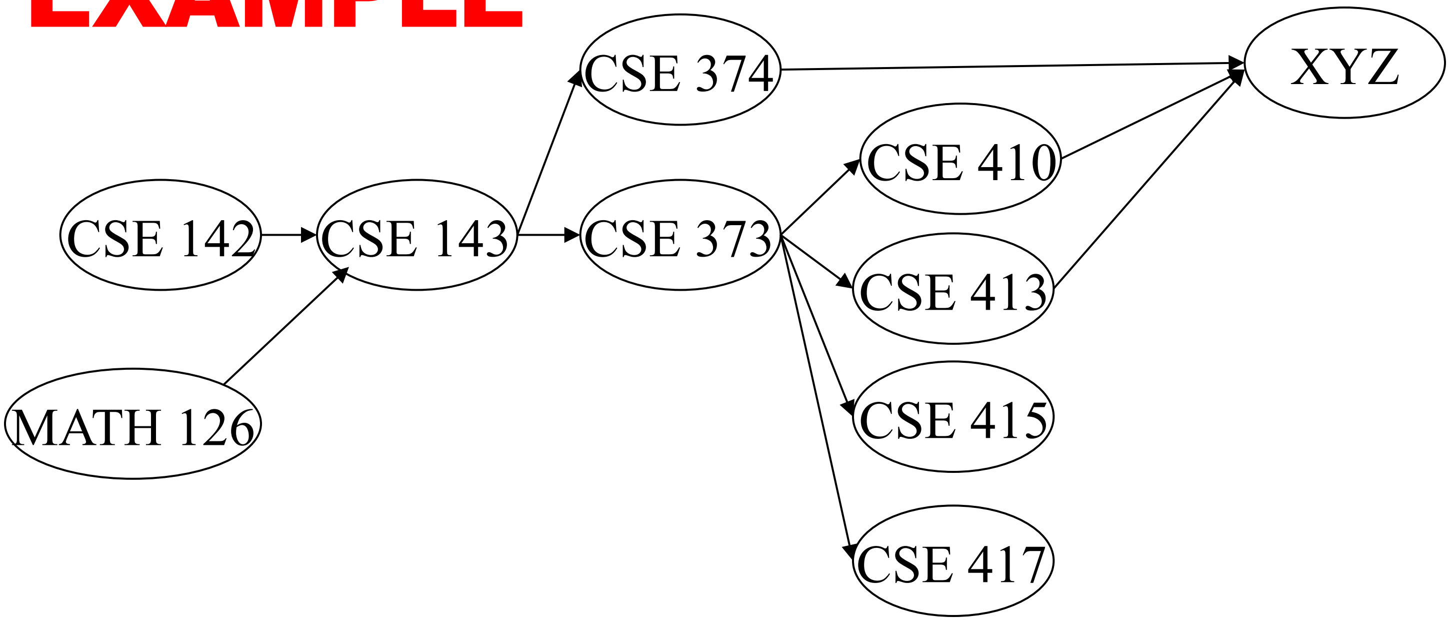
143

374

373

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x					
In-degree:	0	0	~~2~~ ~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~3~~ 2

EXAMPLE



Output:

