CSE373: Data Structures & Algorithms

Lecture 20: Minimum Spanning Trees

Linda Shapiro

Spring 2016

Announcements

- HW 4 due Wednesday, May 18
- HW 5 will be due June 1. Ben Jones is our expert in Ezgi's absence. benjones@cs

Minimum Spanning Trees

The minimum-spanning-tree problem

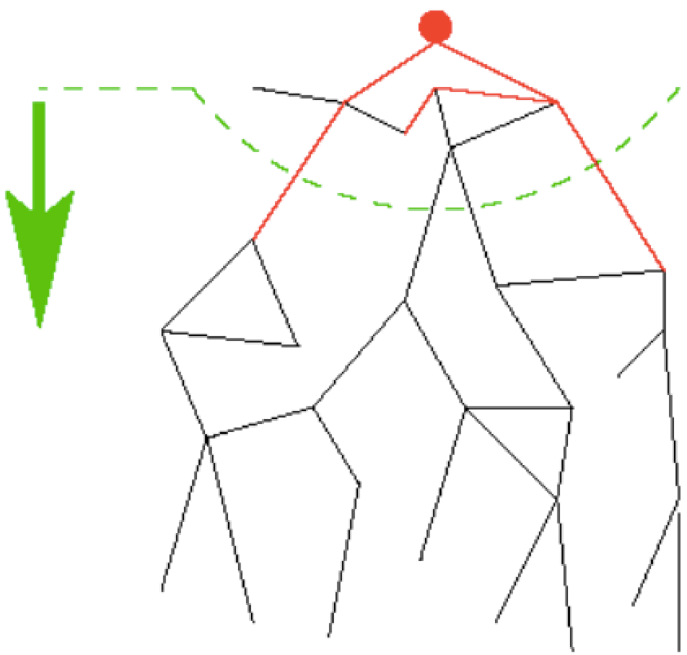
– Given a weighted undirected graph, compute a spanning tree of minimum weight

Given an undirected graph G=(V,E), find a graph G'=(V, E') such that:

– E' is a subset of E
– |E'| = |V| - 1
– G' is connected

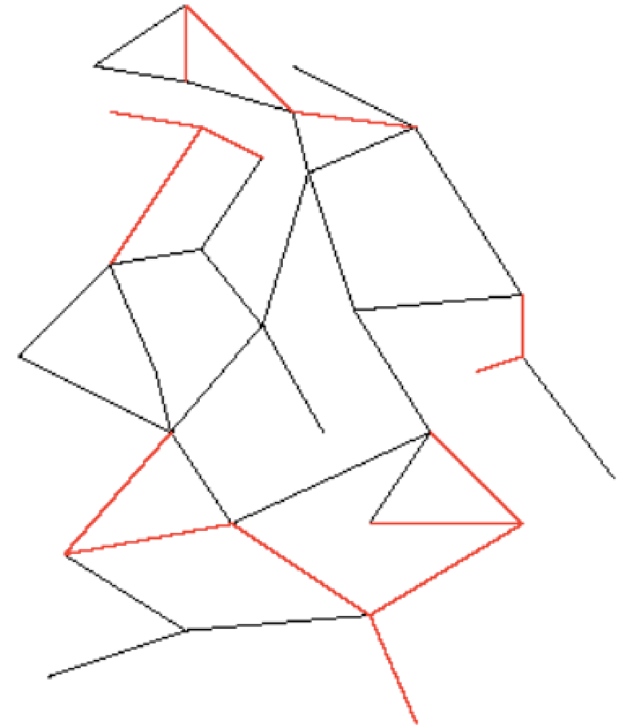
G' is a minimum spanning tree.

Two different approaches



Prim's Algorithm

Almost identical to Dijkstra's



Kruskals's Algorithm

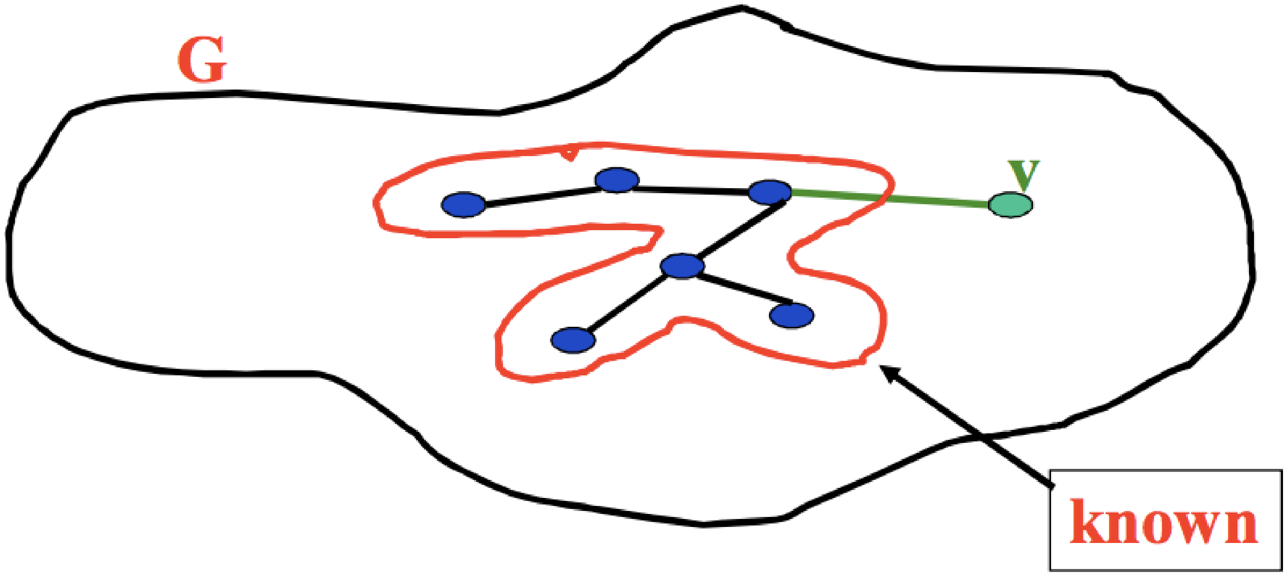
Completely different!

Prim's Algorithm Idea

Idea: Grow a tree by picking a vertex from the unknown set that has the smallest cost. Here cost = cost of the edge that connects that vertex to the known set. *Pick the vertex with the smallest cost that connects "known" to "unknown."*

A *node-based* greedy algorithm

Builds MST by greedily adding nodes



Prim's vs. Dijkstra's

Recall:

Dijkstra picked the unknown vertex with smallest cost where
cost = distance to the source.

