CSE373: Data Structures & Algorithms

Priority Queues

Hunter Zahn

Summer 2016

A Quick Note:

- Homework 3 out! Start early!

A new ADT: Priority Queue

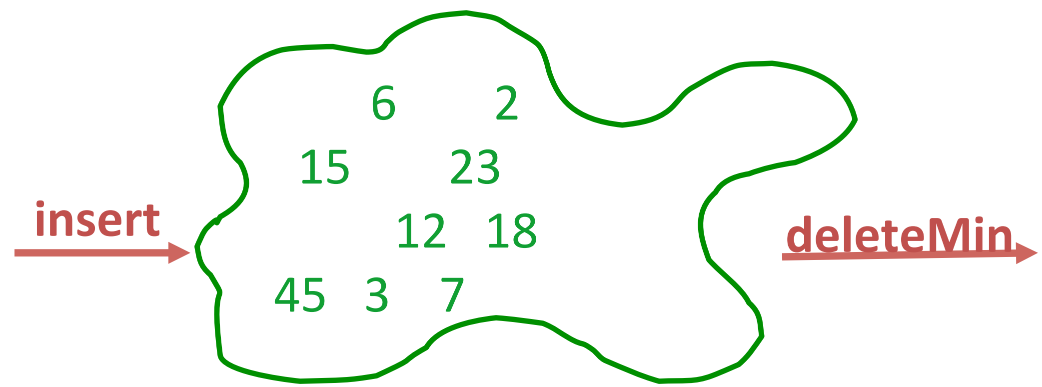
- Textbook Chapter 6
  – Nice to see a new and surprising data structure

- A **priority queue** holds *compare-able data*
  – Like dictionaries and unlike stacks and queues, need to *compare items*
    - Given *x* and *y*, is *x* less than, equal to, or greater than *y*
    - Meaning of the ordering can depend on your data
    - Many data structures require this: dictionaries, sorting
  – Integers are comparable, so will use them in examples
    - But the priority queue ADT is much more general
    - Typically two fields, the *priority* and the *data*

Priorities

- Each item has a "priority"
  - The *lesser* item is the one with the *greater* priority
  - So "priority 1" is more important than "priority 4"
  - (Just a convention, think "first is best")

- Operations:
  - **`insert`**
  - **`deleteMin`**
  - **`is_empty`**



- Key property: **`deleteMin`** *returns* and *deletes* the item with greatest priority (lowest priority value)
  - Can resolve ties arbitrarily

Example

`insert` *e1* with priority *5*
`insert` *e2* with priority *3*
`insert` *e3* with priority *4*
a = **`deleteMin`** **`// a = e2`**
b = **`deleteMin`** **`// b = e3`**
`insert` *e4* with priority *2*
`insert` *e5* with priority *6*
c = **`deleteMin`** **`// c = e4`**
d = **`deleteMin`** **`// d = e1`**

- Analogy: **`insert`** is like **`enqueue`**, **`deleteMin`** is like **`dequeue`**
  - But the whole point is to use priorities instead of FIFO

Applications

Like all good ADTs, the priority queue arises often

- Sometimes blatant, sometimes less obvious

- Run multiple programs in the operating system
  - "critical" before "interactive" before "compute-intensive"
  - Maybe let users set priority level
- Treat hospital patients in order of severity (or triage)
- Select print jobs in order of decreasing length?

More applications

- “Greedy” algorithms
  - May see an example when we study graphs in a few weeks
- Forward network packets in order of urgency
- Select most frequent symbols for data compression (cf. CSE143)
- Sorting (first **`insert`** all, then repeatedly **`deleteMin`**)
  - Much like Homework 1 uses a stack to implement reverse

Finding a good data structure

- Will show an efficient, non-obvious data structure for this ADT
  - But first let's analyze some "obvious" ideas for *n* data items
  - All times worst-case; assume arrays "have room"

data	*insert algorithm / time*	*deleteMin algorithm / time*
unsorted array		
unsorted linked list		
sorted array		
sorted linked list		
binary search tree		
AVL tree		
(our) hash table		

Need a good data structure!

