CSE373: Data Structures and Algorithms

Shortest Paths and Dijkstra's Algorithm

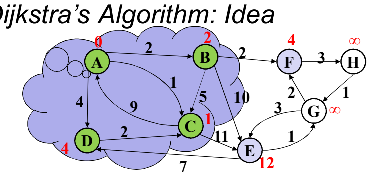
Steve Tanimoto
Winter 2016

This lecture material represents the work of multiple instructors at the University of Washington. Thank you to all who have contributed!

Dijkstra's Algorithm

- The idea: reminiscent of BFS, but adapted to handle weights
  - Grow the set of nodes whose shortest distance has been computed
  - Nodes not in the set will have a "best distance so far"
  - A priority queue will turn out to be useful for efficiency
- An example of a greedy algorithm
  - A series of steps
  - At each one the locally optimal choice is made

Dijkstra's Algorithm: Idea



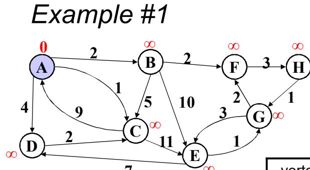
- Initially, start node has cost 0 and all other nodes have cost ∞
- At each step:
  - Pick closest unknown vertex **v**
  - Add it to the "cloud" of known vertices
  - Update distances for nodes with edges from **v**
- That's it! (But we need to prove it produces correct answers)

The Algorithm

1. For each node `v`, set `v.cost = ∞` and `v.known = false`
2. Set `source.cost = 0`
3. While there are unknown nodes in the graph
   a) Select the unknown node `v` with lowest cost
   b) Mark `v` as known
   c) For each edge `(v,u)` with weight `w`,

```
c1 = v.cost + w   // cost of best path through v to u
c2 = u.cost       // cost of best path to u previously known
if(c1 < c2){      // if the path through v is better
  u.cost = c1
  u.path = v      // for computing actual paths
```

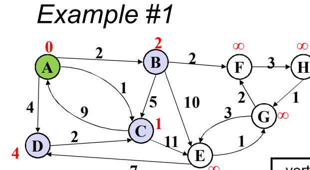

Example #1



Order Added to Known Set:

vertex	known?	cost	path
A		0	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	
H		??	

Example #1

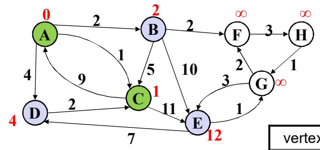


Order Added to Known Set:

A

vertex	known?	cost	path
A	Y	0	
B		≤ 2	A
C		≤ 1	A
D		≤ 4	A
E		??	
F		??	
G		??	
H		??	

Example #1

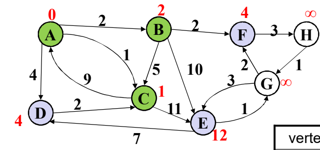


vertex	known?	cost	path
A	Y	0	
B		≤ 2	A
C	Y	1	A
D		≤ 4	A
E		≤ 12	C
F		??	
G		??	
H		??	

Order Added to Known Set:

A, C

Example #1

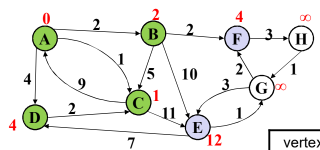


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D		≤ 4	A
E		≤ 12	C
F		≤ 4	B
G		??	
H		??	

Order Added to Known Set:

A, C, B

Example #1

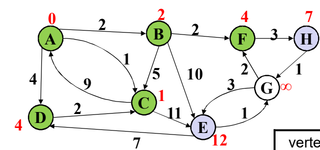


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F		≤ 4	B
G		??	
H		??	

Order Added to Known Set:

A, C, B, D

Example #1

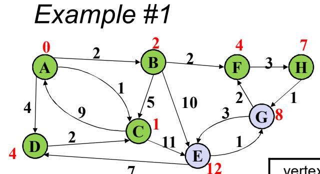


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F	Y	4	B
G		??	
H		≤ 7	F

Order Added to Known Set:

A, C, B, D, F

Example #1

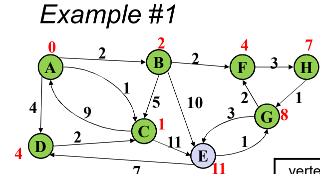


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F	Y	4	B
G		≤ 8	H
H	Y	7	F

Order Added to Known Set:

A, C, B, D, F, H

Example #1

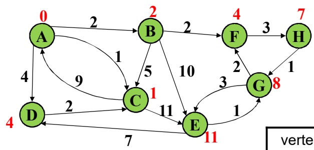


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Order Added to Known Set:

A, C, B, D, F, H, G

Example #1



Order Added to Known Set:

A, C, B, D, F, H, G, E

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

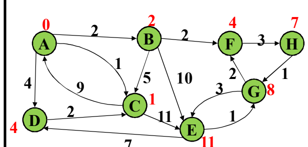
Features

- When a vertex is marked known, the cost of the shortest path to that node is known
  - The path is also known by following back-pointers
- While a vertex is still not known, another shorter path to it *might* still be found

Note: The "Order Added to Known Set" is not important
- A detail about how the algorithm works (client doesn't care)
- Not used by the algorithm (implementation doesn't care)
- It is sorted by path-cost, resolving ties in some way
  - Helps give intuition of why the algorithm works

Interpreting the Results

- Now that we're done, how do we get the path from, say, A to E?



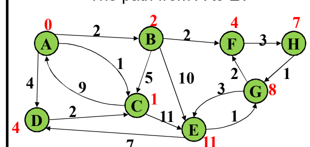
Order Added to Known Set:

A, C, B, D, F, H, G, E

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Stopping Short

- How would this have worked differently if we were only interested in:
  - The path from A to G?
  - The path from A to E?

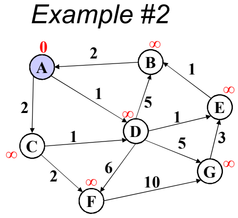


Order Added to Known Set:

A, C, B, D, F, H, G, E

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

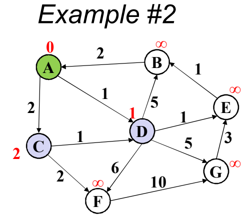
Example #2



Order Added to Known Set:

vertex	known?	cost	path
A		0	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	

Example #2

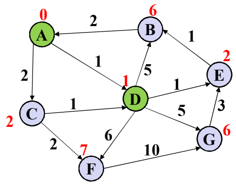


Order Added to Known Set:

A

vertex	known?	cost	path
A	Y	0	
B		??	
C		≤ 2	A
D		≤ 1	A
E		??	
F		??	
G		??	

Example #2

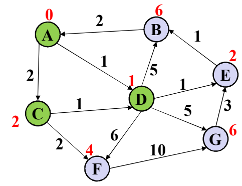


Order Added to Known Set:

A, D

vertex	known?	cost	path
A	Y	0	
B		≤ 6	D
C		≤ 2	A
D	Y	1	A
E		≤ 2	D
F		≤ 7	D
G		≤ 6	D

Example #2

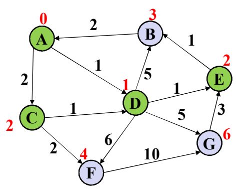


Order Added to Known Set:

A, D, C

vertex	known?	cost	path
A	Y	0	
B		≤ 6	D
C	Y	2	A
D	Y	1	A
E		≤ 2	D
F		≤ 4	C
G		≤ 6	D

Example #2

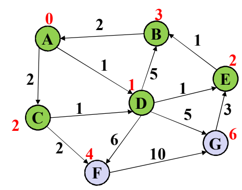


Order Added to Known Set:

A, D, C, E

vertex	known?	cost	path
A	Y	0	
B		≤ 3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F		≤ 4	C
G		≤ 6	D

Example #2

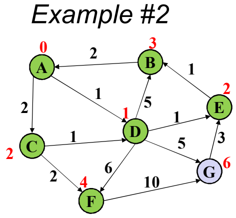


Order Added to Known Set:

A, D, C, E, B

vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F		≤ 4	C
G		≤ 6	D

Example #2

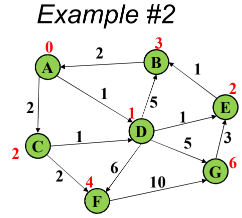


Order Added to Known Set:

A, D, C, E, B, F

vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F	Y	4	C
G		≤ 6	D

Example #2

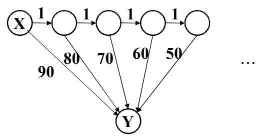


Order Added to Known Set:

A, D, C, E, B, F, G

vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F	Y	4	C
G	Y	6	D

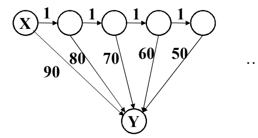
Example #3



How will the best-cost-so-far for Y proceed?

Is this expensive?

