CSE 373: Data Structures & Algorithms

Topological Sorting and Graph Traversals

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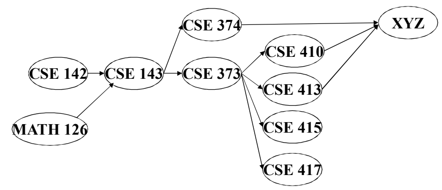
Winter 2016

This lecture material represents the work of multiple instructors at the University of Washington. Thank you to all who have contributed!

Topological Sort

Disclaimer: Do not use for official advising purposes !

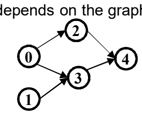
Problem: Given a DAG `G=(V,E)`, output all vertices in an order such that no vertex appears before another vertex that has an edge to it



One example output:
126, 142, 143, 374, 373, 417, 410, 413, XYZ, 415

Questions and comments

- Why do we perform topological sorts only on DAGs?
  - Because a cycle means there is no correct answer
- Is there always a unique answer?
  - No, there can be 1 or more answers; depends on the graph
- Do some DAGs have exactly 1 answer?
  - Yes, including all lists



- Terminology: A DAG represents a partial order and a topological sort produces a total order that is consistent with it

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Uses

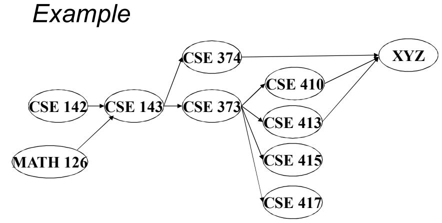
- Figuring out how to graduate
- Computing an order in which to recompute cells in a spreadsheet
- Determining an order to compile files using a Makefile
- In general, taking a dependency graph and finding an order of execution
- …

A First Algorithm for Topological Sort

1. Label ("mark") each vertex with its in-degree
   – Think "write in a field in the vertex"
   – Could also do this via a data structure (e.g., array) on the side
2. While there are vertices not yet output:
   a) Choose a vertex **v** with labeled with in-degree of 0
   b) Output **v** and *conceptually* remove it from the graph
   c) For each vertex **u** adjacent to **v** (i.e. **u** such that (**v**,**u**) in `E`), decrement the in-degree of **u**

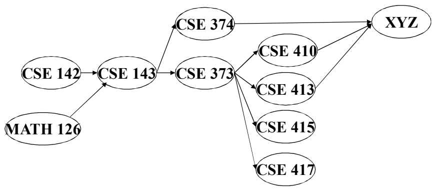
Example

Output:



Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?										
In-degree:	0	0	2	1	1	1	1	1	1	3

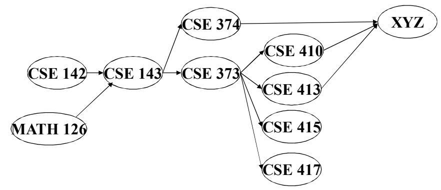
Example



Output:
126

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x									
In-degree:	0	0	2	1	1	1	1	1	1	3
			1							

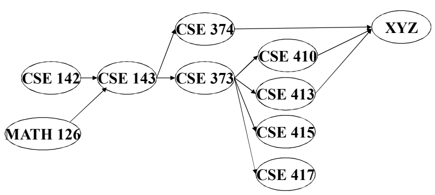
Example



Output:
126
142

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x								
In-degree:	0	0	2	1	1	1	1	1	1	3
			1							
			0							

Example



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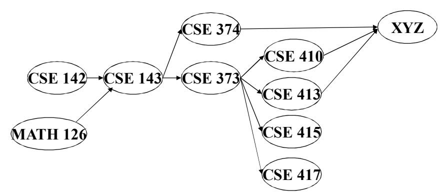
Example



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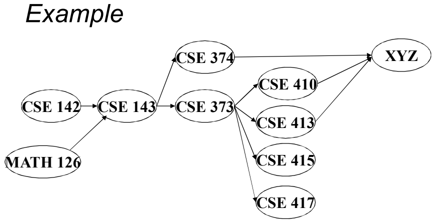
			0							

Example



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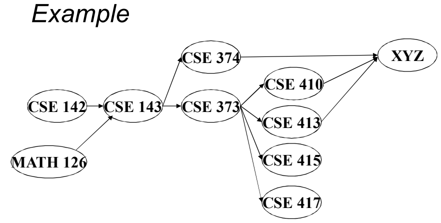
			1	0	0	0	0	0	0	2
			0							

Example



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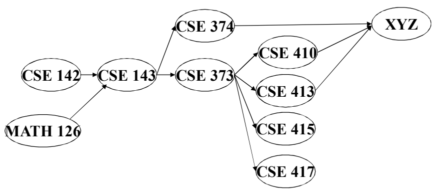
			0							

Example



Output:
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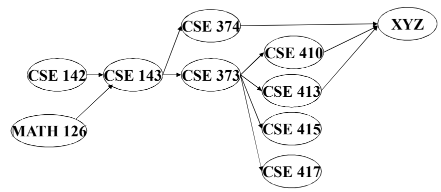
Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x	x			x	
In-degree:	0	0	2	1	1	1	1	1	1	3
			1	0	0	0	0	0	0	2
			0							1

