Topological Sort

Problem: Given a DAG \( G = (V, E) \), output all vertices in an order such that no vertex appears before another vertex that has an edge to it.

One example output:
126, 142, 143, 374, 373, 417, 410, 413, XYZ, 415

Questions and comments

• Why do we perform topological sorts only on DAGs?
  – Because a cycle means there is no correct answer.

• Is there always a unique answer?
  – No, there can be 1 or more answers; depends on the graph.

• Do some DAGs have exactly 1 answer?
  – Yes, including all lists.

• Terminology: A DAG represents a partial order and a topological sort produces a total order that is consistent with it.

A First Algorithm for Topological Sort

1. Label (“mark”) each vertex with its in-degree
   – Think “write in a field in the vertex”
   – Could also do this via a data structure (e.g., array) on the side.

2. While there are vertices not yet output:
   a) Choose a vertex \( v \) with labeled in-degree of 0.
   b) Output \( v \) and conceptually remove it from the graph.
   c) For each vertex \( u \) adjacent to \( v \) (i.e., \( u \) such that \((v, u)\) in \( E \)), decrement the in-degree of \( u \).

Example

Node: 126 142 143 374 373 410 413 415 XYZ
Removed?: 0 0 2 1 1 1 1 1 3
In-degree: 0 0 2 1 1 1 1 1 3

Output:
Example

Output: 126

Node: 126 142 143 374 373 410 413 415 417 XYZ
Removed? x x x x x x
In-degree: 0 0 2 1 1 1 1 1 1 3

Example

Output: 126 142

Node: 126 142 143 374 373 410 413 415 417 XYZ
Removed? x x x x x x
In-degree: 0 0 2 1 1 1 1 1 1 3

Example

Output: 126 142 143

Node: 126 142 143 374 373 410 413 415 417 XYZ
Removed? x x x x x x
In-degree: 0 0 2 1 1 1 1 1 1 3

Example

Output: 126 142 143 374 373 410 413 415 417 XYZ
Removed? x x x x x x
In-degree: 0 0 2 1 1 1 1 1 1 3
Notice

- Needed a vertex with in-degree 0 to start
  - Will always have at least 1 because no cycles
- Ties among vertices with in-degrees of 0 can be broken arbitrarily
  - Can be more than one correct answer, by definition, depending on the graph

Running time?

```python
labelEachVertexWithItsInDegree()
for ctc in range(numVertices):
    v = findNewVertexOfDegreeZero()
    put v next in output
    for each w adjacent to v:
        w.indegree -= 1
```

- What is the worst-case running time?
  - Initialization $O(|V|+|E|)$ (assuming adjacency list)
  - Sum of all find-new-vertex $O(|V|^2)$ (because each $O(|V|)$)
  - Sum of all decrements $O(|E|)$ (assuming adjacency list)
  - So total is $O(|V|^2)$ - not good for a sparse graph!
Doing better

The trick is to avoid searching for a zero-degree node every time!
- Keep the “pending” zero-degree nodes in a list, stack, queue, bag, table, or something
- Order we process them affects output but not correctness or efficiency provided add/remove are both O(1)

Using a queue:
1. Label each vertex with its in-degree, enqueue 0-degree nodes
2. While queue is not empty
   a) $v = \text{dequeue}()$
   b) Output $v$ and remove it from the graph
   c) For each vertex $u$ adjacent to $v$ (i.e. $u$ such that $(v, u)$ in $E$), decrement the in-degree of $u$. If new degree is 0, enqueue it

Running time?

```python
def labelAllAndEnqueueZeros():
    for ctr in range(numVertices):
        v = dequeue()
        put v next in output
        for each w adjacent to v:
            w.indegree -= 1
            if w.indegree==0:
                enqueue(v)
```

• What is the worst-case running time?
  - Initialization: $O(|V|+|E|)$ (assuming adjacency list)
  - Sum of all enqueues and dequeues: $O(|V|)$
  - Sum of all decrements: $O(|E|)$ (assuming adjacency list)
  - So total is $O(|E| + |V|)$ – much better for sparse graph!

Topological Sort

Problem: Given a DAG $G=(V,E)$, output all vertices in an order such that no vertex appears before another vertex that has an edge to it

We interrupt this program for an exciting announcement. CSE 415 will be offered next quarter! We will be using the Python language!

Graph Traversals

Next problem: For an arbitrary graph and a starting node $v$, find all nodes reachable from $v$ (i.e., there exists a path from $v$)
- Possibly “do something” for each node
- Examples: print to output, set a field, etc.