Prim's pick the unknown vertex with smallest cost where
cost = distance from this vertex to the known set
(in other words, the cost of the smallest edge connecting this vertex to the known set)

Otherwise identical ☺

Prim's Algorithm

1. For each node **`v`**, set **`v.cost = ∞`** and **`v.known = false`**
2. Choose any node **`v`**
   a) Mark **`v`** as known
   b) For each edge **`(v,u)`** with weight **`w`**, set **`u.cost=w`** and **`u.prev=v`**
3. While there are unknown nodes in the graph
   a) Select the unknown node **`v`** with lowest cost
   b) Mark **`v`** as known and add **`(v, v.prev)`** to output
   c) For each edge **`(v,u)`** with weight **`w`**,

```
if(w < u.cost) {
  u.cost = w;
  u.prev = v;
}
```

Prim's Example

Select the unknown node v with lowest cost
Mark v as known and add (v, v.prev) to output
For each edge (v,u) with weight w,
    if(w < u.cost) {
        u.cost = w;
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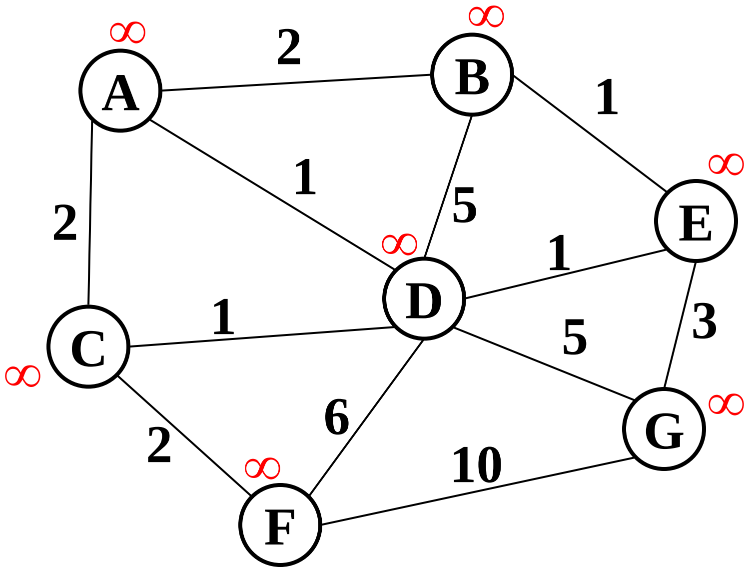
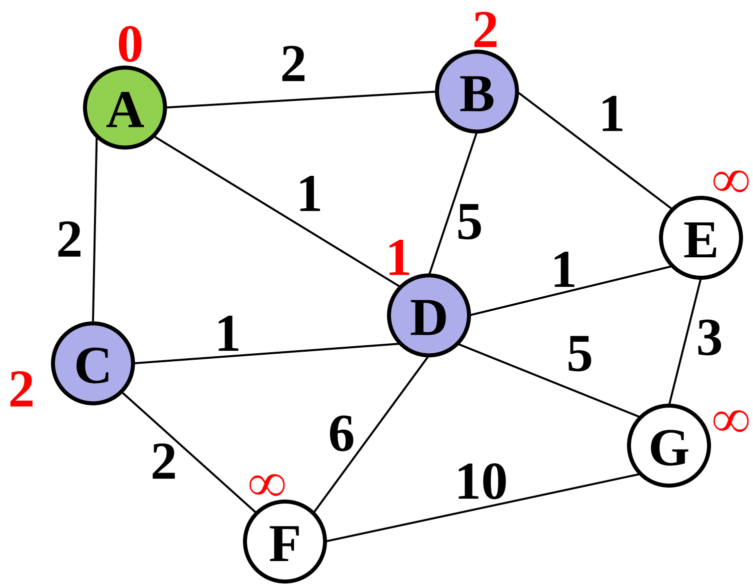


vertex	known?	cost	prev
A		??	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	

Prim's Example



Select the unknown node v with lowest cost
Mark v as known and add (v, v.prev) to output
For each edge (v,u) with weight w,
if(w < u.cost) {
u.cost = w;
u.prev = v;}

vertex	known?	cost	prev
A	Y	0	
B		2	A
C		2	A
D		1	A
E		??	
F		??	
G		??	

Prim's Example

Select the unknown node v with lowest cost
Mark v as known and add (v, v.prev) to output
For each edge (v,u) with weight w,

```
if(w < u.cost) {
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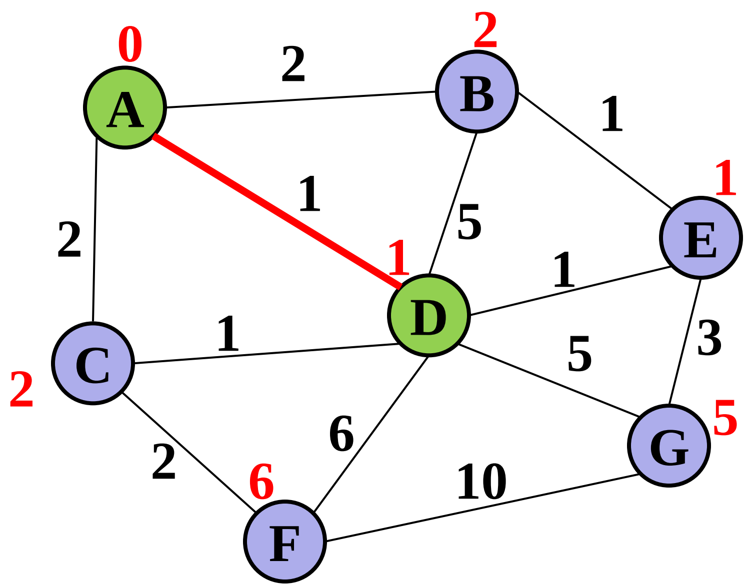

vertex	known?	cost	prev
A	Y	0	
B		2	A
C		1	D
D	Y	1	A
E		1	D
F		6	D
G		5	D

Prim's Example

Select the unknown node v with lowest cost
Mark v as known and add (v, v.prev) to output
For each edge (v,u) with weight w,

```
if(w < u.cost) {
  u.cost = w;
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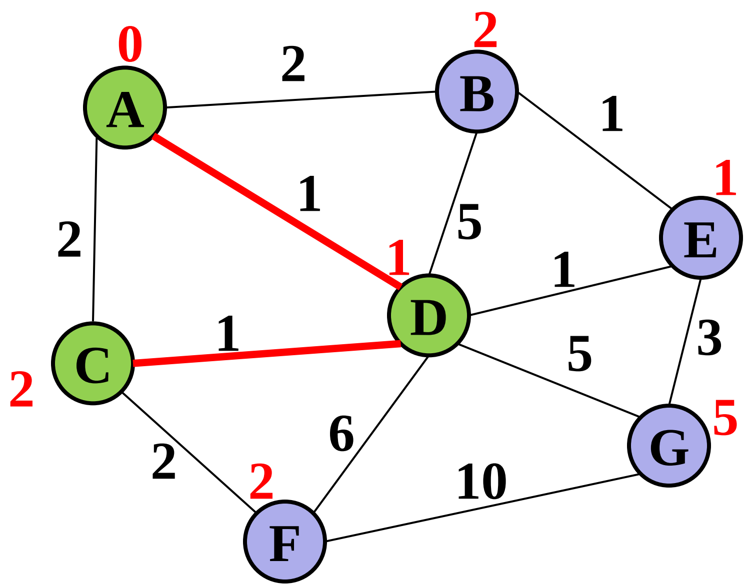