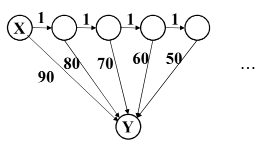
Example #3



How will the best-cost-so-far for Y proceed? 90, 81, 72, 63, 54, …

Is this expensive?

Example #3



How will the best-cost-so-far for Y proceed? 90, 81, 72, 63, 54, …

Is this expensive? No, each *edge* is processed only once

A Greedy Algorithm

- Dijkstra's algorithm
  - For single-source shortest paths in a weighted graph (directed or undirected) with no negative-weight edges
- An example of a *greedy algorithm*:
  - At each step, always does what seems best at that step
    - A locally optimal step, not necessarily globally optimal
  - Once a vertex is known, it is not revisited
    - Turns out to be globally optimal

Where are We?

- Had a problem: Compute shortest paths in a weighted graph with no negative weights
- Learned an algorithm: Dijkstra's algorithm
- What should we do after learning an algorithm?
  - Prove it is correct
    - Not obvious!
    - We will sketch the key ideas
  - Analyze its efficiency
    - Will do better by using a data structure we learned earlier!

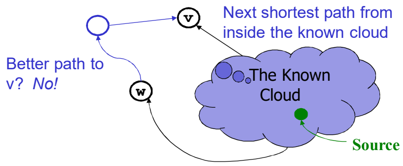
Correctness: Intuition

Rough intuition:

All the "known" vertices have the correct shortest path
- True initially: shortest path to start node has cost 0
- If it stays true every time we mark a node "known", then by induction this holds and eventually everything is "known"

Key fact we need: When we mark a vertex "known" we won't discover a shorter path later!
- This holds only because Dijkstra's algorithm picks the node with the next shortest path-so-far
- The proof is by contradiction…

Correctness: The Cloud (Rough Sketch)



Suppose **v** is the next node to be marked known ("added to the cloud")

- The best-known path to **v** must have only nodes "in the cloud"
  - Else we would have picked a node closer to the cloud than **v**
- Suppose the actual shortest path to **v** is different
  - It won't use only cloud nodes, or we would know about it
  - So it must use non-cloud nodes. Let **w** be the *first* non-cloud node on this path. The part of the path up to **w** is already known and must be shorter than the best-known path to **v**. So **v** would not have been picked. Contradiction.

Efficiency, first approach

Use pseudocode to determine asymptotic run-time
- Notice each edge is processed only once

```
Dijkstra(V, E, vStart):
  for v in V:
    v.cost=infinity; v.known=False
  vStart.cost = 0
  while not all nodes are known:
    b = find unknown node with smallest cost
    b.known = True
    for edge = (b,a) in E:
     if not a.known:
       if b.cost + weight((b,a)) < a.cost:
         a.cost = b.cost + weight((b,a))
         a.path = b
```


Efficiency, first approach

Use pseudocode to determine asymptotic run-time
- Notice each edge is processed only once

```
Dijkstra(V, E, vStart):
  for v in V:
    v.cost=infinity; v.known=False
  vStart.cost = 0
  while not all nodes are known:
    b = find unknown node with smallest cost
    b.known = True
    for edge = (b,a) in E:
     if not a.known:
       if b.cost + weight((b,a)) < a.cost:
         a.cost = b.cost + weight((b,a))
         a.path = b
```

O(|V|)

Efficiency, first approach

Use pseudocode to determine asymptotic run-time
- Notice each edge is processed only once

```
Dijkstra(V, E, vStart):
  for v in V:
    v.cost=infinity; v.known=False
  vStart.cost = 0
  while not all nodes are known:
    b = find unknown node with smallest cost
    b.known = True
    for edge = (b,a) in E:
     if not a.known:
       if b.cost + weight((b,a)) < a.cost:
         a.cost = b.cost + weight((b,a))
         a.path = b
```

O(|V|)

O(|V|^2)

Efficiency, first approach

Use pseudocode to determine asymptotic run-time
- Notice each edge is processed only once

```
Dijkstra(V, E, vStart):
  for v in V:
    v.cost=infinity; v.known=False
  vStart.cost = 0
  while not all nodes are known:
    b = find unknown node with smallest cost
    b.known = True
    for edge = (b,a) in E:
     if not a.known:
       if b.cost + weight((b,a)) < a.cost:
         a.cost = b.cost + weight((b,a))
         a.path = b
```

O(|V|)

O(|V|^2)

O(|E|)