126

142

143

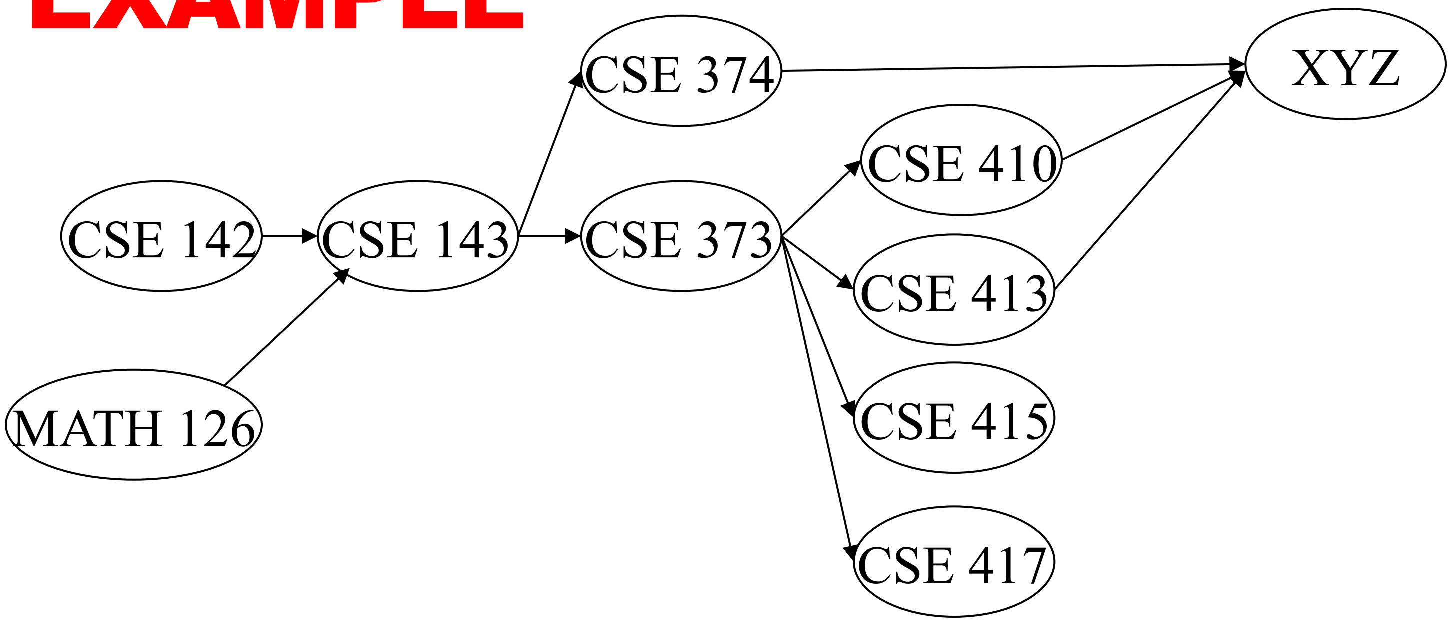
374

373

417

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x				x	
In-degree:	0	0	2	1	1	1	1	1	1	3
			1	0	0	0	0	0	0	2
			0							

EXAMPLE



Output:

126

142

143

374

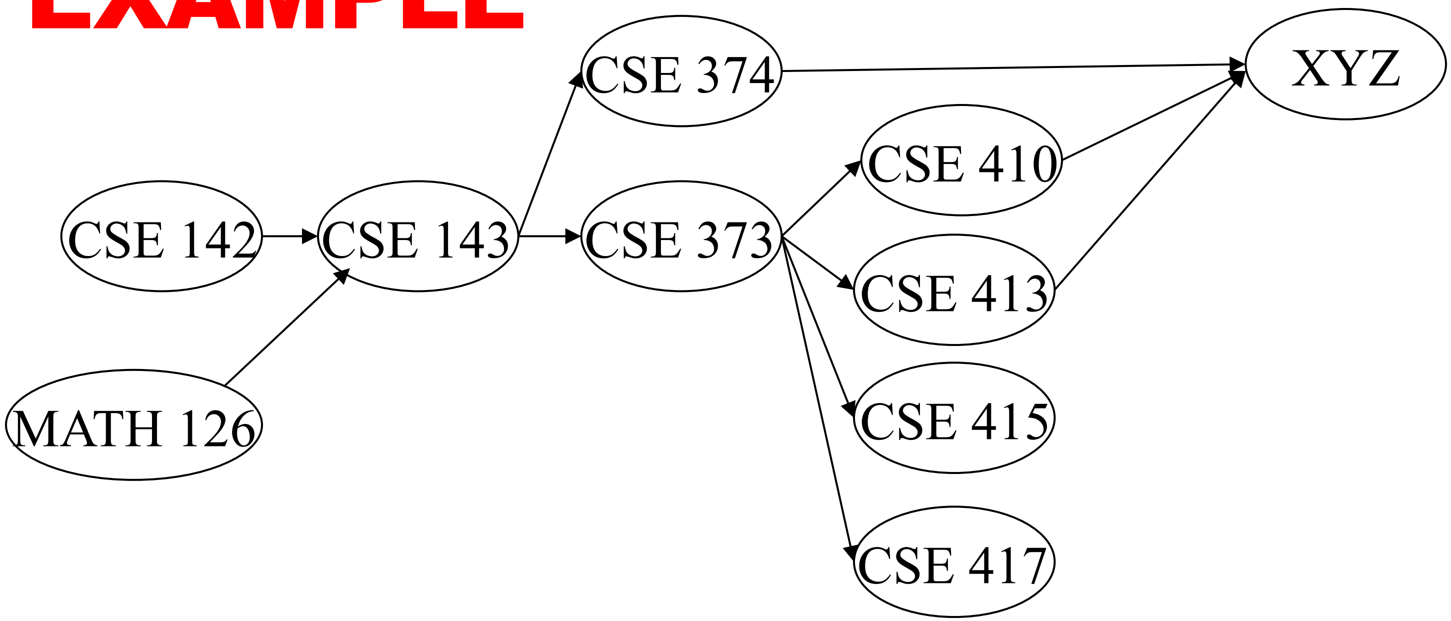
373

417

410

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x	x			x	
In-degree:	0	0	~~2~~	~~1~~	~~1~~	~~1~~	~~1~~	~~1~~	~~1~~	~~3~~
			~~1~~	0	0	0	0	0	0	~~2~~
			0							1

EXAMPLE



Output:

126

142

143

374

373

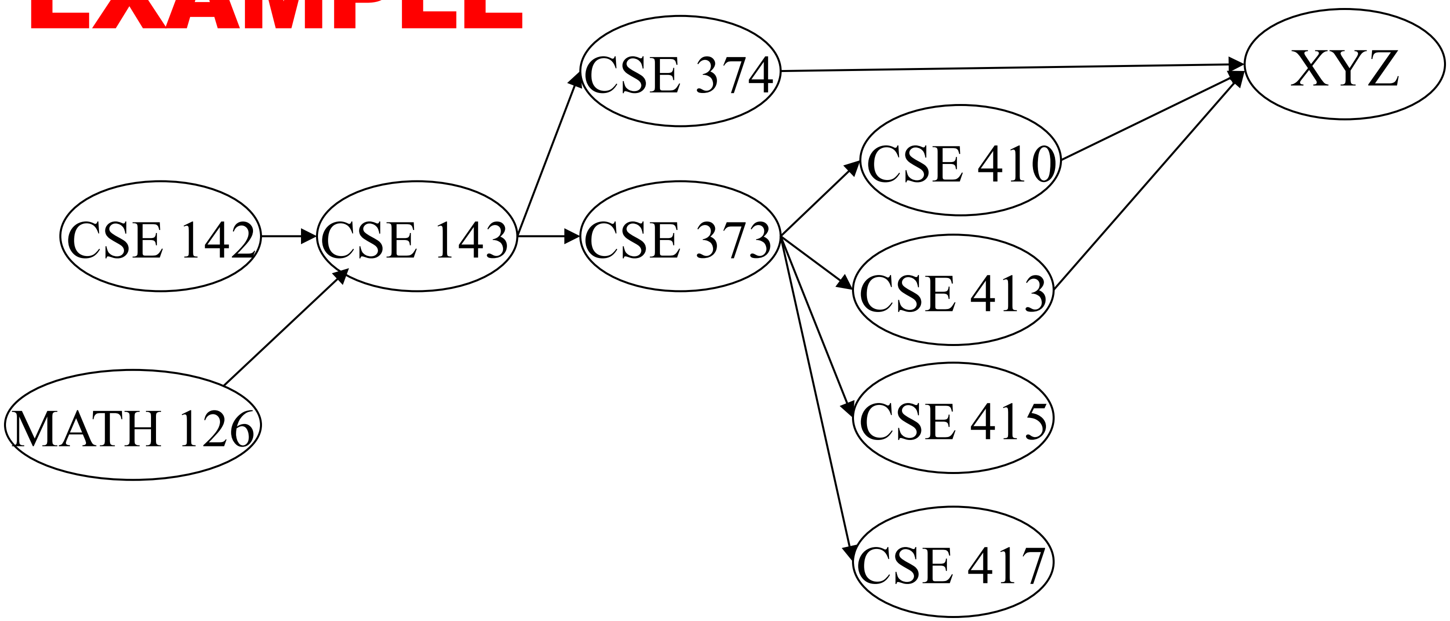
417

410

413

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x	x	x		x	
In-degree:	0	0	~~2~~ ~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~3~~ ~~2~~ ~~1~~ 0

