CSE373: Data Structures & Algorithms

Lecture 9: Priority Queues and Binary Heaps

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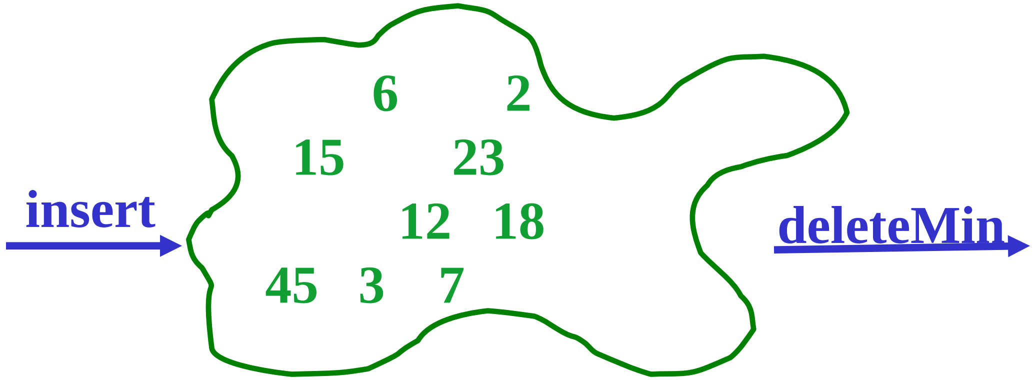
A new ADT: Priority Queue

- A **priority queue** holds *compare-able data*
  - Like dictionaries, we need to *compare items*
    - Given *x* and *y*, is *x* less than, equal to, or greater than *y*
    - Meaning of the ordering can depend on your data
  - Integers are comparable, so will use them in examples
    - But the priority queue ADT is much more general
    - Typically two fields, the *priority* and the *data*

Priorities

- Each item has a "priority"
  - In our examples, the *lesser* item is the one with the *greater* priority
  - So "priority 1" is more important than "priority 4"
  - (Just a convention, think "first is best")

- Operations:
  - **`insert`**
  - **`deleteMin`**
  - **`is_empty`**



- Key property: **`deleteMin`** *returns* and *deletes* the item with greatest priority (lowest priority value)
  - Can resolve ties arbitrarily

Example

`insert` *x1* with priority *5*	*(x1,5)*
`insert` *x2* with priority *3*	*(x1,5) (x2,3)*
`insert` *x3* with priority *4*	*(x1,5) (x3,4) (x2,3)*
a = **`deleteMin // x2`**	*(x1,5) (x3,4)*
b = **`deleteMin // x3`**	*(x1,5)*
`insert` *x4* with priority *2*	
`insert` *x5* with priority *6*	
c = **`deleteMin // x4`**	
d = **`deleteMin // x1`**	

- Analogy: **`insert`** is like **`enqueue`**, **`deleteMin`** is like **`dequeue`**
  - But the whole point is to use priorities instead of FIFO

Applications

Like all good ADTs, the priority queue arises often

- Sometimes blatant, sometimes less obvious

- Run multiple programs in the operating system
  - "critical" before "interactive" before "compute-intensive"
  - Maybe let users set priority level
- Treat hospital patients in order of severity (or triage)
- Select print jobs in order of decreasing length?
- Forward network packets in order of urgency
- Select most frequent symbols for data compression
- Sort (first **`insert`** all, then repeatedly **`deleteMin`**)

Finding a good data structure

- Will show an efficient, non-obvious data structure for this ADT
  - But first let's analyze some "obvious" ideas for *n* data items
  - All times worst-case; assume arrays "have room"

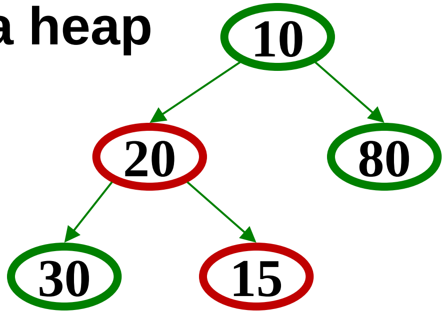
data	*insert algorithm / time*		*deleteMin algorithm / time*	
unsorted array	add at end	$O(1)$	search	$O(n)$
unsorted linked list	add at front	$O(1)$	search	$O(n)$
sorted circular array	search / shift	$O(n)$	move front	$O(1)$
sorted linked list	put in right place	$O(n)$	remove at front	$O(1)$
binary search tree	put in right place	$O(n)$	leftmost	$O(n)$
AVL tree	put in right place	$O(\mathbf{log}\ n)$	leftmost	$O(\mathbf{log}\ n)$

Our data structure: the Binary Heap

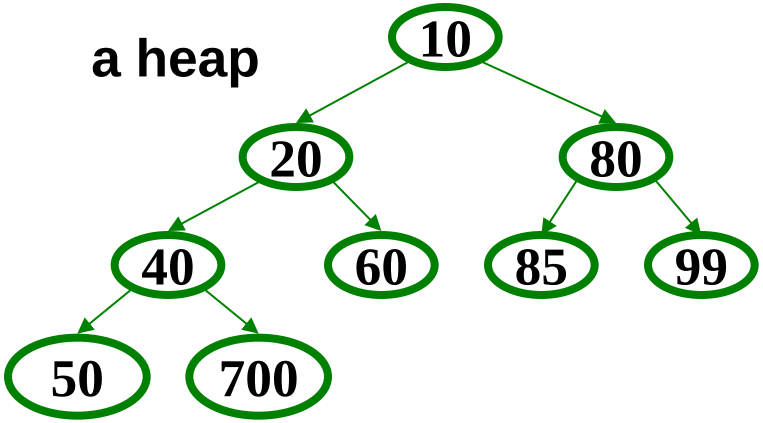
A *binary min-heap* (or just *binary heap* or just *heap*) has:

- Structure property: A *complete* binary tree
- Heap property: The priority of every (non-root) node is less than the priority of its parent
  - ***Not** a binary search tree*

not a heap



a heap

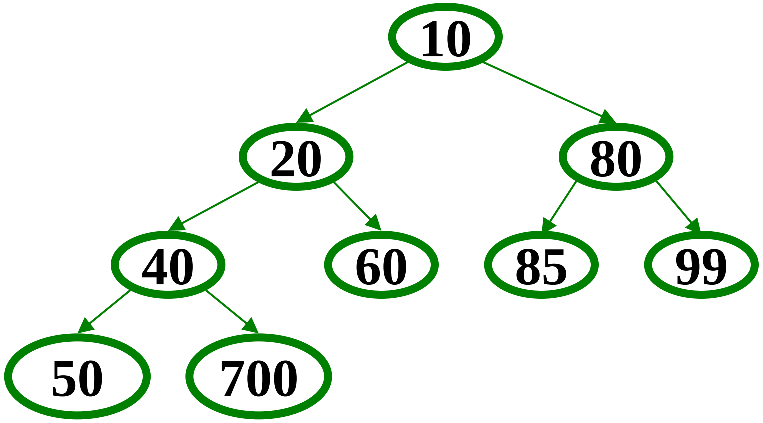


So:

- Where is the most important item?
- What is the height of a heap with *n* items?