Example



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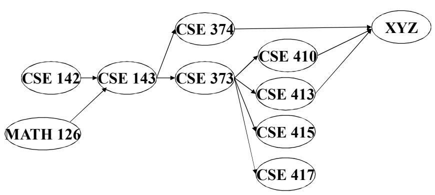
In-degree:	0	0	2	1	1	1	1	1	1	3
			1	0	0	0	0	0	0	2
			0							1
										0

Example



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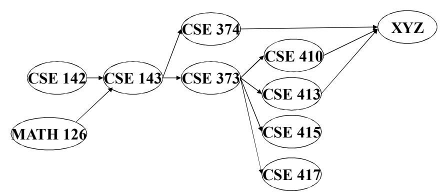
Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x	x	x		x	x
In-degree:	0	0	2	1	1	1	1	1	1	3
			1	0	0	0	0	0	0	2
			0							1
										0

Example



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In-degree:	0	0	2	1	1	1	1	1	1	3
			1	0	0	0	0	0	0	2
			0							1
										0

Notice

- Needed a vertex with in-degree 0 to start
  - Will always have at least 1 because no cycles
- Ties among vertices with in-degrees of 0 can be broken arbitrarily
  - Can be more than one correct answer, by definition, depending on the graph

Running time?

```
labelEachVertexWithItsInDegree()
for ctr in range(numVertices):
  v = findNewVertexOfDegreeZero()
  put v next in output
  for each w adjacent to v:
    w.indegree -=1
```

- What is the worst-case running time?
  - Initialization $O(|V|+|E|)$ (assuming adjacency list)
  - Sum of all find-new-vertex $O(|V|^2)$ (because each $O(|V|)$)
  - Sum of all decrements $O(|E|)$ (assuming adjacency list)
  - So total is $O(|V|^2)$ – not good for a sparse graph!

Doing better

The trick is to avoid searching for a zero-degree node every time!

- Keep the "pending" zero-degree nodes in a list, stack, queue, bag, table, or something
- Order we process them affects output but not correctness or efficiency provided add/remove are both $O(1)$

Using a queue:

1. Label each vertex with its in-degree, enqueue 0-degree nodes
2. While queue is not empty
   a) **v** = dequeue()
   b) Output **v** and remove it from the graph
   c) For each vertex **u** adjacent to **v** (i.e. **u** such that (**v**,**u**) in **E**), decrement the in-degree of **u**, if new degree is 0, enqueue it

Running time?

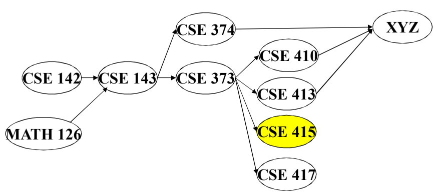
```
labelAllAndEnqueueZeros()
 for ctr in range(numVertices):
   v = dequeue()
   put v next in output
   for each w adjacent to v:
     w.indegree -= 1
     if w.indegree==0:
       enqueue(v)
```

- What is the worst-case running time?
  - Initialization: $O(|V|+|E|)$ (assuming adjacency list)
  - Sum of all enqueues and dequeues: $O(|V|)$
  - Sum of all decrements: $O(|E|)$ (assuming adjacency list)
  - So total is $O(|E| + |V|)$ – much better for sparse graph!

Topological Sort

We interrupt this program for an exciting announcement. CSE 415 will be offered next quarter! We will be using the Python language!

Problem: Given a DAG `G=(V,E)`, output all vertices in an order such that no vertex appears before another vertex that has an edge to it



One example output:

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Graph Traversals

Next problem: For an arbitrary graph and a starting node **v**, find all nodes *reachable* from **v** (i.e., there exists a path from **v**)

- Possibly "do something" for each node
- Examples: print to output, set a field, etc.

- Subsumed problem: Is an undirected graph connected?
- Related but different problem: Is a directed graph strongly connected?
  - Need cycles back to starting node

Basic idea:

- Keep following nodes
- But "mark" nodes after visiting them, so the traversal terminates and processes each reachable node exactly once

Abstract Idea

```
traverseGraph(startNode):
   Set pending = emptySet()
   pending.add(startNode)
   mark startNode as visited
   while pending is not empty:
     next = pending.remove()
     for each node u adjacent to next:
        if(u is not marked):
          mark u
          pending.add(u)
```


Running Time and Options

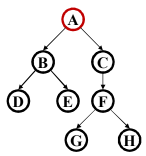
- Assuming **add** and **remove** are $O(1)$, entire traversal is $O(|E|)$
  - Use an adjacency list representation

- The order we traverse depends entirely on **add** and **remove**
  - Popular choice: a stack "depth-first graph search" "DFS"
  - Popular choice: a queue "breadth-first graph search" "BFS"

- DFS and BFS are "big ideas" in computer science
  - Depth: recursively explore one part before going back to the other parts not yet explored
  - Breadth: explore areas closer to the start node first