EXAMPLE



Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x	x	x		x	x
In-degree:	0	0	~~2~~ → ~~1~~ → 0	~~1~~ → 0	~~1~~ → 0	~~1~~ → 0	~~1~~ → 0	~~1~~ → 0	~~1~~ → 0	~~3~~ → ~~2~~ → ~~1~~ → 0

Output:

126

142

143

374

373

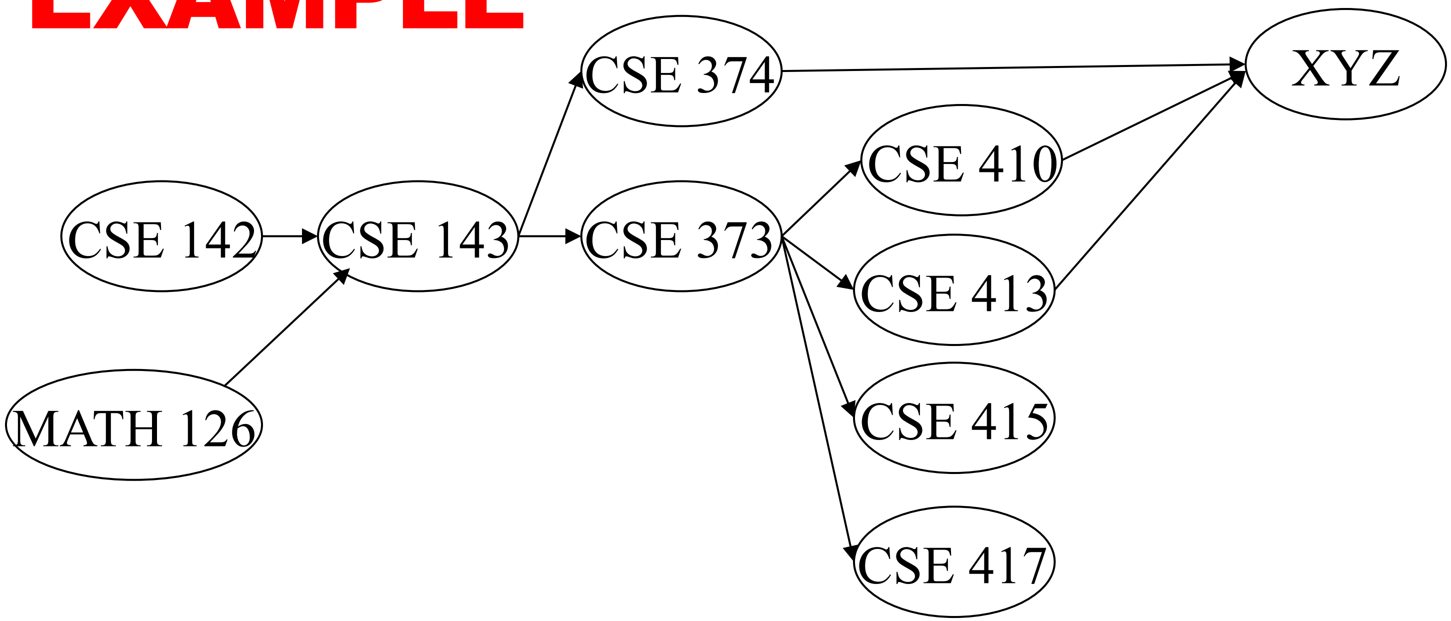
417

410

413

XYZ

EXAMPLE



Output:

126

142

143

374

373

417

410

413

XYZ

415

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x	x	x	x	x	x
In-degree:	0	0	~~2~~	~~1~~	~~1~~	~~1~~	~~1~~	~~1~~	~~1~~	~~3~~
			~~1~~	0	0	0	0	0	0	~~2~~
			0							~~1~~
										0