Operations: basic idea

- **`deleteMin`**:
  1. Remove root node
  2. Move right-most node in last row to root to restore structure property
  3. "Percolate down" to restore heap property
- **`insert:`**
  1. Put new node in next position on bottom row to restore structure property
  2. "Percolate up" to restore heap property

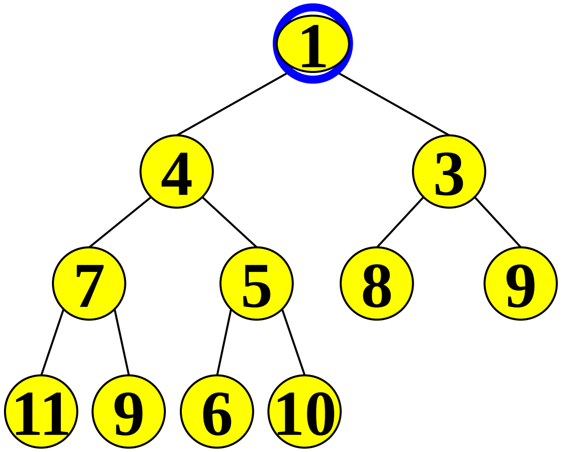


Overall strategy:

- *Preserve structure property*
- *Break and restore heap property*

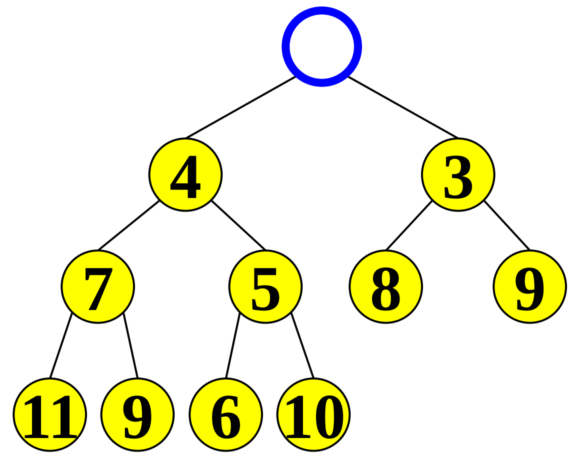
DeleteMin

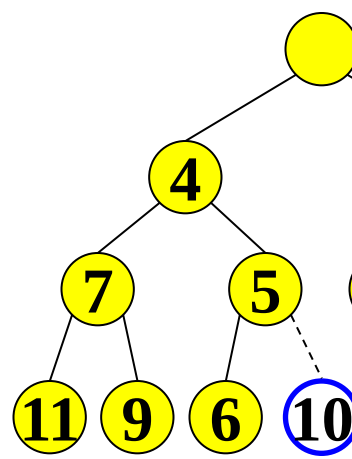
Delete (and later return) value at root node



DeleteMin: Keep the Structure Property

- We now have a "hole" at the root
  - Need to fill the hole with another value
- Keep structure property: When we are done, the tree will have one less node and must still be complete
- Pick the last node on the bottom row of the tree and move it to the "hole"

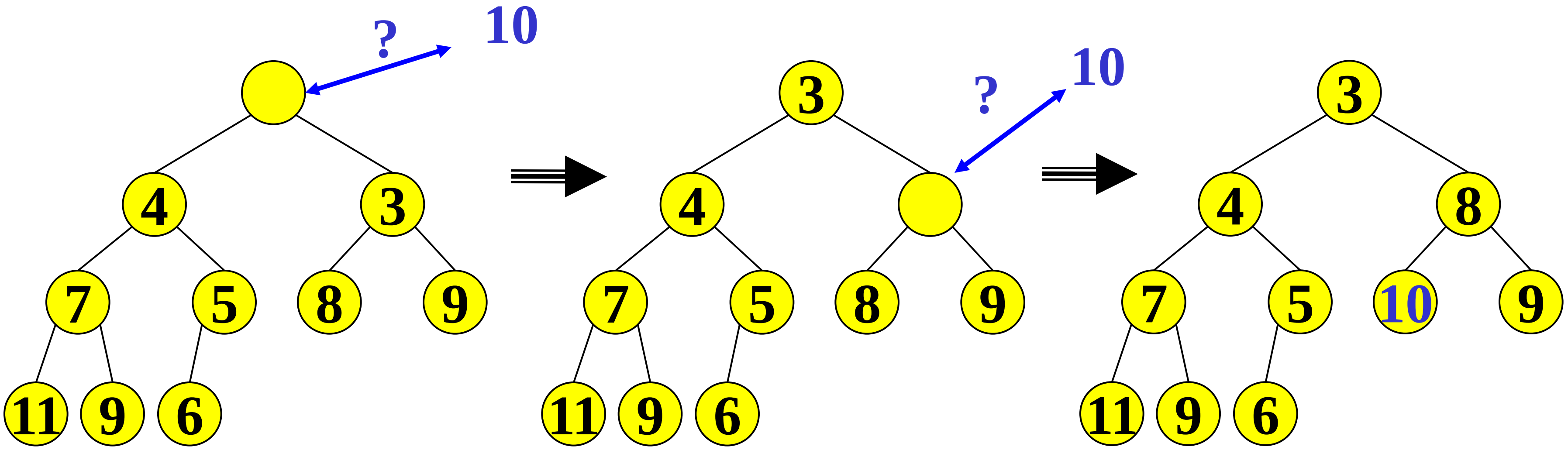




DeleteMin: Restore the Heap Property

Percolate down:

- Keep comparing priority of item with both children
- If priority is less important, swap with the most important child and go down one level
- Done if both children are less important than the item or we've reached a leaf node



Run time?

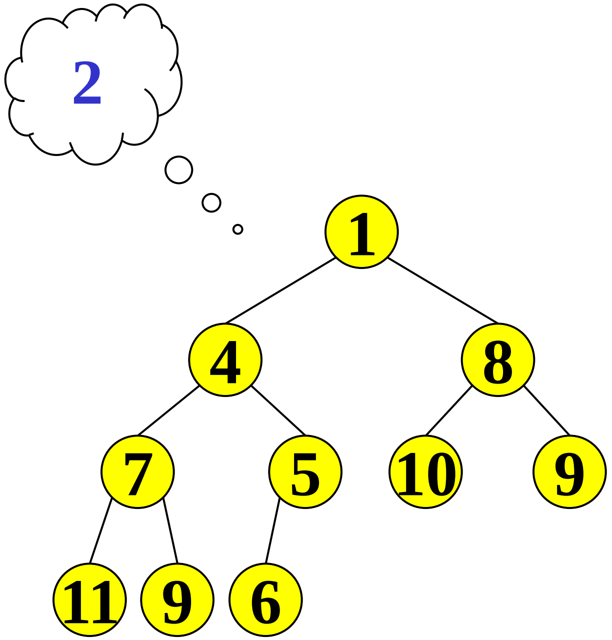
Runtime is O(height of heap) $O(\log n)$

