Minimum Spanning Trees and Kruskal's Algorithm

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Thank you to all who have contributed!

Minimum Spanning Trees

The minimum-spanning-tree problem
– Given a weighted undirected graph, compute a spanning tree of minimum weight

Given an undirected graph \( G = (V, E) \), find a graph \( G' = (V, E') \) such that:
– \( E' \) is a subset of \( E \)
– \( |E'| = |V| - 1 \)
– \( G' \) is connected

G' is a minimum spanning tree.

Minimum Spanning Tree Algorithms

• Kruskal's Algorithm for Minimum Spanning Tree construction
  – A greedy algorithm.
  – Uses a priority queue.
  – Uses the UNION-FIND technique.

• Prim's Algorithm for Minimum Spanning Tree
  – Related to Dijkstra's Algorithm for shortest paths.
  – Both based on expanding cloud of known vertices (basically using a priority queue instead of a DFS stack)

Kruskal's Algorithm

Idea: Grow a forest out of edges that do not create a cycle. Pick an edge with the smallest weight.

An edge-based greedy algorithm
Builds MST by greedily adding edges

Kruskal's Algorithm Pseudocode

1. Sort edges by weight (better: put in min-heap)
2. Put each node in its own subset (of a UNION-FIND instance)
3. While output size < \( |V| - 1 \)
   – Consider next smallest edge \( (u,v) \)
   – if \( \text{find}(u) \) and \( \text{find}(v) \) indicate \( u \) and \( v \) are in different sets
     • output \( (u,v) \)
     • Perform \( \text{union}(\text{find}(u), \text{find}(v)) \)

Recall invariant:
\( u \) and \( v \) in same set if and only if connected in output-so-far

Kruskal's Example

Edges in sorted order:
1: (A,D), (C,D), (B,E), (D,E)
2: (A,B), (C,F), (A,C)
3: (E,G)
5: (D,G), (B,D)
6: (D,F)
10: (F,G)

Output:
Note: At each step, the UNION-FIND subsets correspond to the trees in a forest.
Kruskal's Example

Edges in sorted order:
1: (A,D), (C,D), (B,E), (D,E)
2: (A,B), (C,F), (A,C)
3: (E,G)
5: (D,G), (B,D)
6: (D,F)
10: (F,G)

Output: (A,D)

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Example

Edges in sorted order:
1: (A,D), (C,D), (B,E), (D,E)
2: (A,B), (C,F), (A,C)
3: (E,G)
5: (D,G), (B,D)
6: (D,F)
10: (F,G)

Output: (A,D), (C,D)

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Example

Edges in sorted order:
1: (A,D), (C,D), (B,E), (D,E)
2: (A,B), (C,F), (A,C)
3: (E,G)
5: (D,G), (B,D)
6: (D,F)
10: (F,G)

Output: (A,D), (C,D), (B,E)

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Example

Edges in sorted order:
1: (A,D), (C,D), (B,E), (D,E)
2: (A,B), (C,F), (A,C)
3: (E,G)
5: (D,G), (B,D)
6: (D,F)
10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E)

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Example

Edges in sorted order:
1: (A,D), (C,D), (B,E), (D,E)
2: (A,B), (C,F), (A,C)
3: (E,G)
5: (D,G), (B,D)
6: (D,F)
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Output: (A,D), (C,D), (B,E), (D,E)

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Example

Edges in sorted order:
1: (A,D), (C,D), (B,E), (D,E)
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Output: (A,D), (C,D), (B,E), (D,E)

Note: At each step, the union/find sets are the trees in the forest
### Kruskal's Example

Edges in sorted order:
1: (A,D), (C,D), (B,E), (D,E)
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Output: (A,D), (C,D), (B,E), (D,E), (C,F)

Note: At each step, the union-find sets are the trees in the forest.

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### Kruskal's Example

Edges in sorted order:
1: (A,D), (C,D), (B,E), (D,E)
2: (A,B), (C,F), (A,C)
3: (E,G)
5: (D,G), (B,D)
6: (D,F)
10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E), (C,F), (E,G)

Note: At each step, the union-find sets are the trees in the forest.

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### Kruskal's Algorithm Analysis

Idea: Grow a forest out of edges that do not grow a cycle. (This is similar to the maze-construction problem: knocking down a wall was essentially adding an edge that connected adjacent cells.)

- But now consider the edges in order by weight

So:
- Sort edges: $O(|E|\log |E|)$
- Iterate through edges using union-find for cycle detection almost $O(|E|)$

Somewhat better:
- Floyd's algorithm to build min-heap with edges $O(|E|)$
- Iterate through edges using UNION-FIND for cycle prevention and deleteMin to get next edge $O(|E|\log |E|)$
- Not better worst-case asymptotically, but often stops long before considering all edges.

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### Kruskal's Algorithm

List the edges in order of size:
- ED 2
- AB 3
- AE 4
- CD 4
- BC 5
- EF 5
- CF 6
- AF 7
- BF 8
- CF 8

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### Kruskal's Algorithm

Select the edge with min cost:

**ED 2**

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### Kruskal's Algorithm

Select the next minimum cost edge that does not create a cycle:

**ED 2**, **AB 3**
Select the next minimum cost edge that does not create a cycle:

- ED 2
- AB 3
- CD 4 (or AE 4)

BC 5 – forms a cycle

EF 5

All vertices have been connected. The solution is:

- ED 2
- AB 3
- CD 4
- AE 4
- EF 5

Total weight of tree: 18