Efficiency, first approach

Use pseudocode to determine asymptotic run-time
- Notice each edge is processed only once

```
Dijkstra(V, E, vStart):
  for v in V:
    v.cost=infinity; v.known=False
  vStart.cost = 0
  while not all nodes are known:
    b = find unknown node with smallest cost
    b.known = True
    for edge = (b,a) in E:
     if not a.known:
       if b.cost + weight((b,a)) < a.cost:
         a.cost = b.cost + weight((b,a))
         a.path = b
```

O(|V|)

O(|V|^2)

O(|E|)

O(|V|^2)

Improving asymptotic running time

- So far: $O(|V|^2)$
- We had a similar "problem" with topological sort being $O(|V|^2)$ due to each iteration looking for the node to process next
  - We solved it with a queue of zero-degree nodes
  - But here we need the lowest-cost node and costs can change as we process edges
- Solution?

Improving (?) asymptotic running time

- So far: $O(|V|^2)$
- We had a similar "problem" with topological sort being $O(|V|^2)$ due to each iteration looking for the node to process next
  - We solved it with a queue of zero-degree nodes
  - But here we need the lowest-cost node and costs can change as we process edges
- Solution?
  - A priority queue holding all unknown nodes, sorted by cost
  - But must support **`decreaseKey`** operation
    - Must maintain a reference from each node to its current position in the priority queue
    - Conceptually simple, but can be a pain to code up

Efficiency, second approach

Use pseudocode to determine asymptotic run-time

```
Dijkstra(V, E, vStart):
  for v in V:
    v.cost=infinity; v.known=False
  vStart.cost = 0
  build-heap with all vertices
  while heap is not empty:
    b = deleteMin()
    b.known = True
    for edge = (b,a) in E:
     if not a.known:
       if b.cost + weight((b,a)) < a.cost:
         decreaseKey(a,"new cost – old cost")
         a.path = b
```


Efficiency, second approach

Use pseudocode to determine asymptotic run-time

```
Dijkstra(V, E, vStart):
  for v in V:
    v.cost=infinity; v.known=False
  vStart.cost = 0
  build-heap with all vertices
  while heap is not empty:
    b = deleteMin()
    b.known = True
    for edge = (b,a) in E:
     if not a.known:
       if b.cost + weight((b,a)) < a.cost:
         decreaseKey(a,"new cost – old cost")
         a.path = b
```

O(|V|)

Efficiency, second approach

Use pseudocode to determine asymptotic run-time

```
Dijkstra(V, E, vStart):
  for v in V:
    v.cost=infinity; v.known=False
  vStart.cost = 0
  build-heap with all vertices
  while heap is not empty:
    b = deleteMin()
    b.known = True
    for edge = (b,a) in E:
     if not a.known:
       if b.cost + weight((b,a)) < a.cost:
         decreaseKey(a,"new cost – old cost")
         a.path = b
```

O(|V|)

O(|V|log|V|)

Efficiency, second approach

Use pseudocode to determine asymptotic run-time

```
Dijkstra(V, E, vStart):
  for v in V:
    v.cost=infinity; v.known=False
  vStart.cost = 0
  build-heap with all vertices
  while heap is not empty:
    b = deleteMin()
    b.known = True
    for edge = (b,a) in E:
     if not a.known:
       if b.cost + weight((b,a)) < a.cost:
         decreaseKey(a,"new cost – old cost")
         a.path = b
```

O(|V|)

O(|V|log|V|)

O(|E|log|V|)

Efficiency, second approach

Use pseudocode to determine asymptotic run-time

```
Dijkstra(V, E, vStart):
  for v in V:
    v.cost=infinity; v.known=False
  vStart.cost = 0
  build-heap with all vertices
  while heap is not empty:
    b = deleteMin()
    b.known = True
    for edge = (b,a) in E:
     if not a.known:
       if b.cost + weight((b,a)) < a.cost:
         decreaseKey(a,"new cost – old cost")
         a.path = b
```

O(|V|)

O(|V|log|V|)

O(|E|log|V|)

O(|V|log|V|+|E|log|V|)

Dense vs. sparse again

- First approach: *O*(|V|^2)
- Second approach: *O*(|V|log|V|+|E|log|V|)
- So which is better?
  - Sparse: *O*(|V|log|V|+|E|log|V|) (if |E| > |V|, then *O*(|E|log|V|))
  - Dense: *O*(|V|^2)
- But, remember these are worst-case and asymptotic
  - Priority queue might have slightly worse constant factors
  - On the other hand, for "normal graphs", we might call **`decreaseKey`** rarely (or not percolate far), making |E|log|V| more like |E|