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A	Y	0	
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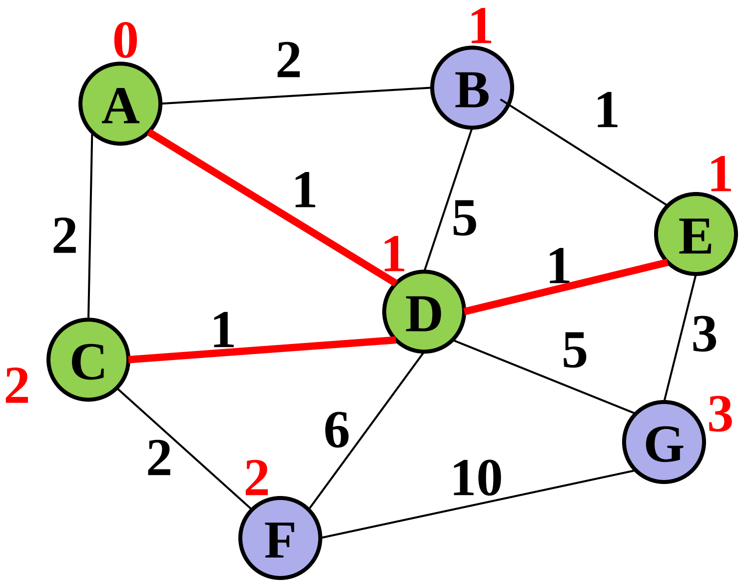
G		5	D

Prim's Example



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vertex	known?	cost	prev
A	Y	0	
B		1	E
C	Y	1	D
D	Y	1	A
E	Y	1	D
F		2	C
G		3	E

Prim's Example

Select the unknown node v with lowest cost
Mark v as known and add (v, v.prev) to output
For each edge (v,u) with weight w,
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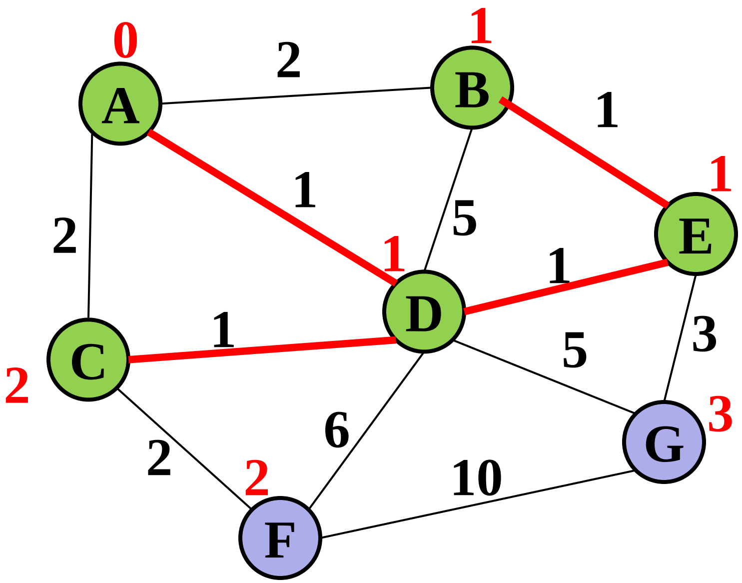

vertex	known?	cost	prev
A	Y	0	
B	Y	1	E
C	Y	1	D
D	Y	1	A
E	Y	1	D
F		2	C
G		3	E

Prim's Example

Select the unknown node v with lowest cost
Mark v as known and add (v, v.prev) to output
For each edge (v,u) with weight w,
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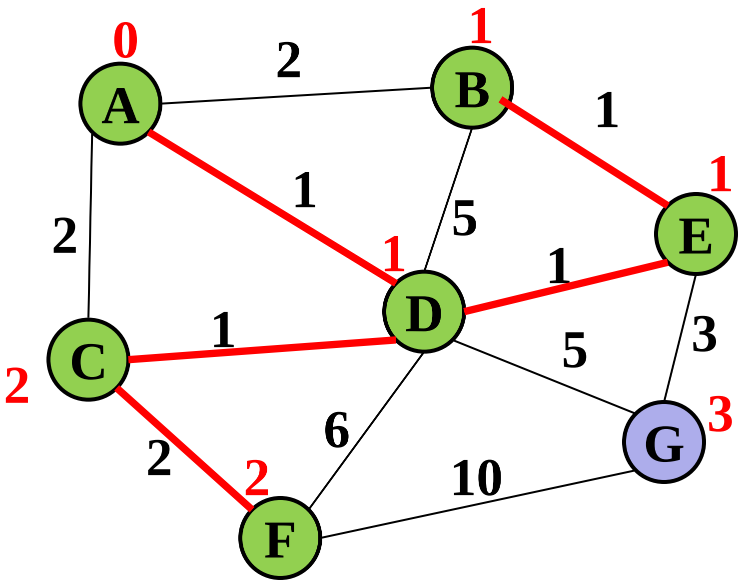

vertex	known?	cost	prev
A	Y	0	
B	Y	1	E
C	Y	1	D
D	Y	1	A
E	Y	1	D
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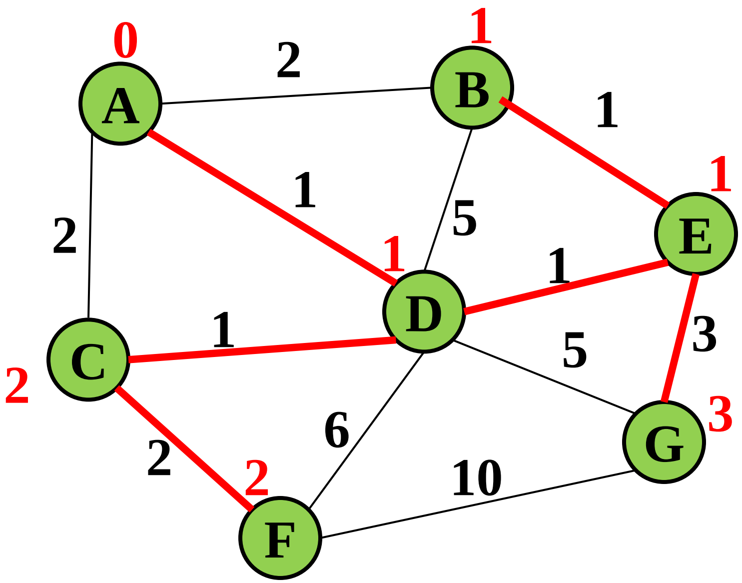
G		3	E

Prim's Example



Select the unknown node v with lowest cost
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D	Y	1	A
E	Y	1	D
F	Y	2	C
G	Y	3	E

