Bounding AVLTree Height

Fibonacci sequence:

$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

$F(0) = 0$

$F(1) = 1$

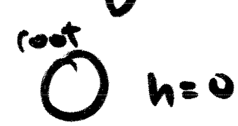
$F(n+1) = F(n-1) + F(n)$

☆ Grows exponentially! ☆

$S(h)$: "the minimum # of nodes in an AVL tree of height h".

defined inductively:

$S(0) = 1$



$S(1) = 2$

$S(-1) = 0$ ⟵ represents "null" tree.

for $h \geq 1$, $S(h) = 1 + S(h-1) + S(h-2)$

intuition: if $S(h)$ grows exponentially in 'h', then the height 'h' grows logarithmically in the number of nodes.

From slides: $S(h)$ seems to be equal to $F(h+3)-1$. will prove by induction.

Let $P(h)$ be $S(h) == F(h+3)-1$. We will prove this for $h \geq 0$

Base Cases

$S(0) = 1 \qquad F(0+3)-1 = 2-1 = 1$ ✓

$S(1) = 2 \qquad F(1+3)-1 = 3-1 = 2$ ✓

* need 2 base cases because

$S(h) = S(h-1) + S(h-2)$.

Inductive Hypothesis

We will assume $P(K)$ and $P(K-1)$, for an arbitrary $K > 1$.

$P(K): S(K) = F(K+3) - 1$

$P(K-1): S(K-1) = F(K+2) - 1$

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* Note, we could also assume that $P(j)$ is true for $1 < j \leq K$, but that is a different type of induction *

Inductive Step:

want to show $P(K+1)$, given $P(K)$ and $P(K-1)$.

$$\boxed{P(K+1) = S(K+1) = F(K+4) - 1.}$$

GOAL

Inductive Step, continued

$S(K+1) = S(K) + S(K-1) + 1$ def of $S(K)$

$S(K+1) = \underline{F(K+3)} - 1 + \underline{F(K+2) - 1} + 1$ by I.H

$= F(K+3) + F(K+2) - 1$ simplify

$= F(K+4) - 1$ def of Fib.

$S(K+1) = F(K+4) - 1$ ✓

$P(K) \wedge P(K-1) \rightarrow P(K+1)$

Conclusion;

we have shown by induction that

$\cancel{S(K+1) = F}$

$S(n) = F(n+3) - 1.$