• Subsumed problem: Is an undirected graph connected?
• Related but different problem: Is a directed graph strongly connected?
  - Need cycles back to starting node

Basic idea:
- Keep following nodes
- But “mark” nodes after visiting them, so the traversal terminates and processes each reachable node exactly once

Abstract Idea

```python
def traverseGraph(startNode):
    Set pending = emptySet()
    pending.add(startNode)
    mark startNode as visited
    while pending is not empty:
        next = pending.remove()
        for each node u adjacent to next:
            if (u is not marked):
                mark u
                pending.add(u)
```

Running Time and Options

• Assuming add and remove are $O(1)$, entire traversal is $O(|E|)$
  - Use an adjacency list representation

• The order we traverse depends entirely on add and remove
  - Popular choice: a stack “depth-first graph search” “DFS”
  - Popular choice: a queue “breadth-first graph search” “BFS”

• DFS and BFS are “big ideas” in computer science
  - Depth: recursively explore one part before going back to the other parts not yet explored
  - Breadth: explore areas closer to the start node first
**Example: Depth First Search**

- A tree is a graph and DFS and BFS are particularly easy to “see”

**DFS**: Mark and process startNode.
   For each node u adjacent to startNode:
   if u is not marked:
      DFS(u)

- A B C D E F G H
- Exactly what we called a “pre-order traversal” for trees
  - The marking is because we support arbitrary graphs and we want to process each node exactly once

**Example: Another Depth First Search**

- A tree is a graph and DFS and BFS are particularly easy to “see”

**DFS2**: Let s = Stack(). s.push(startNode)
   Mark startNode as visited.
   while s is not empty:
      next = s.pop() # and “process”
      For each node u adjacent to next:
      if u is not marked:
         mark u; s.push(u)

- A C F H G B E D
- A different but perfectly fine traversal

**Example: Breadth First Search**

- A tree is a graph and DFS and BFS are particularly easy to “see”

**BFS**: Let q = Queue(); q.enqueue(startNode)
   Mark startNode as visited.
   while q is not empty:
      next = q.dequeue() # and “process”
      For each node u adjacent to next:
      if u is not marked:
         mark u and q.enqueue(u)

- A B C D E F G H
- A “level-order” traversal

**Comparison**

- Breadth-first always finds shortest paths, i.e., “optimal solutions”
  - Better for “what is the shortest path from x to y”
- But depth-first can use less space in finding a path
  - If longest path in the graph is p and highest out-degree is d
    then DFS stack never has more than dp elements
  - But a queue for BFS may hold O(|V|) nodes
- A third approach:
  - Iterative deepening (IDFS):
    - Try DFS but disallow recursion more than K levels deep
    - If that fails, increment K and start the entire search over
      - Like BFS, finds shortest paths. Like DFS, less space.

**Saving the Path**

- Our graph traversals can answer the reachability question:
  - “Is there a path from node x to node y?”
- But what if we want to actually output the path?
  - Like getting driving directions rather than just knowing it’s possible to get there!
- How to do it:
  - Instead of just “marking” a node, store the previous node along the path (when processing u causes us to add v to the search, set v.path field to be u)
  - When you reach the goal, follow path fields back to where you started (and then reverse the answer)
  - If just wanted path length, could put the integer distance at each node instead

**Example using BFS**

- What is a path from Seattle to Tyler
  - Remember marked nodes are not re-enqueued
  - Note shortest paths may not be unique
Single source shortest paths

- Done: BFS to find the minimum path length from \( v \) to \( u \) in \( O(|E|+|V|) \)
- Actually, can find the minimum path length from \( v \) to every node
  - Still \( O(|E|+|V|) \)
  - No faster way for a “distinguished” destination in the worst-case
- Now: Weighted graphs
  - Given a weighted graph and node \( v \), find the minimum-cost path from \( v \) to every node
  - As before, asymptotically no harder than for one destination

Applications

- Driving directions
- Cheap flight itineraries
- Network routing
- Critical paths in project management

Not as easy as BFS

Why BFS won’t work: Shortest path may not have the fewest edges
  - Annoying when this happens with costs of flights

We will assume there are no negative weights
- Problem is ill-defined if there are negative-cost cycles
- Today’s algorithm is wrong if edges can be negative
  - There are other, slower (but not terrible) algorithms

Dijkstra’s Algorithm

- Named after its inventor Edsger Dijkstra (1930-2002)
  - Truly one of the “founders” of computer science; this is just one of his many contributions
  - Many people have a favorite Dijkstra story, even if they never met him

Dijkstra’s Algorithm

- The idea: reminiscent of BFS, but adapted to handle weights
  - Grow the set of nodes whose shortest distance has been computed
  - Nodes not in the set will have a “best distance so far”
  - A priority queue will turn out to be useful for efficiency
- An example of a greedy algorithm
  - A series of steps
  - At each one the locally optimal choice is made