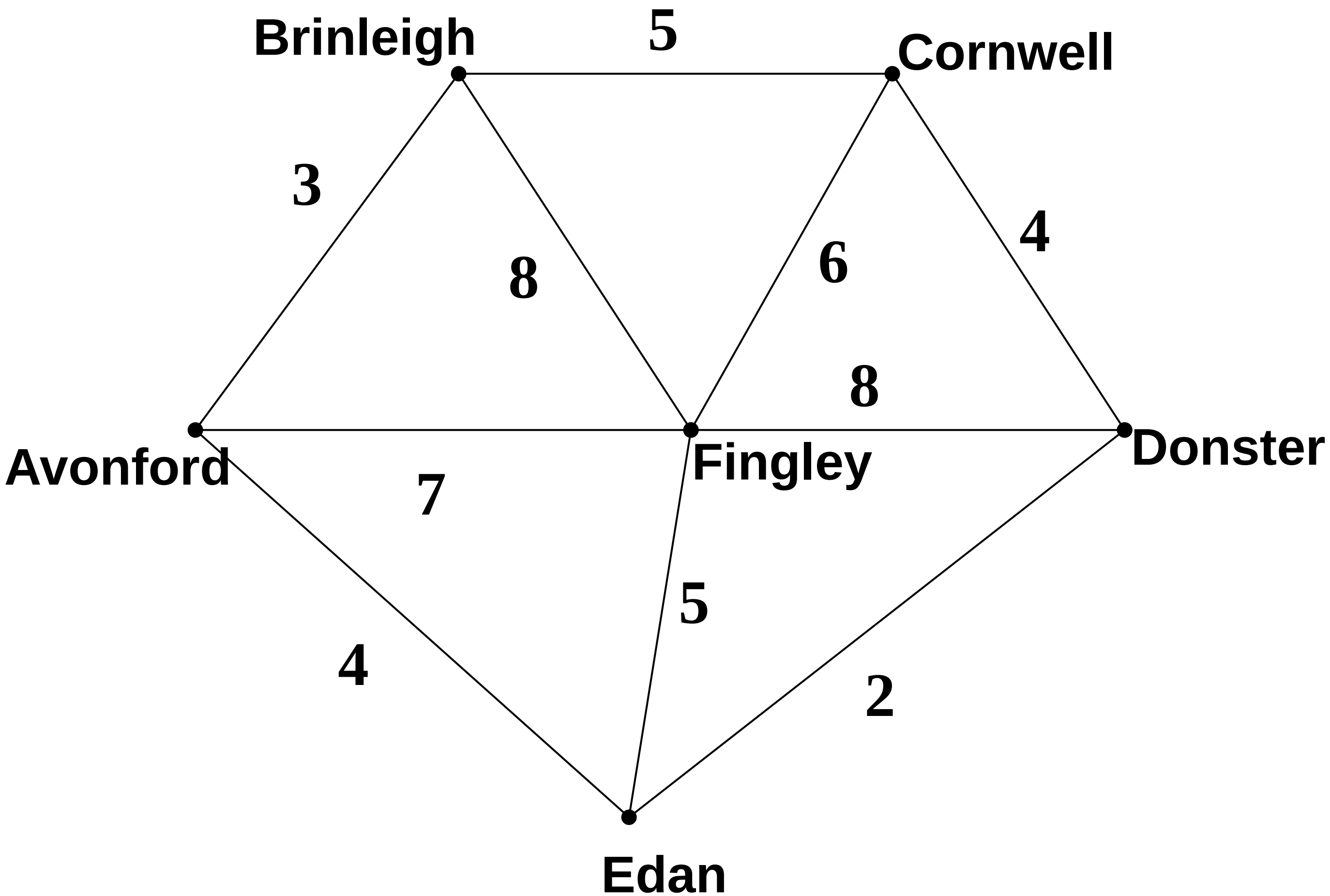
Analysis

- Correctness
  - A bit tricky
  - Intuitively similar to Dijkstra

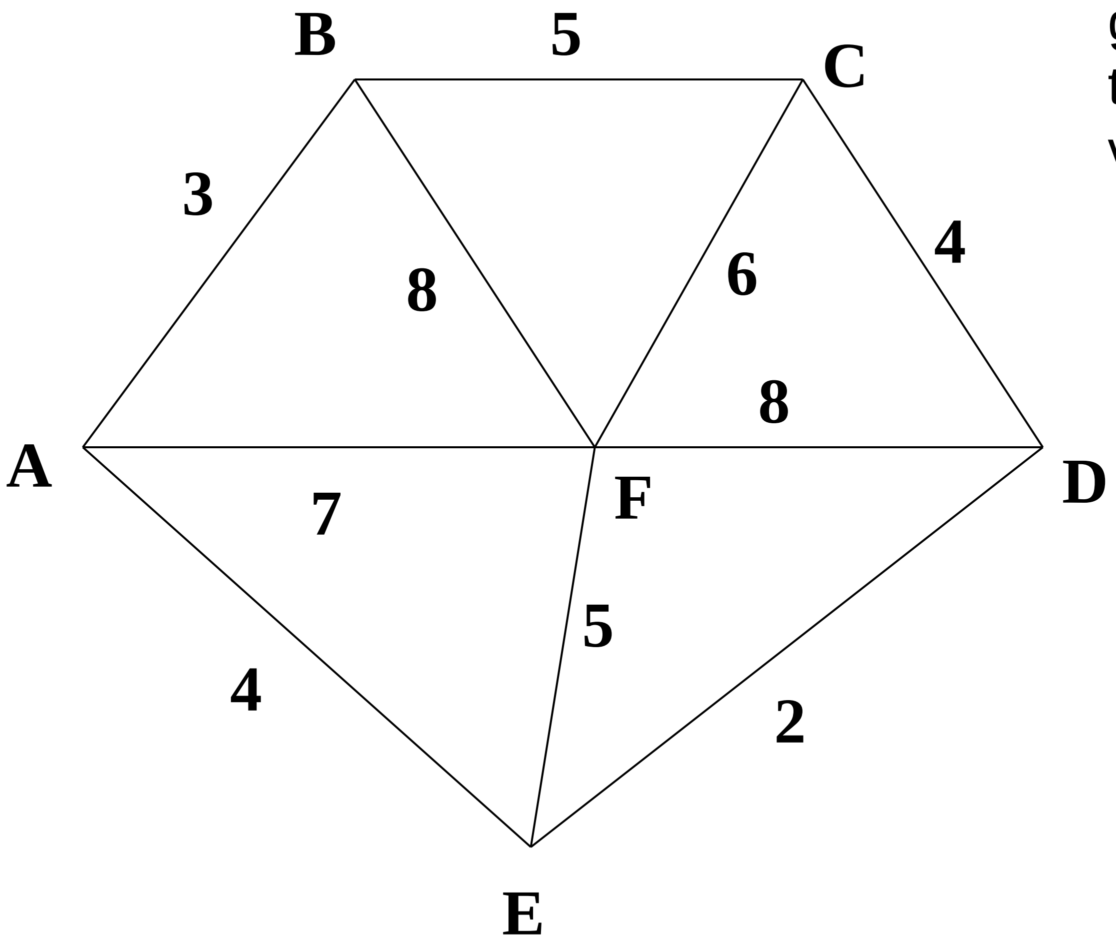
- Run-time
  - Same as Dijkstra
  - $O(|E|\log|V|)$ using a priority queue
    - Costs/priorities are just edge-costs, not path-costs

Another Example

A cable company wants to connect five villages to their network which currently extends to the town of Avonford. What is the minimum length of cable needed?