- Will show an efficient, non-obvious data structure for this ADT
  - But first let's analyze some "obvious" ideas for *n* data items
  - All times worst-case; assume arrays "have room"

data	*insert algorithm / time*		*deleteMin algorithm / time*	
unsorted array	add at end	$O(1)$	search	$O(n)$
unsorted linked list	add at front	$O(1)$	search	$O(n)$
sorted array	search / shift	$O(n)$	stored in reverse	$O(1)$
sorted linked list	put in right place	$O(n)$	remove at front	$O(1)$
binary search tree	put in right place	$O(n)$	leftmost	$O(n)$
AVL tree	put in right place	$O(\texttt{log}\, n)$	leftmost	$O(\texttt{log}\, n)$
(our) hash table	add	$O(1)$	iterate over keys	$O(n)$

More on possibilities

- *If* priorities are random, binary search tree will likely do better
  - *O*(**`log`** *n*) **`insert`** and *O*(**`log`** *n*) **`deleteMin`** on *average*
- One more idea: if priorities are 0, 1, …, *k* can use array of lists
  - **`insert`**: add to front of list at **`arr[priority]`**, *O*(1)
  - **`deleteMin`**: remove from lowest non-empty list *O(k)*
- We are about to see a data structure called a "binary heap"
  - *O*(**`log`** *n*) **`insert`** and *O*(**`log`** *n*) **`deleteMin`** *worst-case*
    - Possible because we don't support unneeded operations; no need to maintain a full sort
  - *If* items arrive in random order, then **`insert`** is *O*(1) on *average*

Our data structure

A *binary min-heap* (or just *binary heap* or just *heap*) is:

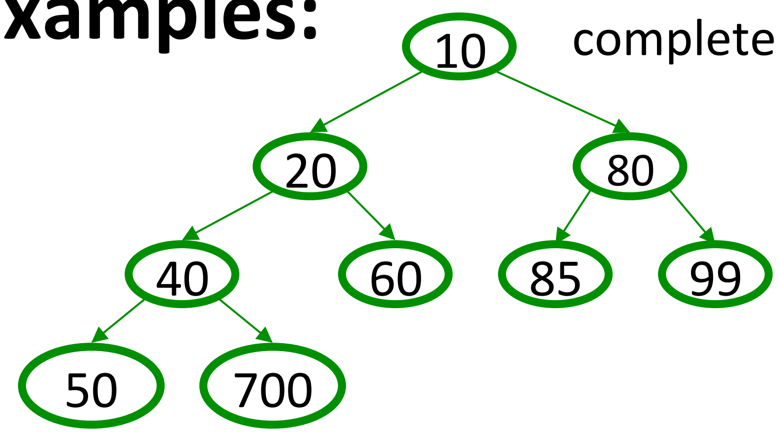
- Structure property: A *complete*e binary tree
- Heap property: The priority of every (non-root) node is greater than the priority of its parent
  - ***Not*** a binary search tree

Structure Property: Completeness

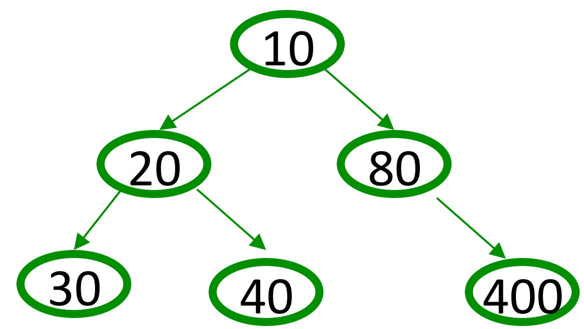
- A Binary Heap is a **complete** binary tree:
  - A binary tree with all levels full, with a possible exception being the bottom level, which is filled left to right