CSE373: Data Structures &

NOTICE

Needed a vertex with in-degree 0 to start

- Will always have at least 1 because no cycles

Ties among vertices with in-degrees of 0 can be broken arbitrarily

- Can be more than one correct answer, by definition, depending on the graph

IMPLEMENTATION

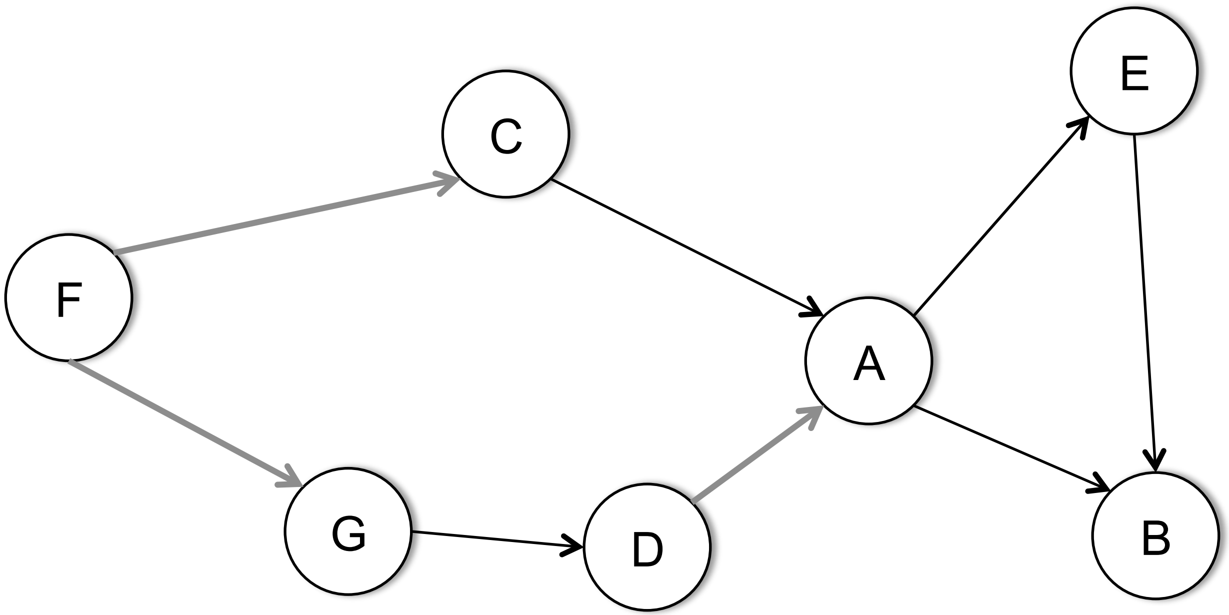
The trick is to avoid searching for a zero-degree node every time!

- Keep the “pending” zero-degree nodes in a list, stack, queue, bag, table, or something
- Order we process them affects output but not correctness or efficiency provided add/remove are both $O(1)$

Using a queue:

1. **Label each vertex with its in-degree, enqueue 0-degree nodes**
2. **While queue is not empty**
   a) **v** = dequeue()
   b) Output **v** and remove it from the graph
   c) For each vertex **u** adjacent to **v** (i.e. **u** such that (**v**,**u**) in **`E`**), decrement the in-degree of **u**, if new degree is 0, enqueue it