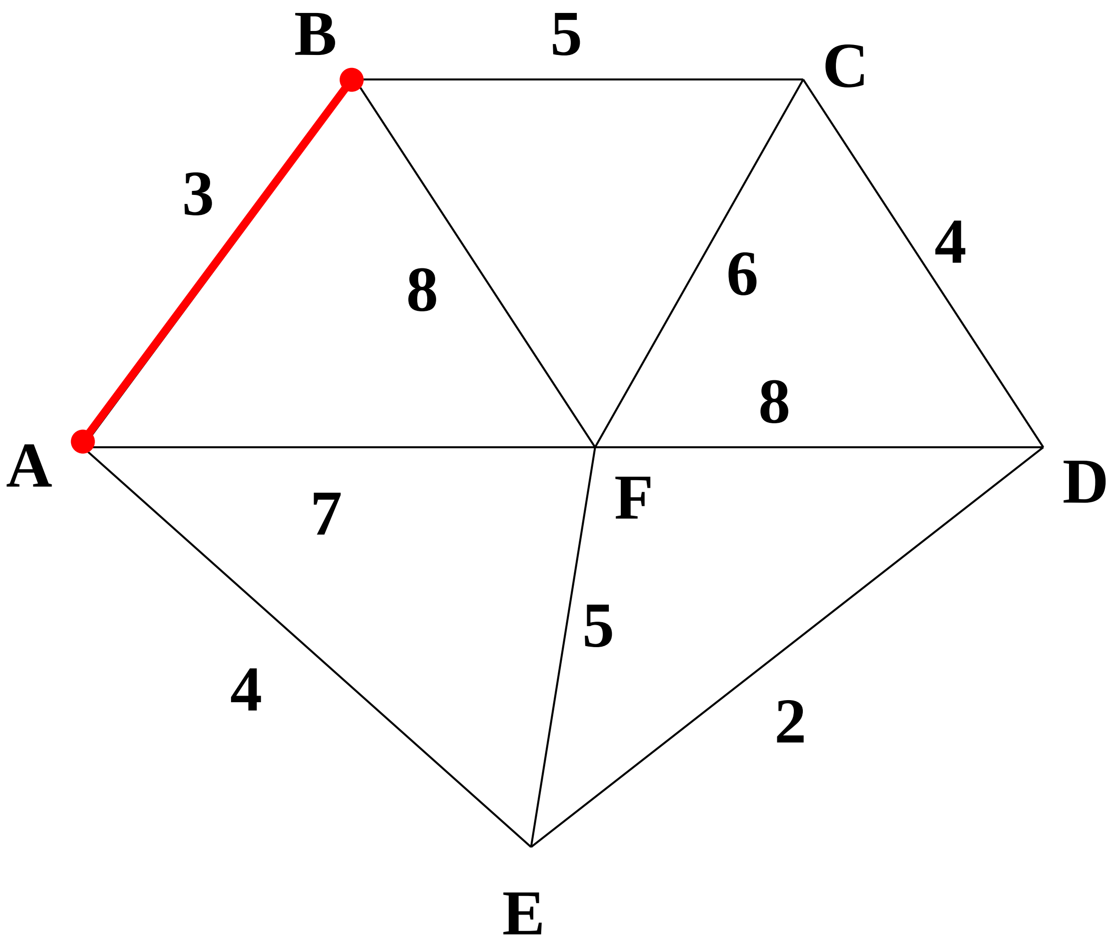


Prim's Algorithm



Model the situation as a graph and find the MST that connects all the villages (nodes).

Prim's Algorithm



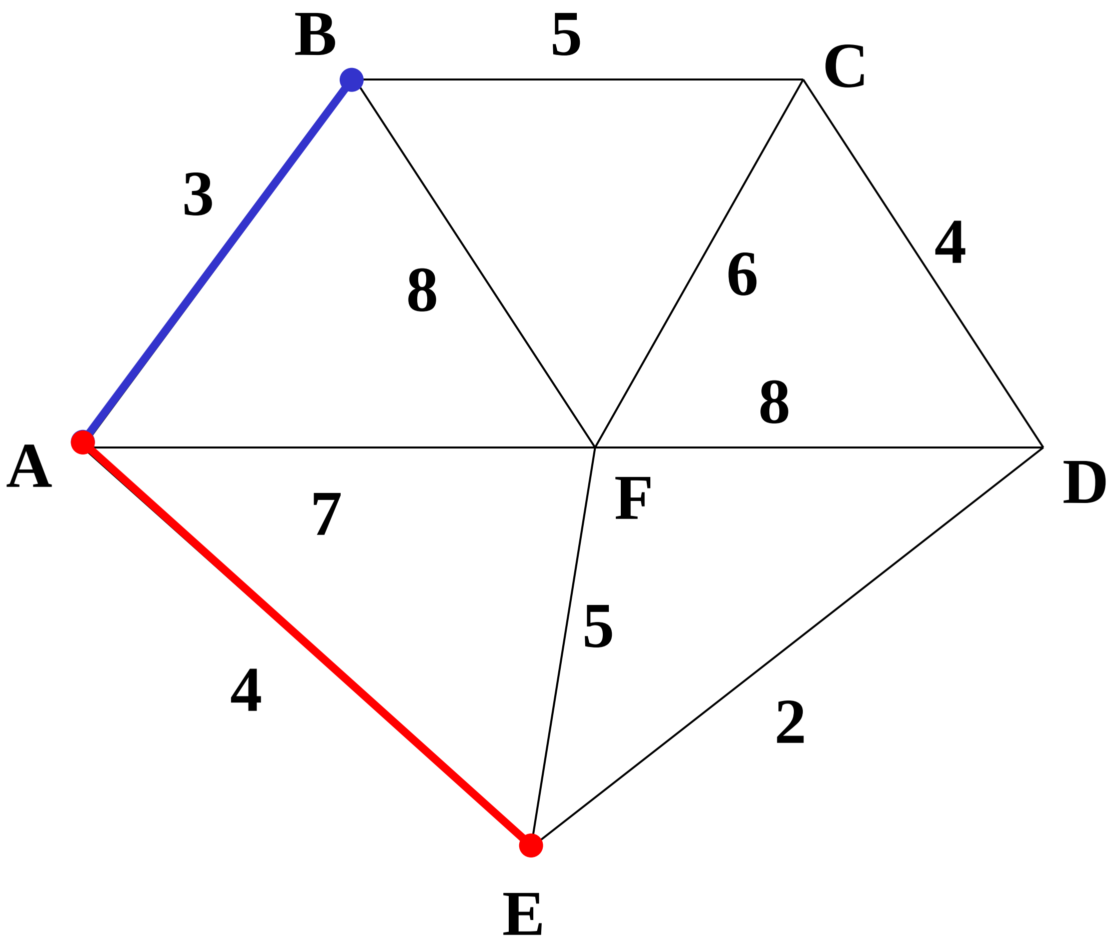
Select any vertex

A

Select the shortest edge connected to that vertex (since it's the only known one)