incomplete

Examples:



complete

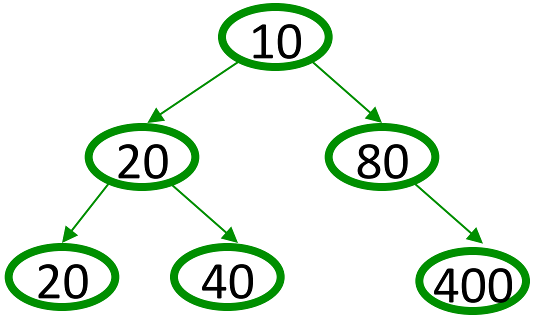


are these trees *complete*?

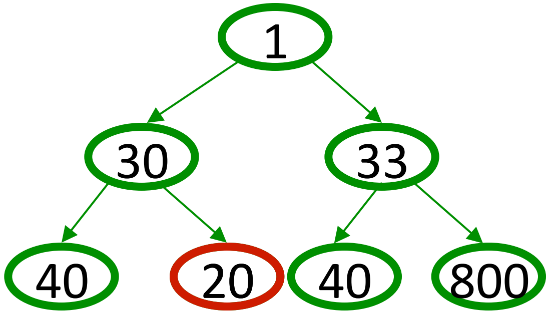
Heap Order Property

- The priority of every (non-root) node is greater than (or equal to) that of it's parent.

Examples: heap



not a heap



which of these are *heaps*?

Our data structure

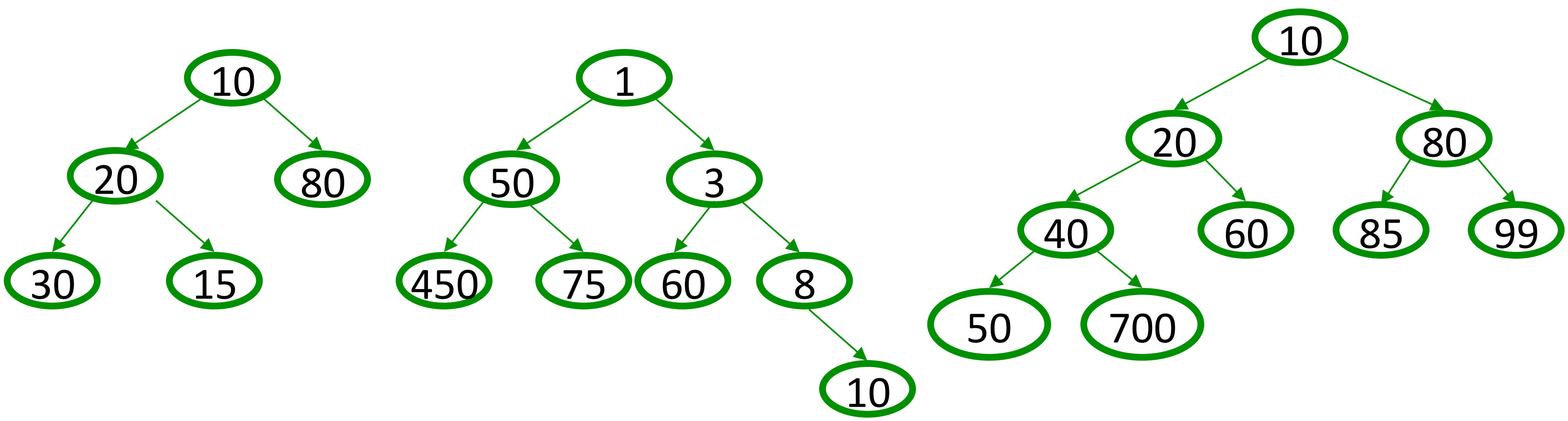
A *binary min-heap* (or just *binary heap* or just *heap*) is:

- Structure property: A *complete*e binary tree
- Heap property: The priority of every (non-root) node is greater than (or equal to) the priority of its parent
  - ***Not*** a binary search tree

Our data structure

A *binary min-heap* (or just *binary heap* or just *heap*) is:

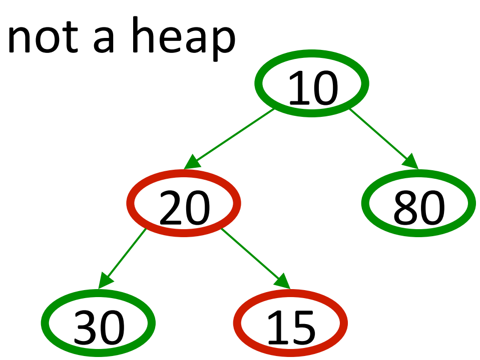
- Structure property: A *complete* binary tree
- Heap property: The priority of every (non-root) node is greater than (or equal to) the priority of its parent
  - ***Not*** a binary search tree

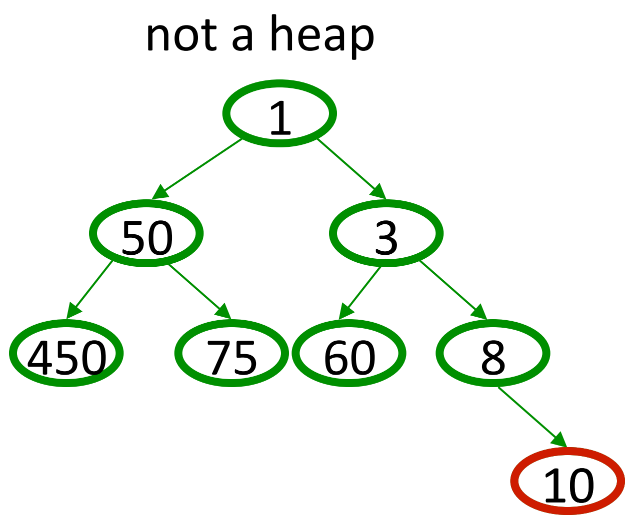


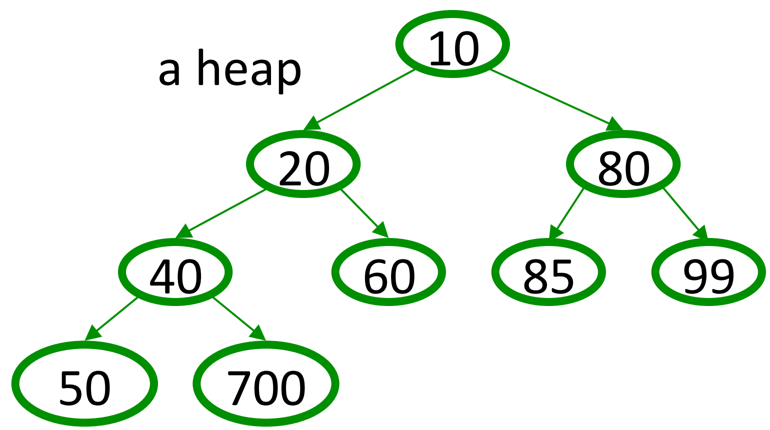
Our data structure

A *binary min-heap* (or just *binary heap* or just *heap*) is:

- Structure property: A *complete* binary tree
- Heap property: The priority of every (non-root) node is greater than (or equal to) the priority of its parent
  - ***Not*** a binary search tree