Height of a complete binary tree of n nodes = $\lfloor \log_2(n) \rfloor$

Insert

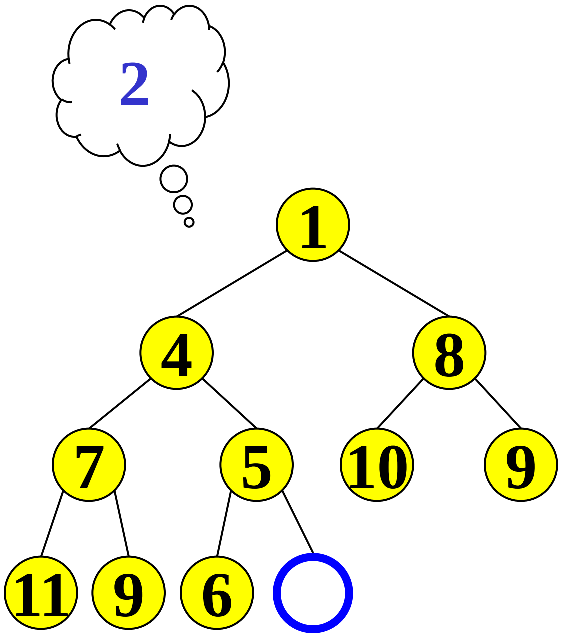
- Add a value to the tree

- Afterwards, structure and heap properties must still be correct



Insert: Maintain the Structure Property

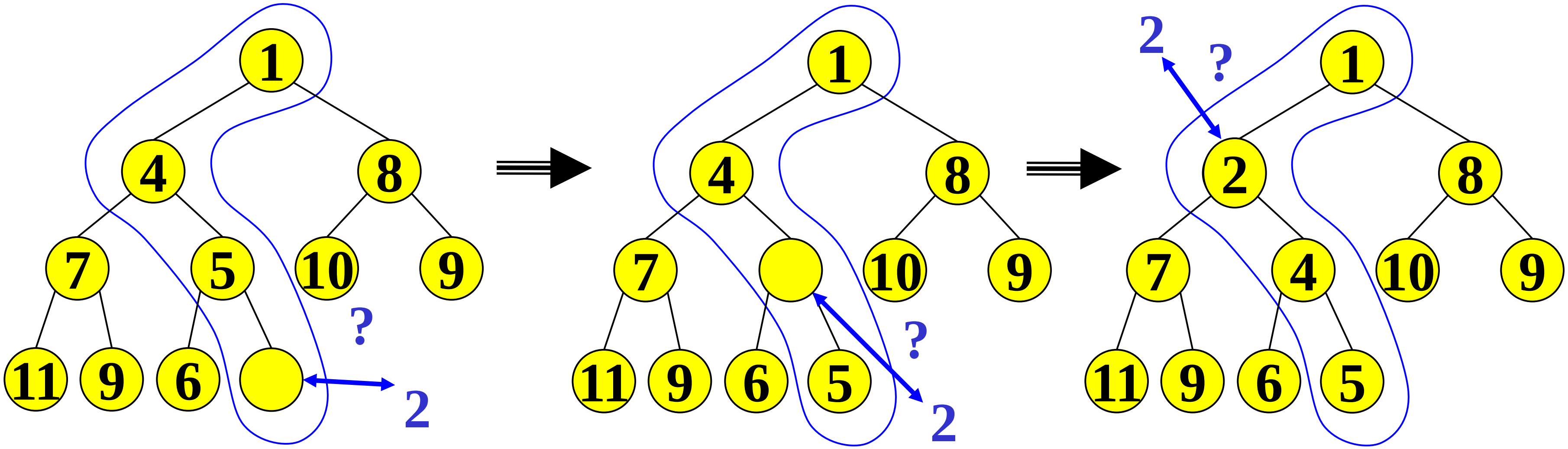
- There is only one valid tree shape after we add one more node
- So put our new data there and then focus on restoring the heap property



Insert: Restore the heap property

Percolate up:

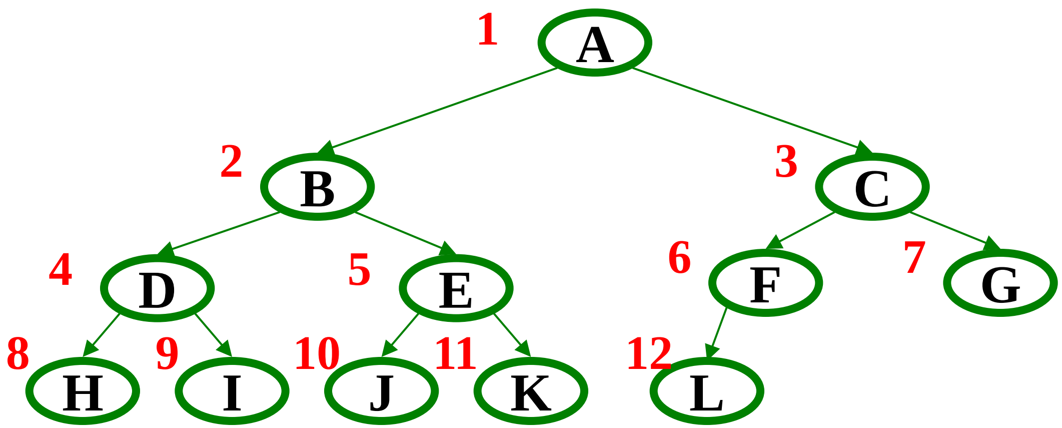
- Put new data in new location
- If parent is less important, swap with parent, and continue
- Done if parent is more important than item or reached root



What is the running time?

Like `deleteMin`, worst-case time proportional to tree height: *O*(log *n*)

Array Representation of Binary Trees



From node **i**:

left child: **i*2**
right child: **i*2+1**
parent: **i/2**

(wasting index 0 is convenient for the index arithmetic)

implicit (array) implementation:

	A	B	C	D	E	F	G	H	I	J	K	L	
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Judging the array implementation

Plusses:

- Non-data space: just index 0 and unused space on right
  - In conventional tree representation, one edge per node (except for root), so *n*-1 wasted space (like linked lists)
  - Array would waste more space if tree were not complete
- Multiplying and dividing by 2 is very fast (shift operations in hardware)
- Last used position is just index **`size`**

Minuses:

- Same might-be-empty or might-get-full problems we saw with stacks and queues (resize by doubling as necessary)

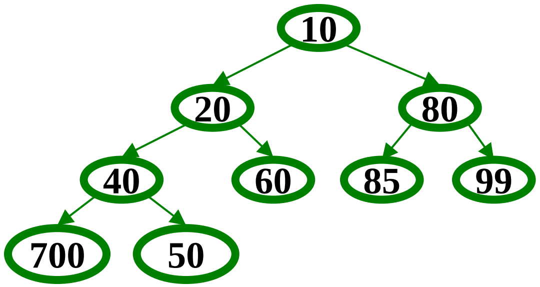
Plusses outweigh minuses: “this is how people do it”

This pseudocode uses ints. In real use, you will have data nodes with priorities.

Pseudocode: insert into binary heap

```
void insert(int val) {
  if(size==arr.length-1)
    resize();
  size++;
  i=percolateUp(size,val);
  arr[i] = val;
}
```

```
int percolateUp(int hole,
                int val) {
  while(hole > 1 &&
        val < arr[hole/2])
    arr[hole] = arr[hole/2];
    hole = hole / 2;
  }
  return hole;
}
```