AB 3

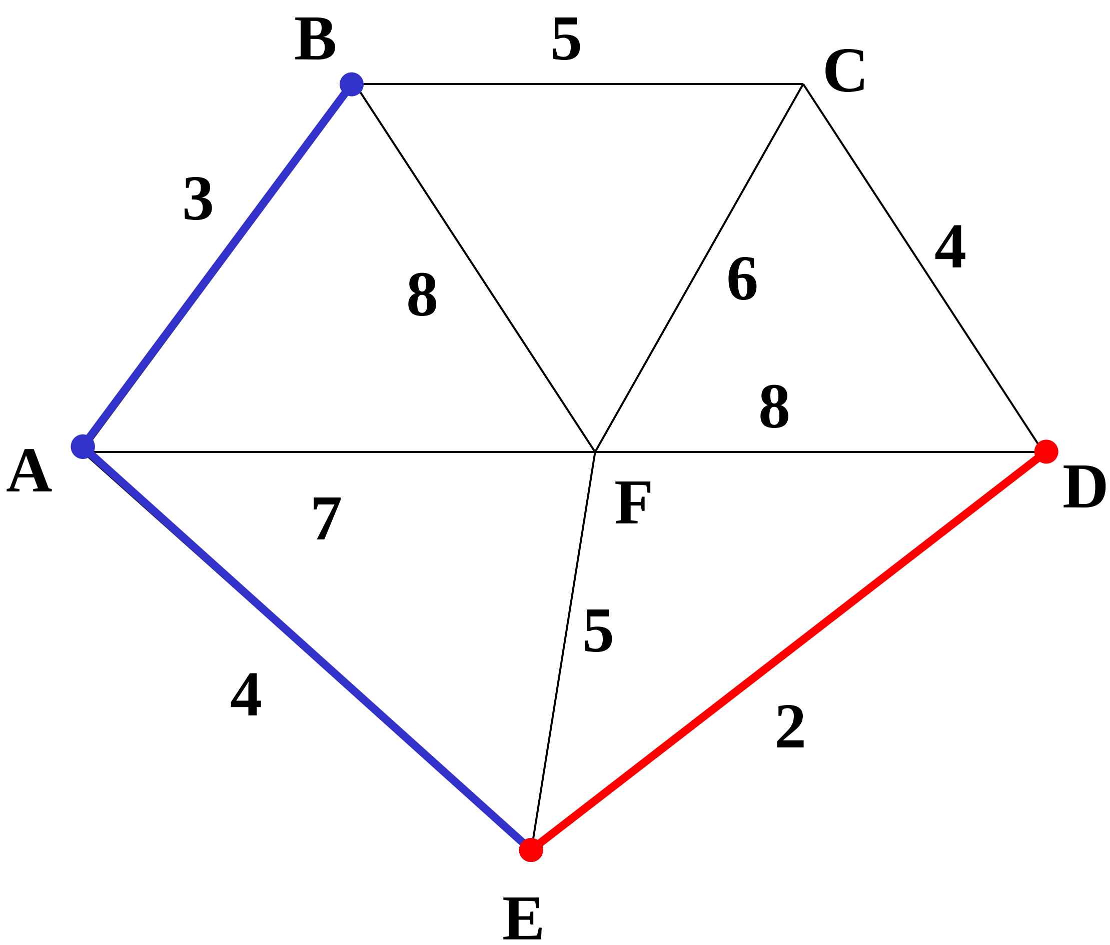
Prim’s Algorithm



Select the shortest edge that connects an unknown vertex to any known vertex.

AE 4

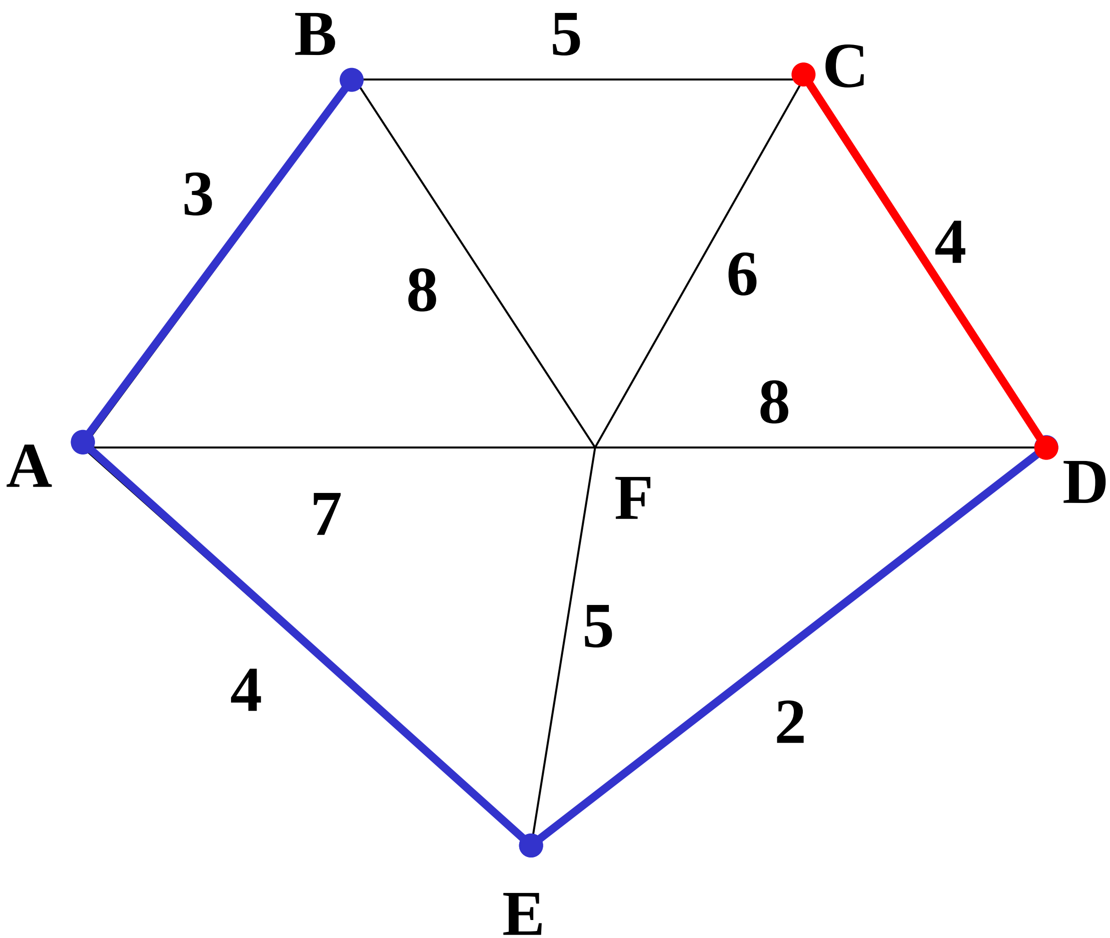
Prim's Algorithm



Select the shortest edge that connects an unknown vertex to any known vertex.