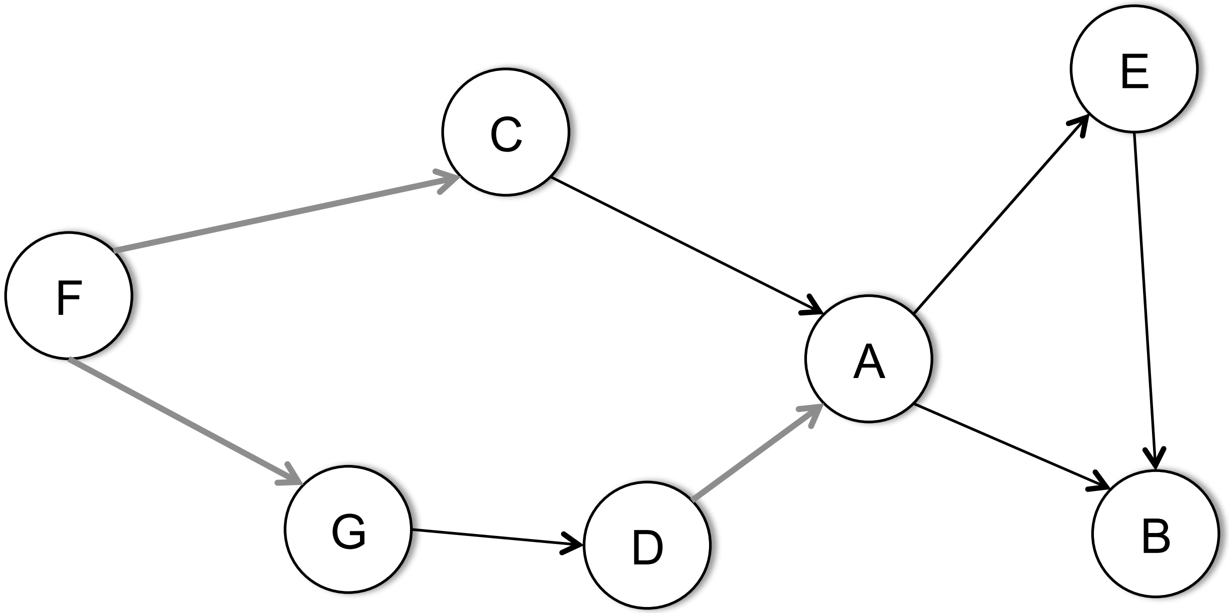
TRAVERSAL



Start with the nodes that have in-degree 0 (no prereqs)

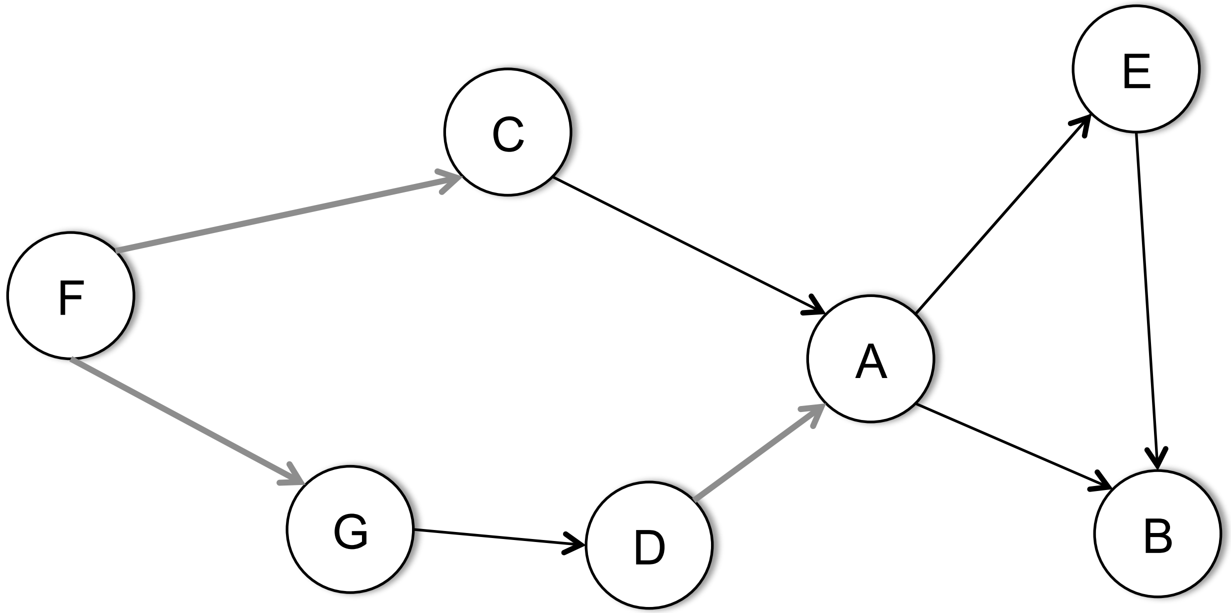
Then eliminate that vertex (print it out) and eliminate its out edges.

TRAVERSAL



What is a valid topological sort of this graph?

TRAVERSAL



What is a valid topological sort of this graph?

F,C,G,D,A,E,B

F,G,D,C,A,E,B

F,G,C,D,A,E,B

Are these all the valid solutions?

TOPOLOGICAL SORT

- **What use does this traversal have?**

TOPOLOGICAL SORT

- **What use does this traversal have?**
  - Good for dependency resolution

TOPOLOGICAL SORT

- **What use does this traversal have?**
  - Good for dependency resolution
  - Can also be used for cycle detection

TOPOLOGICAL SORT

- **What use does this traversal have?**
  - Good for dependency resolution
  - Can also be used for cycle detection
- **How could we find cycles in an undirected graph?**

TOPOLOGICAL SORT

- **What use does this traversal have?**
  - Good for dependency resolution
  - Can also be used for cycle detection
- **How could we find cycles in an undirected graph?**
  - Any traversal that visits a node more than once has a cycle.