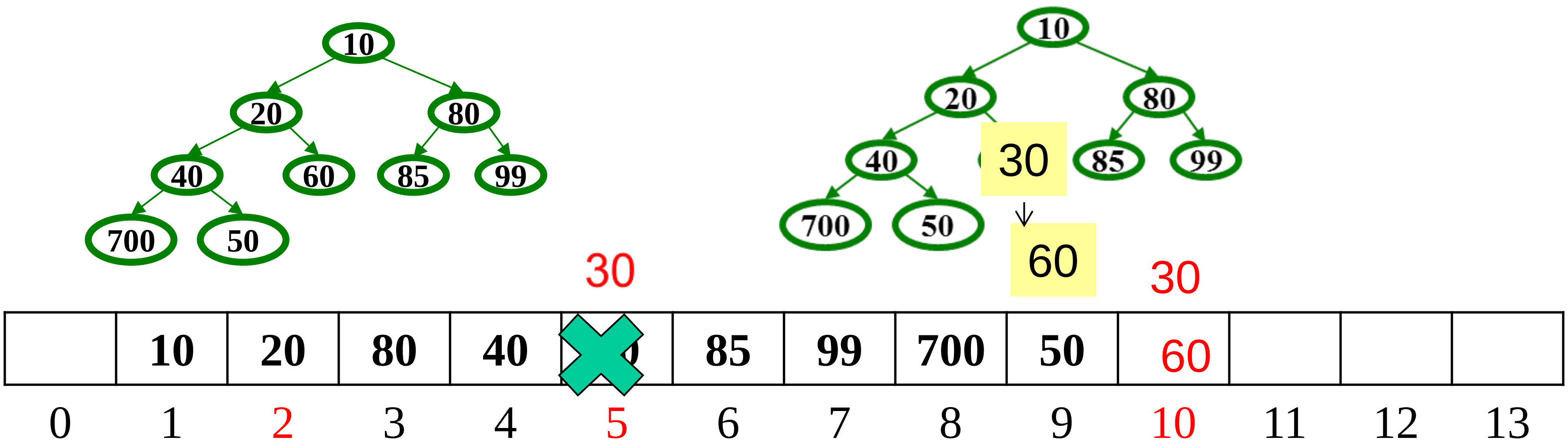
	10	20	80	40	60	85	99	700	50				
0	1	2	3	4	5	6	7	8	9	10	11	12	13

This pseudocode uses ints. In real use, you will have data nodes with priorities.

insert(30) *percolateUp(10, 30)*

```
void insert(int val) {
  if(size==arr.length-1)
    resize();
  size++;
  i=percolateUp(size,val);
  arr[i] = val;
}
```

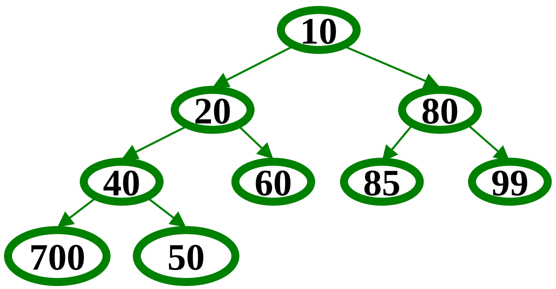
```
int percolateUp(int hole,
                int val) {
  while(hole > 1 &&
        val < arr[hole/2])
    arr[hole] = arr[hole/2];
    hole = hole / 2;
  }
  return hole;
}
```



Pseudocode: deleteMin from binary heap

```
int deleteMin() {
  if(isEmpty()) throw…
  ans = arr[1];
  hole = percolateDown
           (1,arr[size]);
  arr[hole] = arr[size];
  size--;
  return ans;
}
```

```
int percolateDown(int hole,
                  int val) {
 while(2*hole <= size) {
  left  = 2*hole;
  right = left + 1;
  if(right > size ||
     arr[left] < arr[right])
    target = left;
  else
    target = right;
  if(arr[target] < val) {
    arr[hole] = arr[target];
    hole = target;
  } else
      break;
 }
 return hole;
}
```



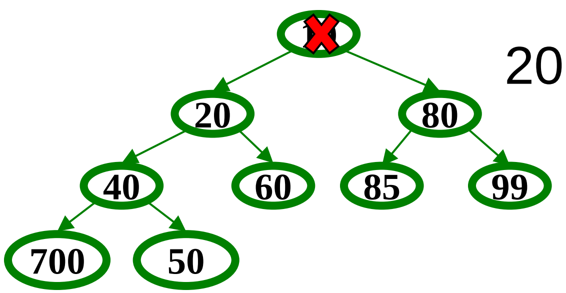
	10	20	80	40	60	85	99	700	50				
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Pseudocode: deleteMin from binary heap

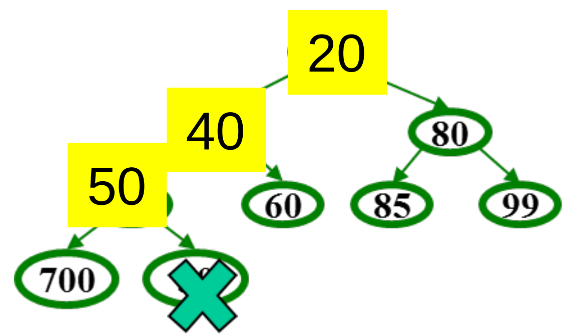
20 40 50

	~~10~~	**20**	**80**	**40**	**60**	**85**	**99**	**700**	**50**				
0	1	2	3	4	5	6	7	8	9	10	11	12	13

50?