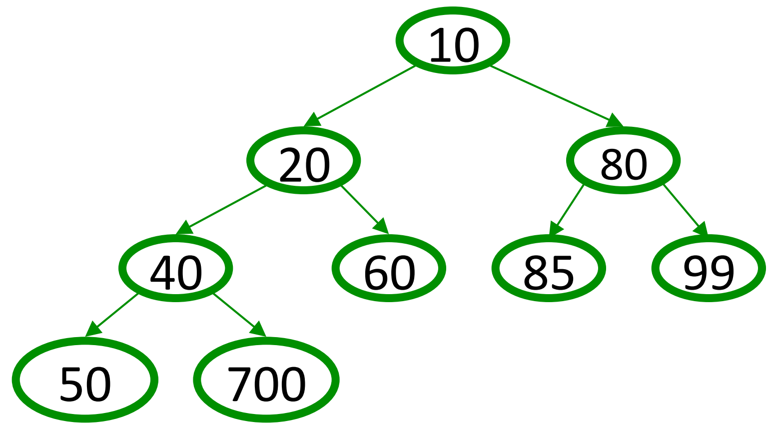


So:

- Where is the highest-priority item?
- What is the height of a heap with *n* items?

Operations: basic idea

- `findMin`: return `root.data`
- `deleteMin`:
  1. `answer = root.data`
  2. Move right-most node in last row to root to restore structure property
  3. "Percolate down" to restore heap property
- `insert:`
  1. Put new node in next position on bottom row to restore structure property
  2. "Percolate up" to restore heap property

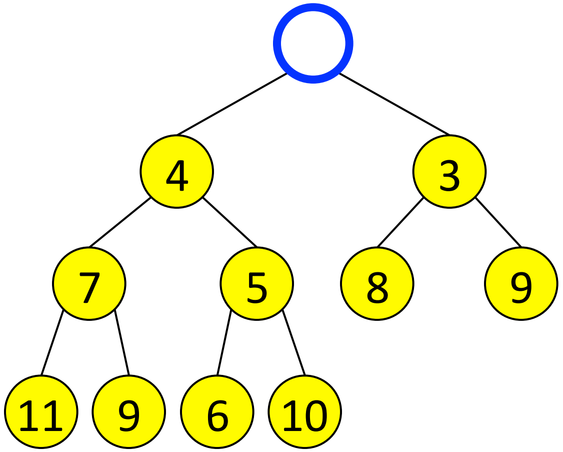


Overall strategy:

- *Preserve structure property*
- *Break and restore heap property*

DeleteMin

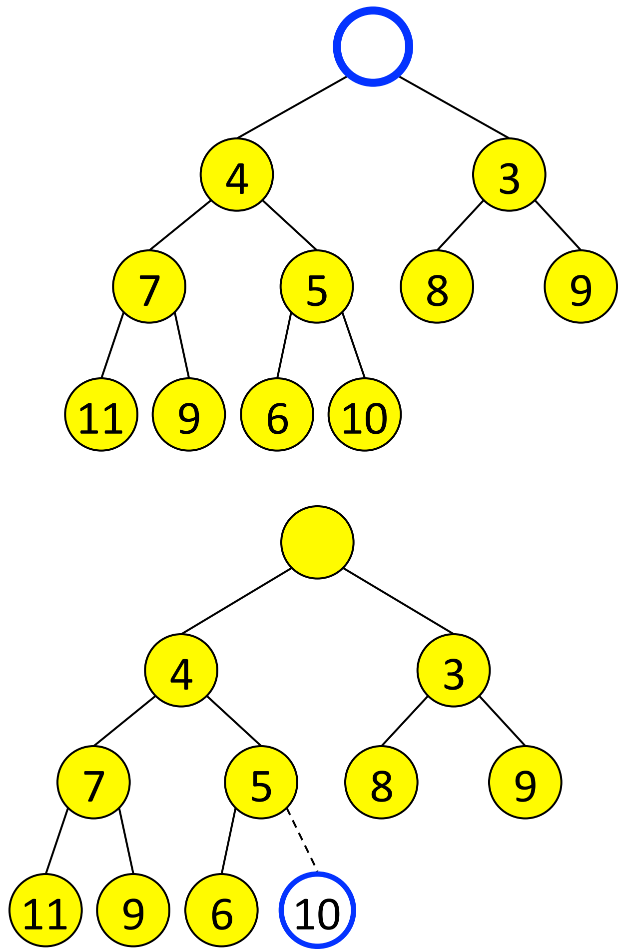
1. Delete (and later return) value at root node