GRAPH PROBLEMS

- When thinking about graphs, it is important to understand what the graph represents

GRAPH PROBLEMS

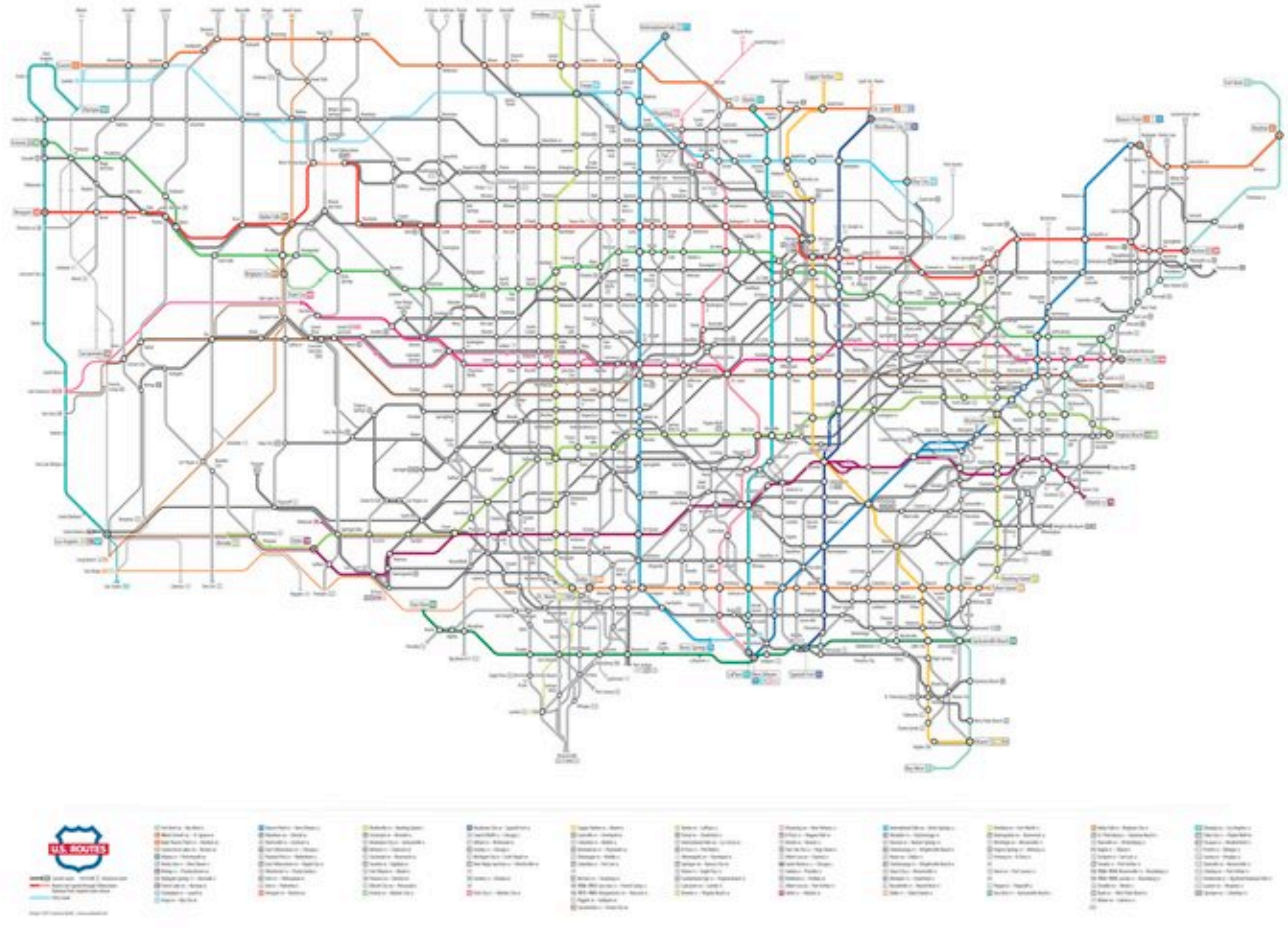
- **When thinking about graphs, it is important to understand what the graph represents**
  - Topological sort:

GRAPH PROBLEMS

- **When thinking about graphs, it is important to understand what the graph represents**
  - Topological sort:
    - Programs and dependencies
    - Courses and prereqs

GRAPH PROBLEMS

- **When thinking about graphs, it is important to understand what the graph represents**
  - Topological sort:
    - Programs and dependencies
    - Courses and prereqs
  - What the vertices and edges are impact what the “solution” is


U.S. ROUTES

GRAPH PROBLEMS

- **What type of problem could we want to solve with a graph of US cities and the freeway distance between them**

GRAPH PROBLEMS

- **What type of problem could we want to solve with a graph of US cities and the freeway distance between them**
  - Same as a lot of network problems

GRAPH PROBLEMS

- **What type of problem could we want to solve with a graph of US cities and the freeway distance between them**
  - Same as a lot of network problems
  - “Traffic” networks

GRAPH PROBLEMS

- **What type of problem could we want to solve with a graph of US cities and the freeway distance between them**
  - Same as a lot of network problems
  - “Traffic” networks
  - What do our edges represent?

SINGLE SOURCE SHORTEST PATH

- Given an undirected, *unweighted* graph G(V,E) and a start vertex *A*, find the shortest path to all connected vertices

SINGLE SOURCE SHORTEST PATH

- **Given an undirected, *unweighted* graph G(V,E) and a start vertex *A*, find the shortest path to all connected vertices**
  - If a graph is unweighted you can treat all of their weights as 1

SINGLE SOURCE SHORTEST PATH

- **Given an undirected, *unweighted* graph G(V,E) and a start vertex *A*, find the shortest path to all connected vertices**
  - If a graph is unweighted you can treat all of their weights as 1
  - Do a BFS traversal of the tree and keep track of paths!

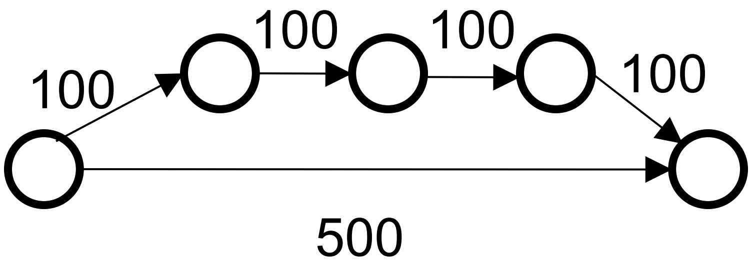
SINGLE SOURCE SHORTEST PATH

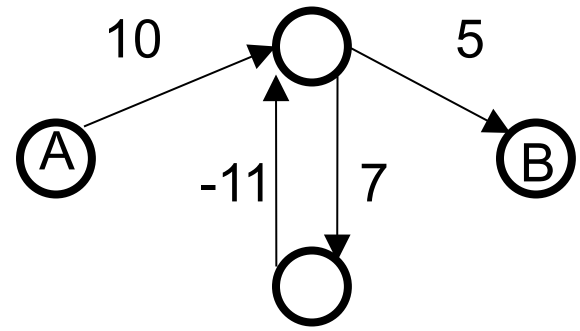
- **Given an undirected, *unweighted* graph G(V,E) and a start vertex *A*, find the shortest path to all connected vertices**
  - If a graph is unweighted you can treat all of their weights as 1
  - Do a BFS traversal of the tree and keep track of paths!
  - Path-keeping is non-trivial, we'll talk about it on Wednesday

SINGLE SOURCE SHORTEST PATH

- **Given an undirected, *unweighted* graph G(V,E) and a start vertex *A*, find the shortest path to all connected vertices**
  - If a graph is unweighted you can treat all of their weights as 1
  - Do a BFS traversal of the tree and keep track of paths!
  - Path-keeping is non-trivial, we'll talk about it on Wednesday
  - What if the graph has weights?

PATH-FINDING





Why BFS won't work: Shortest path may not have the fewest edges

- Annoying when this happens with costs of flights

We will assume there are no negative weights

- *Problem* is *ill-defined* if there are negative-cost *cycles*
- *Wednesday's algorithm* is *wrong* if *edges* can be negative
  - There are other, slower (but not terrible) algorithms

NEXT CLASS

- Dijkstra's algorithm

NEXT CLASS

- Dijkstra's algorithm
- P3 checkpoint 2