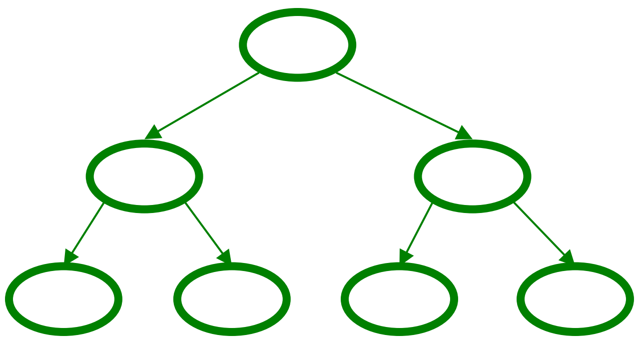
20 is smaller than 80



Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

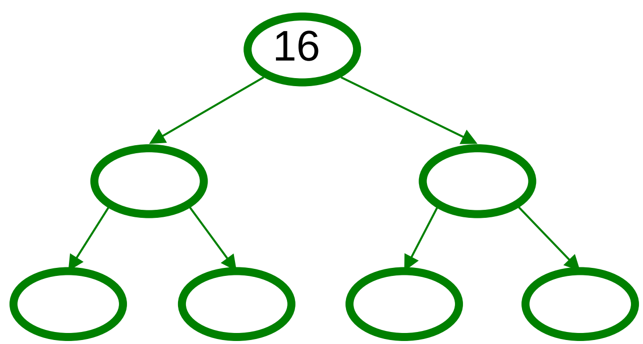
0	1	2	3	4	5	6	7



Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

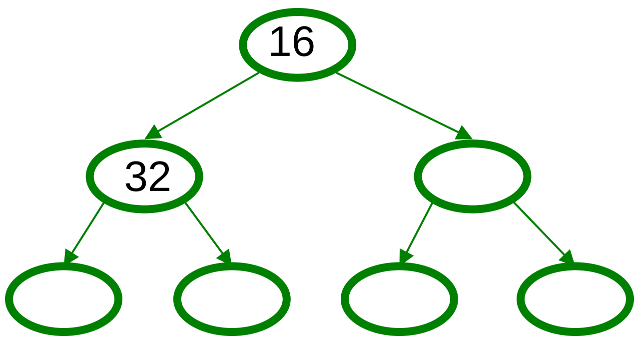
	16						
0	1	2	3	4	5	6	7



Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

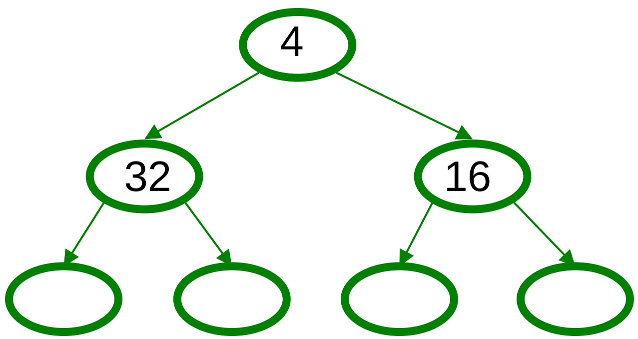
	16	**32**					
0	1	2	3	4	5	6	7



Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

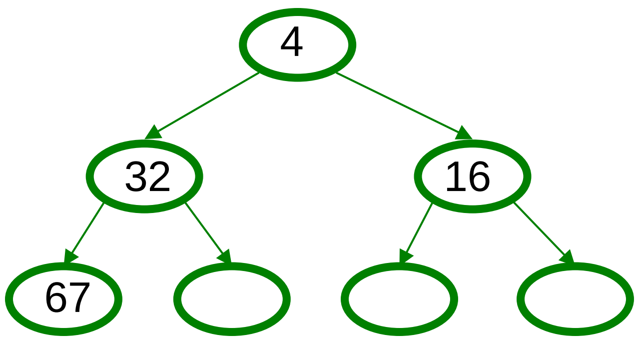
	4	**32**	**16**				
0	1	2	3	4	5	6	7



Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

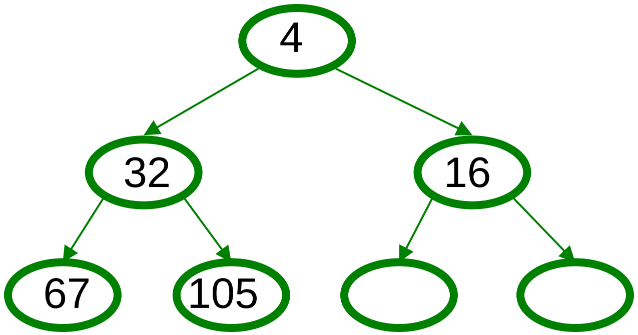
	4	**32**	**16**	**67**			
0	1	2	3	4	5	6	7



Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

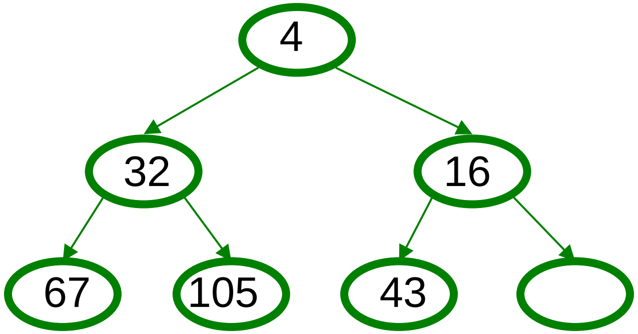
	4	**32**	**16**	**67**	**105**		
0	1	2	3	4	5	6	7



Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

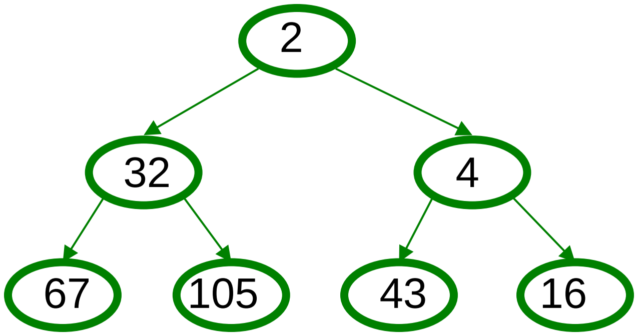
	4	**32**	**16**	**67**	**105**	**43**	
0	1	2	3	4	5	6	7



Exercise

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

	2	**32**	**4**	**67**	**105**	**43**	**16**
0	1	2	3	4	5	6	7



Other operations

- **`decreaseKey`**: given pointer to object in priority queue (e.g., its array index), lower its priority value by *p*
  - Change priority and percolate up

- **`increaseKey`**: given pointer to object in priority queue (e.g., its array index), raise its priority value by *p*
  - Change priority and percolate down

- **`remove`**: given pointer to object in priority queue (e.g., its array index), remove it from the queue
  - **`decreaseKey`** with $p = \infty$, then **`deleteMin`**

Running time for all these operations?

Build Heap

- Suppose you have *n* items to put in a new (empty) priority queue
  - Call this operation **`buildHeap`**

- *n* **`insert`**s works
  - Only choice if ADT doesn't provide **`buildHeap`** explicitly
  - *O*(*n* **`log`** *n*)

- Why would an ADT provide this unnecessary operation?
  - Convenience
  - Efficiency: an *O*(*n*) algorithm called Floyd's Method
  - Common issue in ADT design: how many specialized operations

Floyd's Method

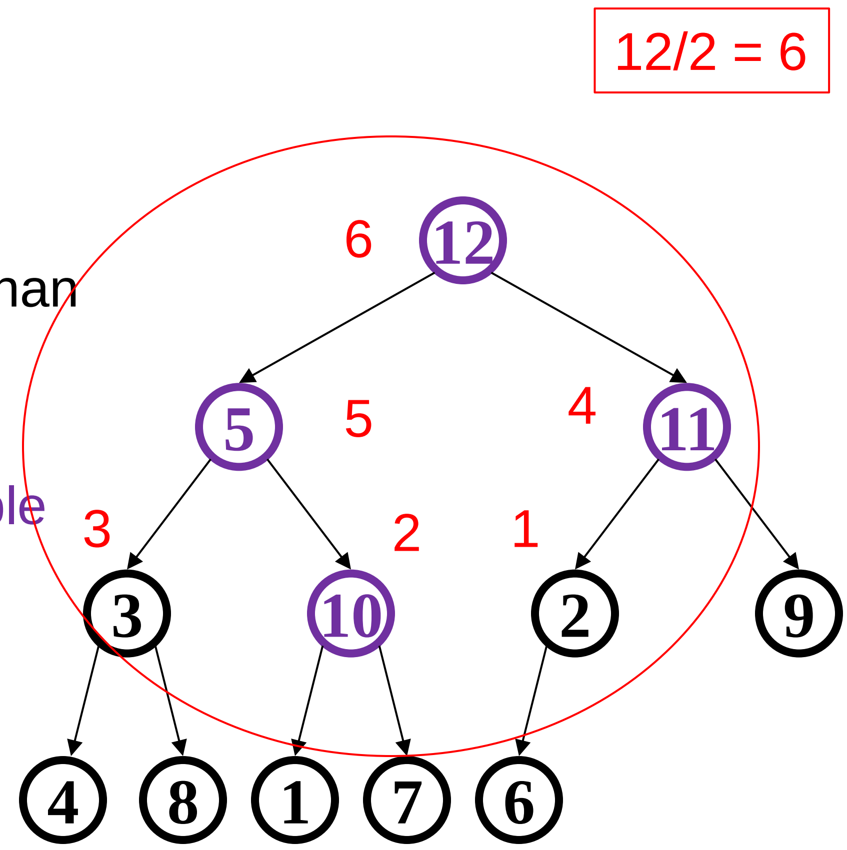
1. Use *n* items to make any complete tree you want
   - That is, put them in array indices 1,…,*n*

2. Treat it as a heap and fix the heap-order property
   - Bottom-up: leaves are already in heap order, work up toward the root one level at a time