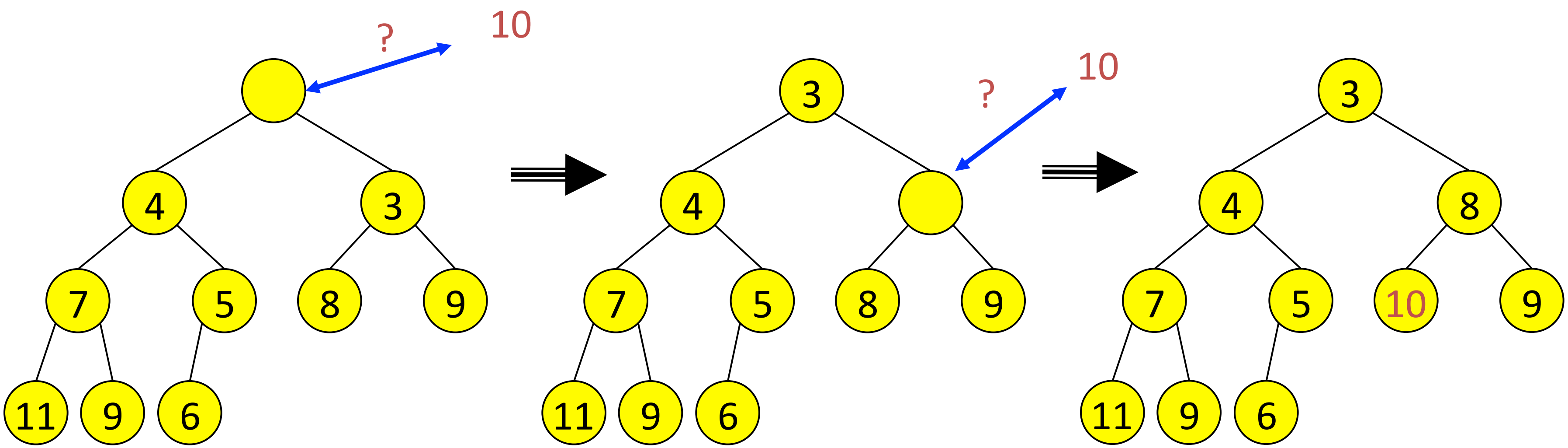
2. Restore the Structure Property

- We now have a "hole" at the root
  - Need to fill the hole with another value

- When we are done, the tree will have one less node and must still be complete



3. Restore the Heap Property



Percolate down:

- Keep comparing with both children
- Swap with lesser child and go down one level
    - What happens if we swap with the larger child?
- Done if both children are ≥ item or reached a leaf node

Why is this correct? What is the run time?

DeleteMin: Run Time Analysis

- We will percolate down at most (height of heap) times
  - So run time is O(height of heap)

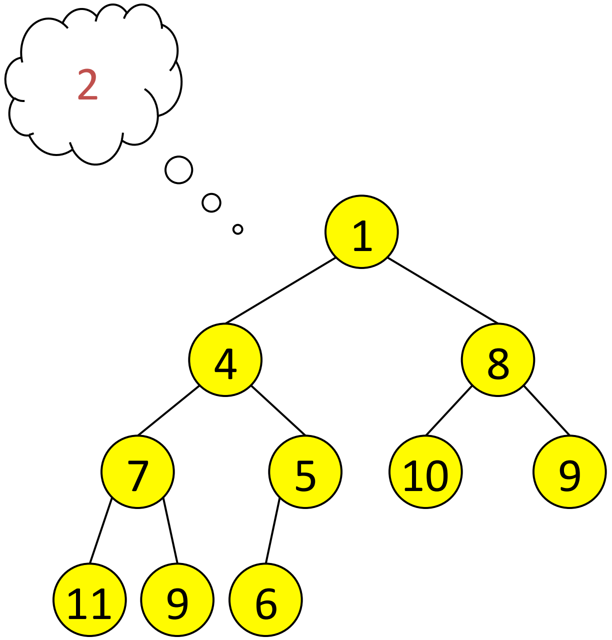
- A heap is a complete binary tree

- Height of a complete binary tree of n nodes?
  - height = $\lfloor \texttt{log}_2(n) \rfloor$