ED 2

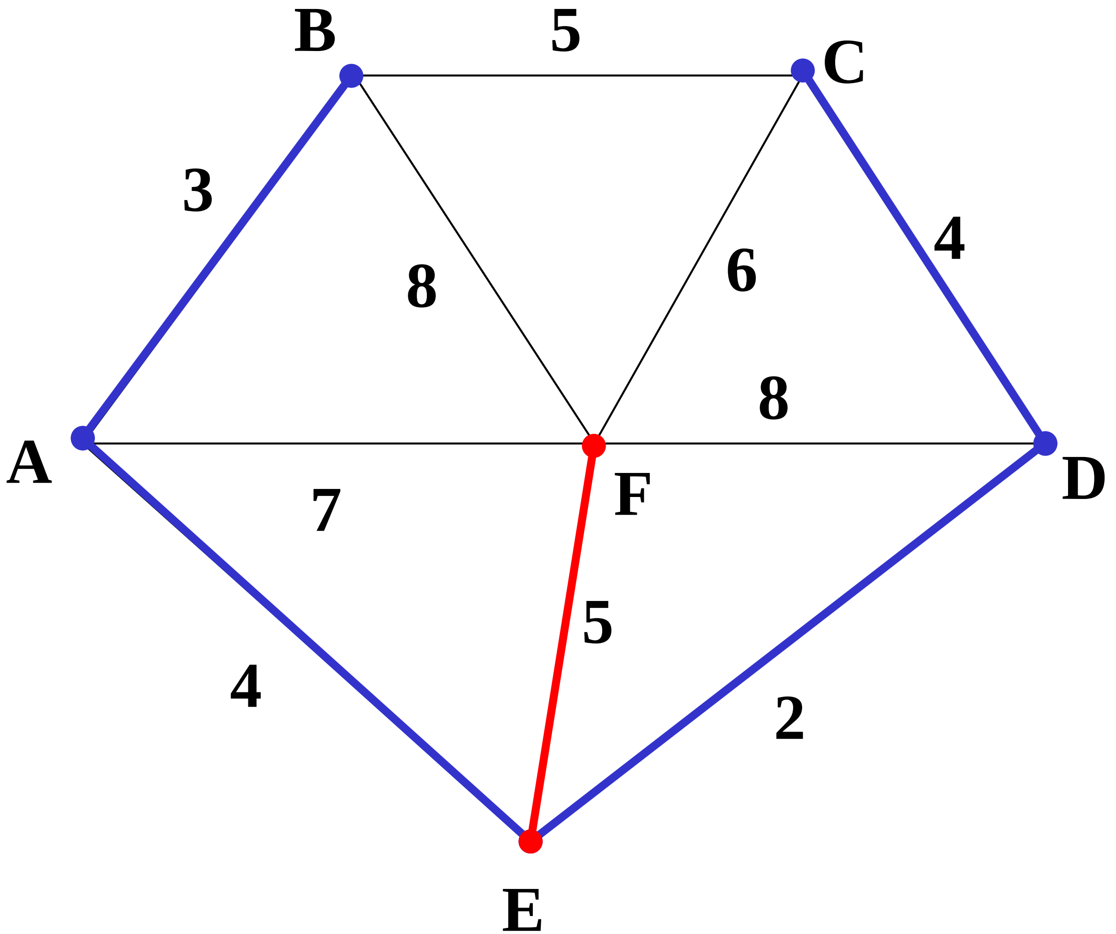
Prim’s Algorithm



Select the shortest edge that connects an unknown vertex to any known vertex.

DC 4

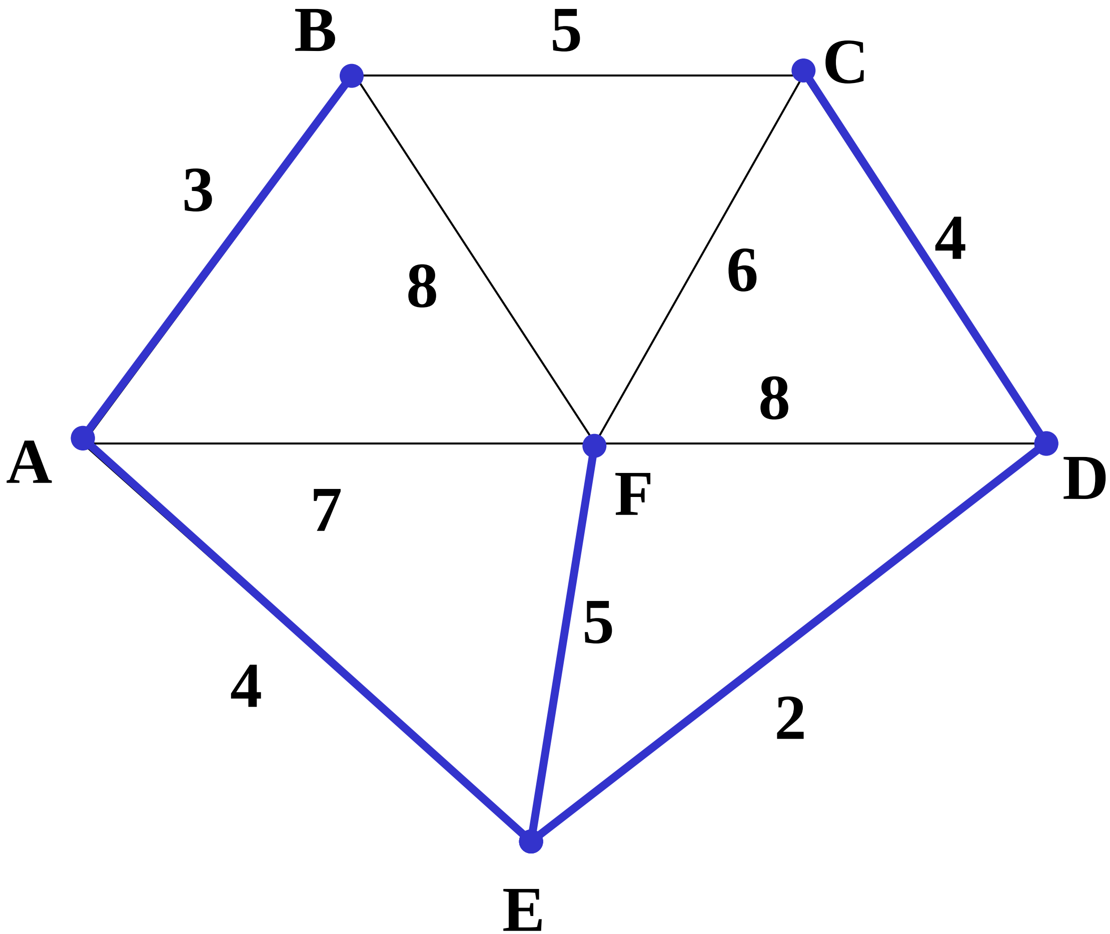
Prim's Algorithm



Select the shortest edge that connects an unknown vertex to any known vertex.