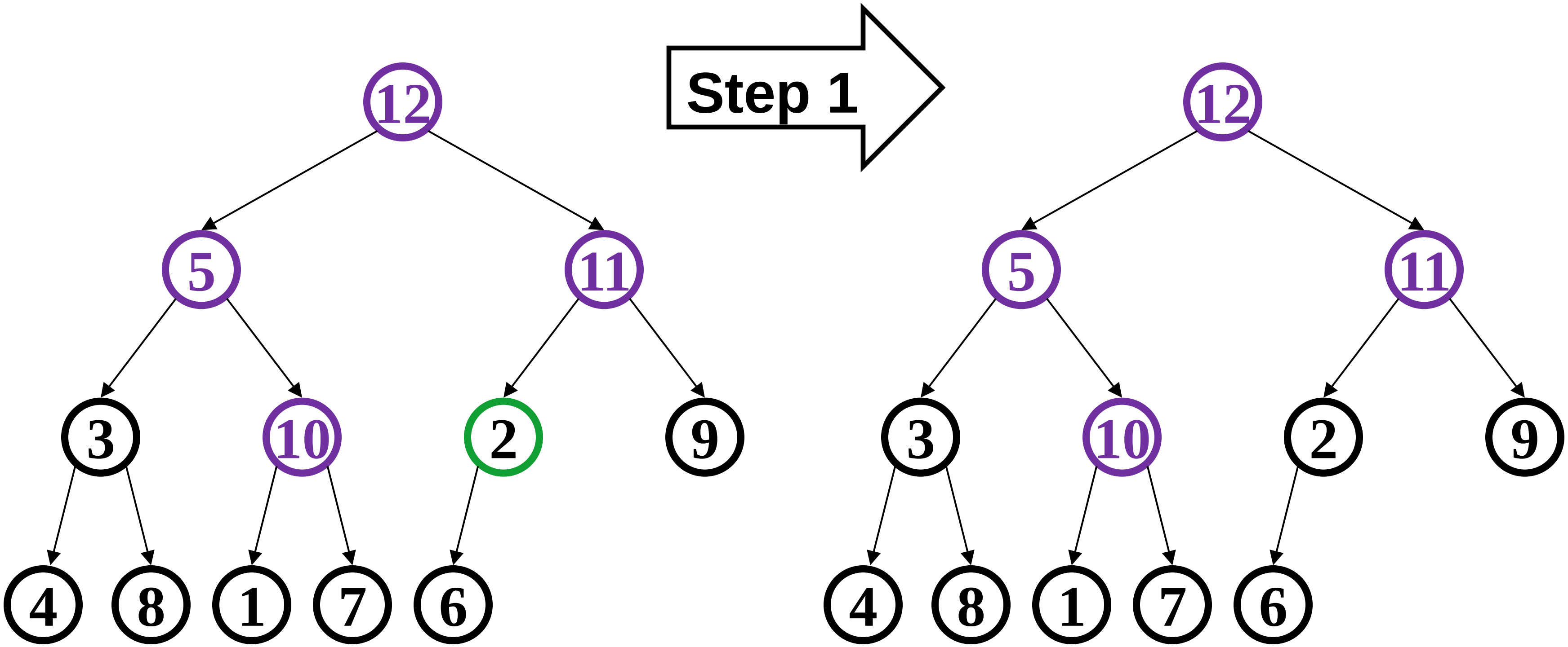
```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Example

- In tree form for readability
  - Purple for node not less than descendants
    - heap-order problem
  - Notice no leaves are purple
  - Check/fix each non-leaf bottom-up (6 steps here)

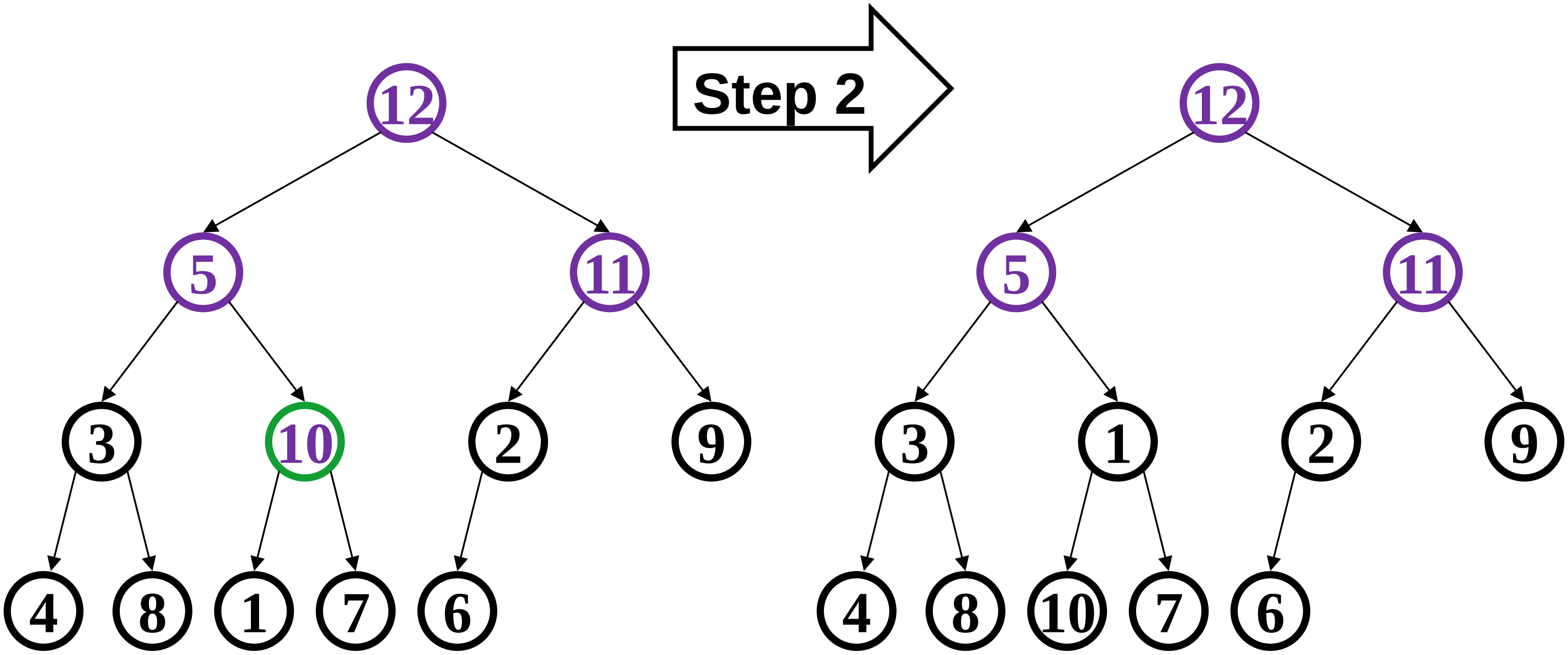


Example



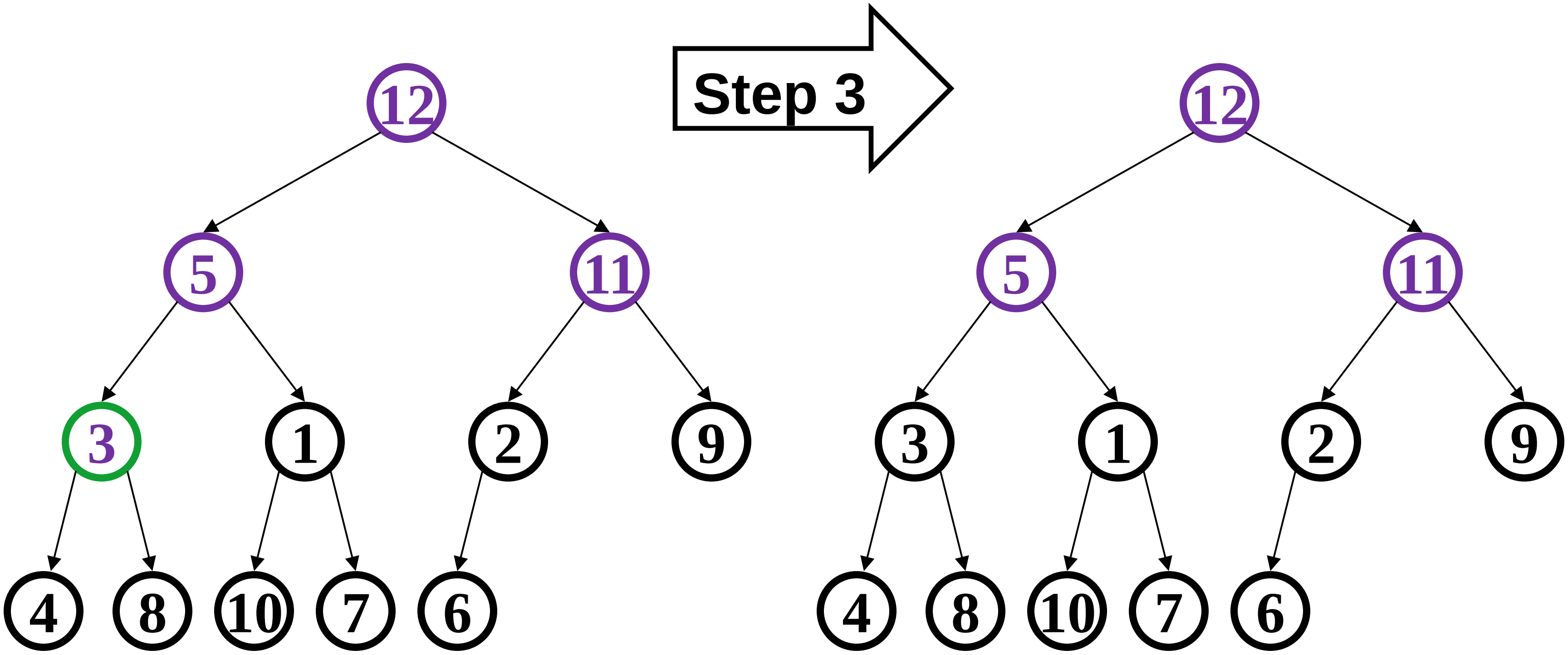
- Happens to already be less than children (er, child)

Example



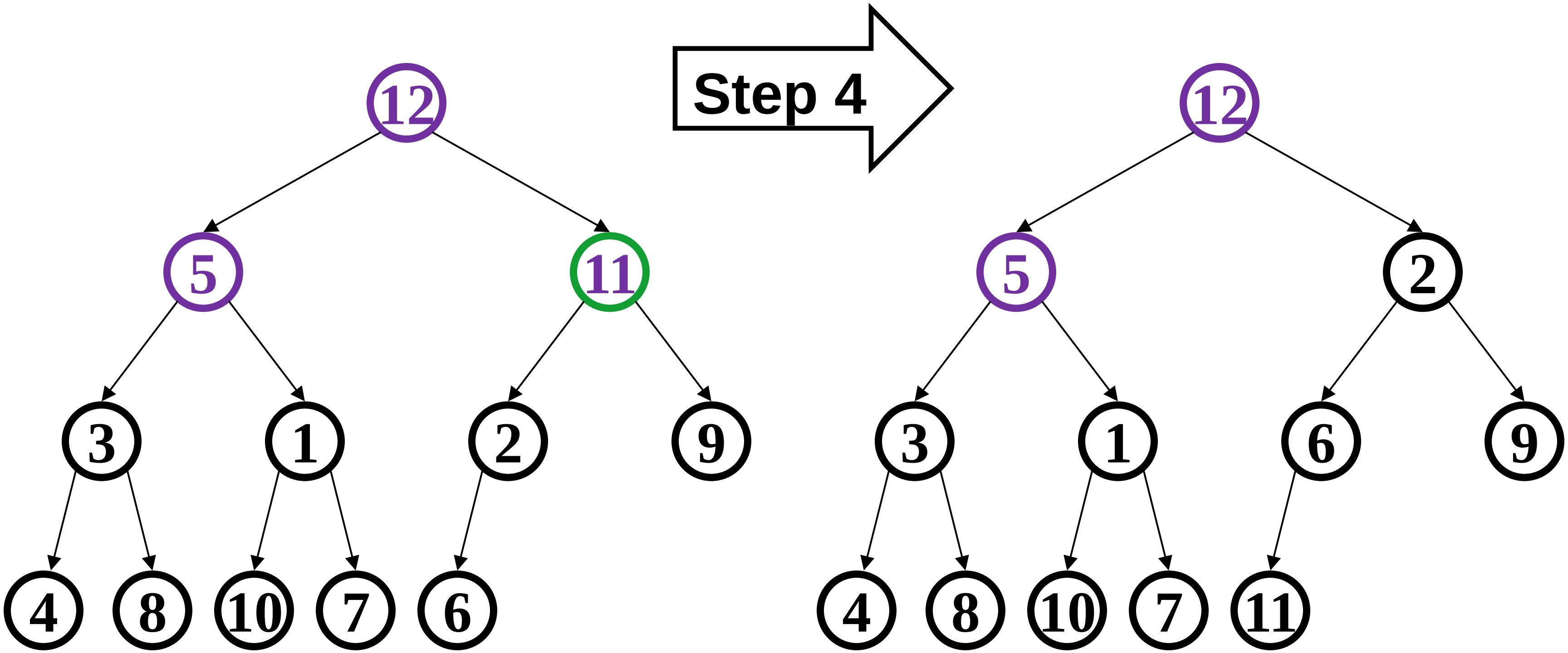
- Percolate down (notice that moves 1 up)

Example



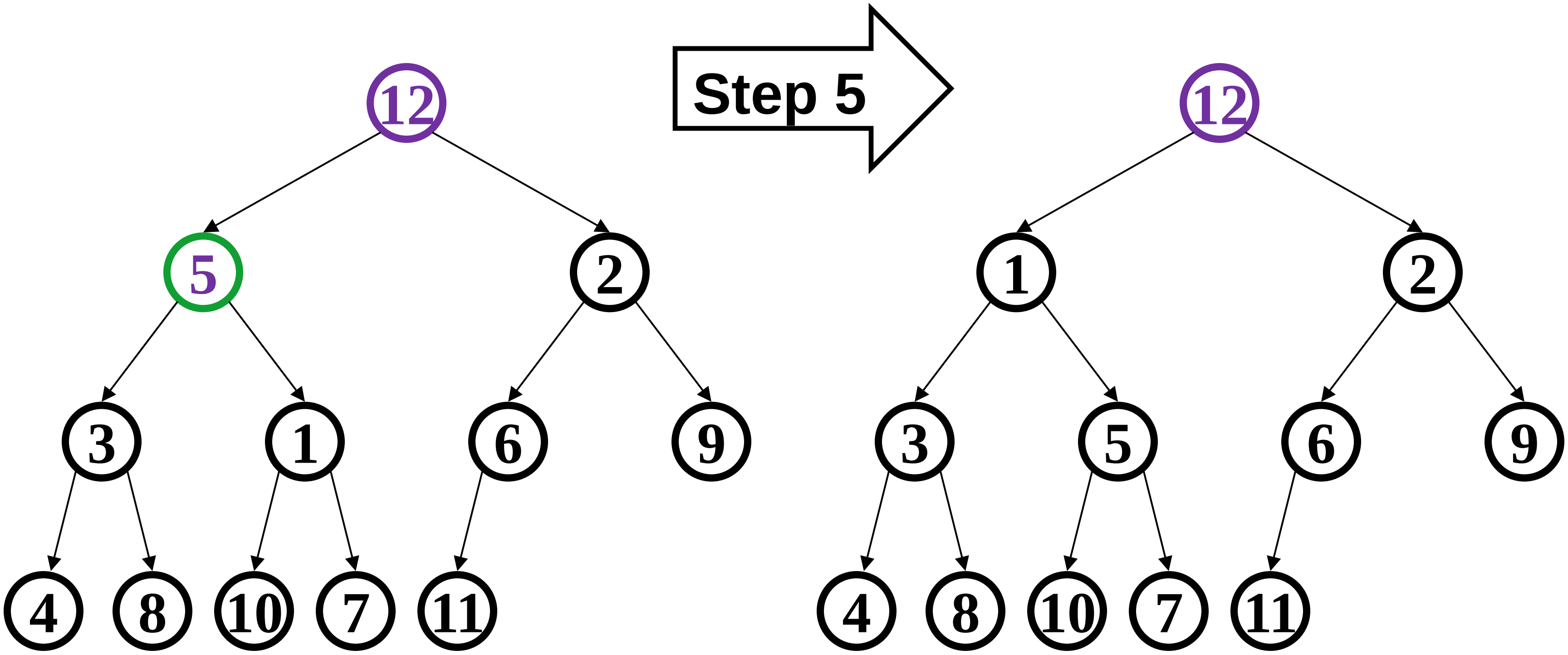
- Another nothing-to-do step

Example

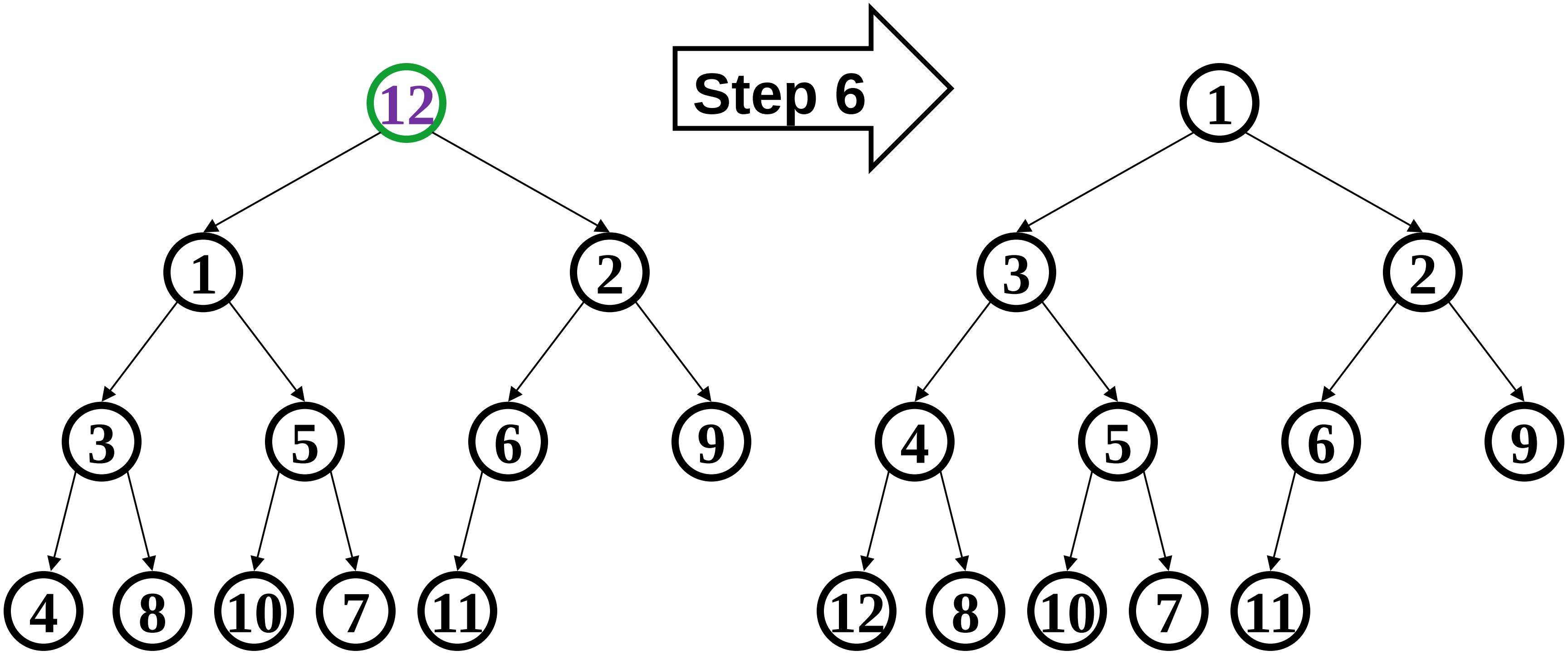


- Percolate down as necessary (steps 4a and 4b)

Example



Example



But is it right?

- "Seems to work"
  - Let's *prove* it restores the heap property (correctness)
  - Then let's *prove* its running time (efficiency)

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Correctness