- Run time of **`deleteMin`** is $O(\texttt{log}\ n)$

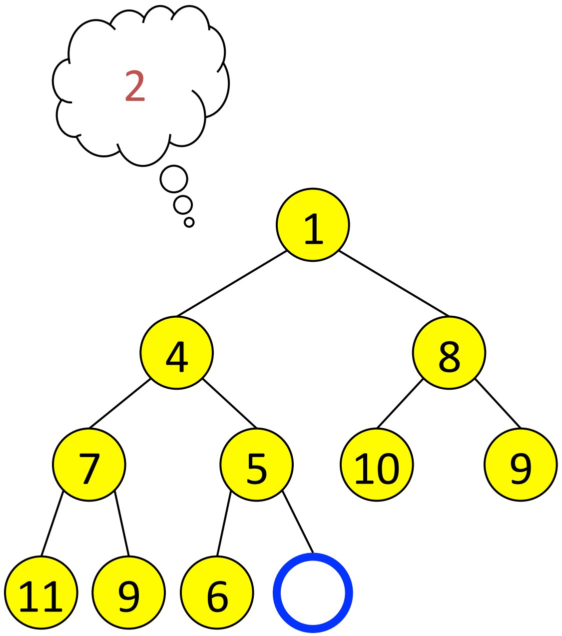
Insert

- Add a value to the tree
- Afterwards, structure and heap properties must still be correct
- Where do we insert the new value?