Example: Depth First Search

- A tree is a graph and DFS and BFS are particularly easy to “see”

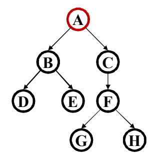



```
DFS(startNode):
  Mark and process startNode.
  For each node u adjacent to startNode:
    if u is not marked:
      DFS(u)
```

- A B D E C F G H
- Exactly what we called a “pre-order traversal” for trees
  - The marking is because we support arbitrary graphs and we want to process each node exactly once

Example: Another Depth First Search

- A tree is a graph and DFS and BFS are particularly easy to “see”

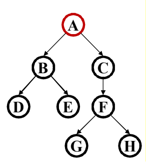



```
DFS2(startNode):
  Let s = Stack(). s.push(startNode)
  Mark startNode as visited.
  while s is not empty:
    next = s.pop() # and "process"
    For each node u adjacent to next:
     if u is not marked:
       mark u; s.push(u)
```

- A C F H G B E D
- A different but perfectly fine traversal

Example: Breadth First Search

- A tree is a graph and DFS and BFS are particularly easy to “see”




```
BFS(startNode):
  Let q = Queue(); q.enqueue(startNode)
  Mark startNode as visited.
  while q is not empty:
    next = q.dequeue() # and "process"
    For each node u adjacent to next:
      if u is not marked:
        mark u and q.enqueue(u)
```

- A B C D E F G H
- A “level-order” traversal

Comparison

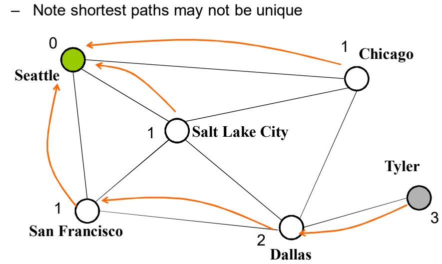
- Breadth-first always finds shortest paths, i.e., “optimal solutions”
  - Better for “what is the shortest path from **x** to **y**”
- But depth-first can use less space in finding a path
  - If *longest path* in the graph is `p` and highest out-degree is `d` then DFS stack never has more than `d*p` elements
  - But a queue for BFS may hold *O*(|V|) nodes
- A third approach:
  - *Iterative deepening (IDFS)*:
    - Try DFS but disallow recursion more than `K` levels deep
    - If that fails, increment `K` and start the entire search over
  - Like BFS, finds shortest paths. Like DFS, less space.

Saving the Path

- Our graph traversals can answer the reachability question:
  - “Is there a path from node x to node y?”
- But what if we want to actually output the path?
  - Like getting driving directions rather than just knowing it’s possible to get there!
- How to do it:
  - Instead of just “marking” a node, store the previous node along the path (when processing **u** causes us to add **v** to the search, set `v.path` field to be **u**)
  - When you reach the goal, follow `path` fields back to where you started (and then reverse the answer)
  - If just wanted path *length*, could put the integer distance at each node instead

Example using BFS

What is a path from Seattle to Tyler
- Remember marked nodes are not re-enqueued
- Note shortest paths may not be unique



Single source shortest paths

- Done: BFS to find the minimum path length from **v** to **u** in $O(|E|+|V|)$
- Actually, can find the minimum path length from **v** to *every node*
  - Still $O(|E|+|V|)$
  - No faster way for a "distinguished" destination in the worst-case
- Now: Weighted graphs

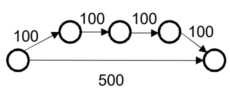
  Given a weighted graph and node **v**,
  find the minimum-cost path from **v** to every node

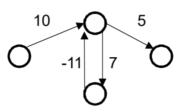
- As before, asymptotically no harder than for one destination

Applications

- Driving directions
- Cheap flight itineraries
- Network routing
- Critical paths in project management

Not as easy as BFS



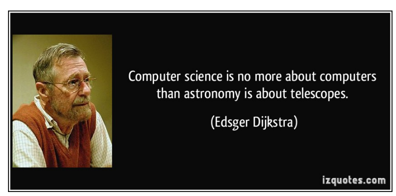


Why BFS won't work: Shortest path may not have the fewest edges
- Annoying when this happens with costs of flights

We will assume there are no negative weights
- *Problem* is *ill-defined* if there are negative-cost *cycles*
- *Today's algorithm* is *wrong* if *edges* can be negative
  - There are other, slower (but not terrible) algorithms

Dijkstra's Algorithm

- Named after its inventor Edsger Dijkstra (1930-2002)
  - Truly one of the "founders" of computer science; this is just one of his many contributions
  - Many people have a favorite Dijkstra story, even if they never met him



Dijkstra's Algorithm

- The idea: reminiscent of BFS, but adapted to handle weights
  - Grow the set of nodes whose shortest distance has been computed
  - Nodes not in the set will have a "best distance so far"
  - A priority queue will turn out to be useful for efficiency
- An example of a greedy algorithm
  - A series of steps
  - At each one the locally optimal choice is made