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Loop Invariant: For all **`j`**>**`i`**, **`arr[j]`** is less than its children

- True initially: If **`j > size/2`**, then **`j`** is a leaf
  - Otherwise its left child would be at position > **`size`**
- True after one more iteration: loop body and **`percolateDown`** make **`arr[i]`** less than children without breaking the property for any descendants

So after the loop finishes, all nodes are less than their children

Efficiency

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Easy argument: **`buildHeap`** is $O(n$ **`log`** $n)$ where n is **`size`**

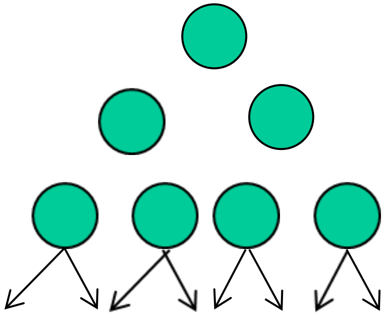
- **`size/2`** loop iterations
- Each iteration does one **`percolateDown`**, each is $O($**`log`** $n)$

This is correct, but there is a more precise ("tighter") analysis of the algorithm…

Efficiency

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Better argument: **`buildHeap`** is *O*(*n*) where *n* is **`size`**



- **`size/2`** total loop iterations: *O*(*n*)
- 1/2 the loop iterations percolate at most 1 step
- 1/4 the loop iterations percolate at most 2 steps
- 1/8 the loop iterations percolate at most 3 steps
- …
- ((1/2) + (2/4) + (3/8) + (4/16) + (5/32) + …) < 2 (page 4 of Weiss)
  – So at most **`2*(size/2)`** *total* percolate steps: *O*(*n*)

Lessons from `buildHeap`

- Without **`buildHeap`**, our ADT already let clients implement their own in $O(n \log n)$ worst case
- By providing a specialized operation internal to the data structure (with access to the internal data), we can do $O(n)$ worst case
  - Intuition: Most data is near a leaf, so better to percolate down
- Can analyze this algorithm for:
  - Correctness:
    - Non-trivial inductive proof using loop invariant
  - Efficiency:
    - First analysis easily proved it was $O(n \log n)$
    - Tighter analysis shows same algorithm is $O(n)$

Exercise: Build the Heap using Floyd's method

3	9	2	6	25	1	80	35
1	2	3	4	5	6	7	8