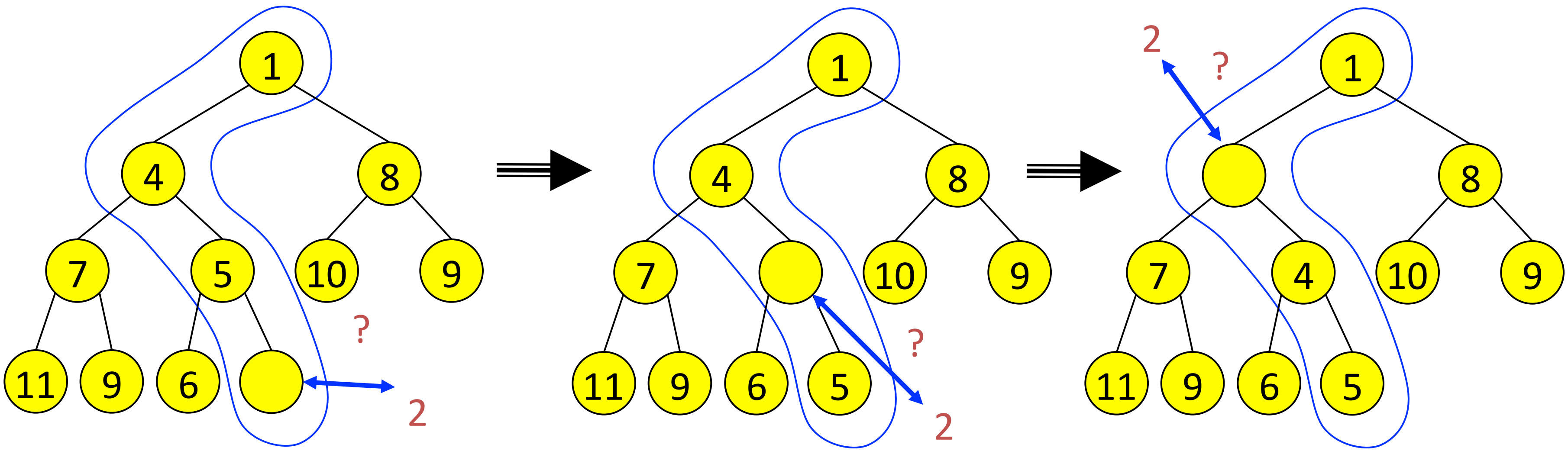


Insert: Maintain the Structure Property

- There is only one valid tree shape after we add one more node
- So put our new data there and then focus on restoring the heap property



Maintain the heap property



Percolate up:

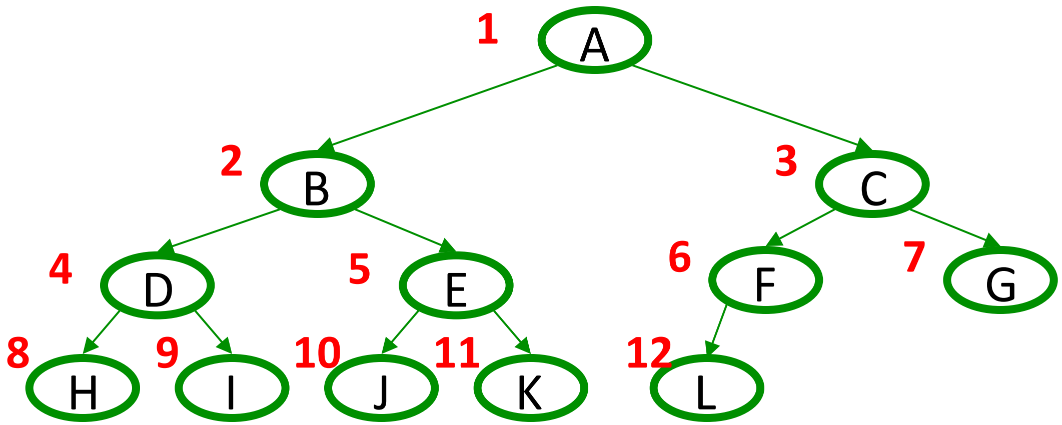
- Put new data in new location
- If parent larger, swap with parent, and continue
- Done if parent ≤ item or reached root

Why is this correct? What is the run time?

Insert: Run Time Analysis

- Like **`deleteMin`**, worst-case time proportional to tree height
  - $O(\texttt{log}\ n)$
- But… **`deleteMin`** needs the "last used" complete-tree position and **`insert`** needs the "next to use" complete-tree position
  - If "keep a reference to there" then **`insert`** and **`deleteMin`** have to adjust that reference: $O(\texttt{log}\ n)$ in worst case
  - Could calculate how to find it in $O(\texttt{log}\ n)$ from the root given the size of the heap
    - But it's not easy
    - And then **`insert`** is always $O(\texttt{log}\ n)$, promised $O(1)$ on average (assuming random arrival of items)
- There's a "trick": don't represent complete trees with explicit edges!

Array Representation of Binary Trees



From node `i`:

left child: `i*2`
right child: `i*2+1`
parent: `i/2`

(wasting index 0 is convenient for the index arithmetic)

implicit (array) implementation:

	A	B	C	D	E	F	G	H	I	J	K	L	
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Judging the array implementation

Plusses:

- Less “wasted” space
  - Just index 0 and unused space on right
  - In conventional tree representation, one edge per node (except for root), so *n*-1 wasted space (like linked lists)
  - Array would waste more space if tree were not complete
- Multiplying and dividing by 2 is very fast (shift operations in hardware)
- Last used position is just index **`size`**

Minuses:

- Same might-be-empty or might-get-full problems we saw with stacks and queues (resize by doubling as necessary)

Plusses outweigh minuses: “this is how people do it”