EF 5

Prim's Algorithm



All vertices have been connected.

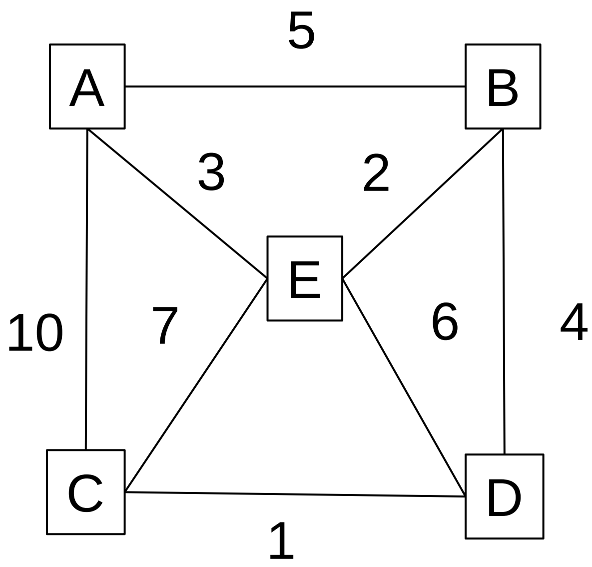
The solution is

AB 3
AE 4
ED 2
DC 4
EF 5

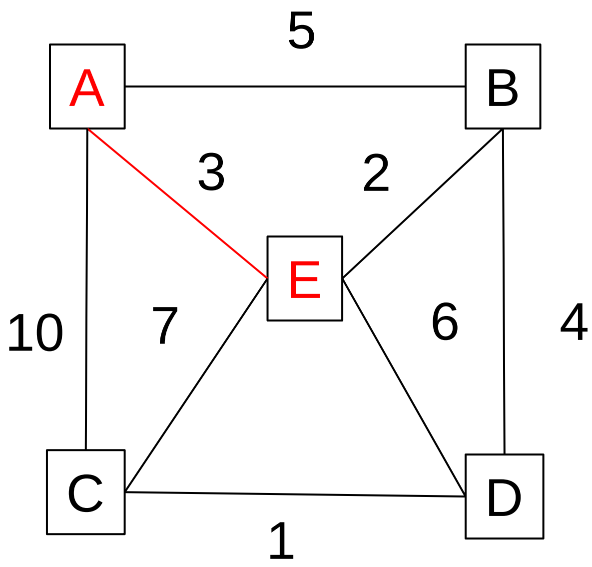
Total weight of tree: 18

Practice

Start with A. Make it Known.
HINT: Choose the least cost edge out of it. Add that node to the Known.

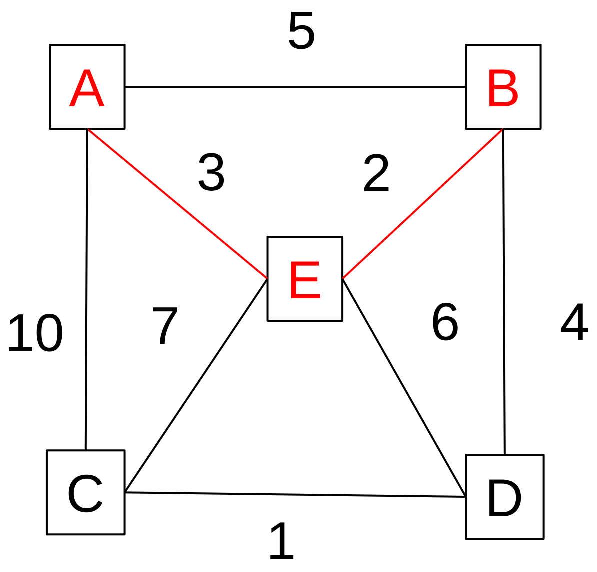


Practice



A connects to B, C, E
Lowest cost is to E

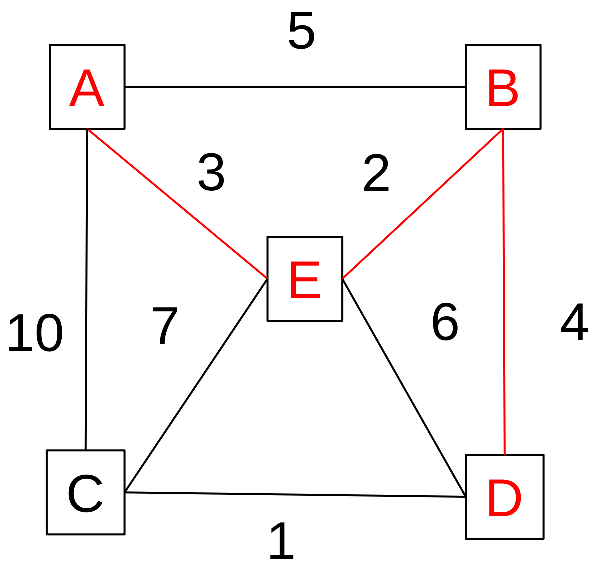
Practice



Now we consider anything connected to A or E.

Nodes considered are B, C, D. Lowest cost to B.

Practice

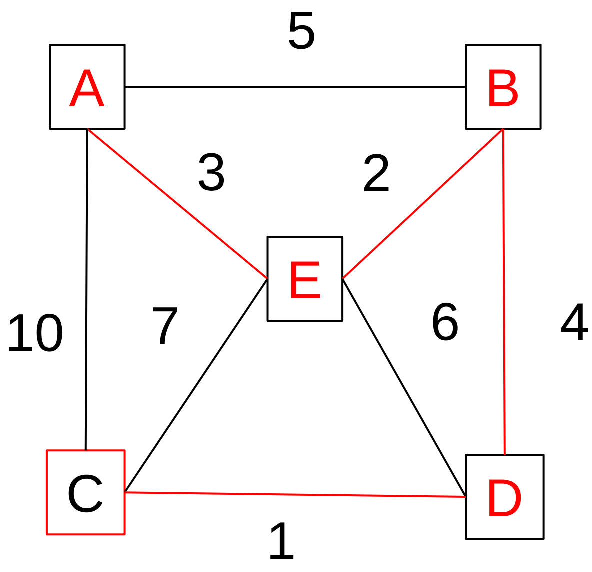


Now we consider anything connected to A, B, or E.

Nodes considered are C, D.

Lowest cost to D.

Practice



Now we consider anything connected to A, B, D, or E.

Nodes considered are C.

Lowest cost is from D to C.

DONE

Minimum Spanning Tree Algorithms

- Prim's Algorithm for Minimum Spanning Tree
  - Similar idea to Dijkstra's Algorithm but for MSTs.
  - Both based on expanding cloud of known vertices
    (basically using a priority queue instead of a DFS stack)

- Kruskal's Algorithm for Minimum Spanning Tree
  - Another, but different, greedy MST algorithm.
  - Uses the Union-Find data structure.

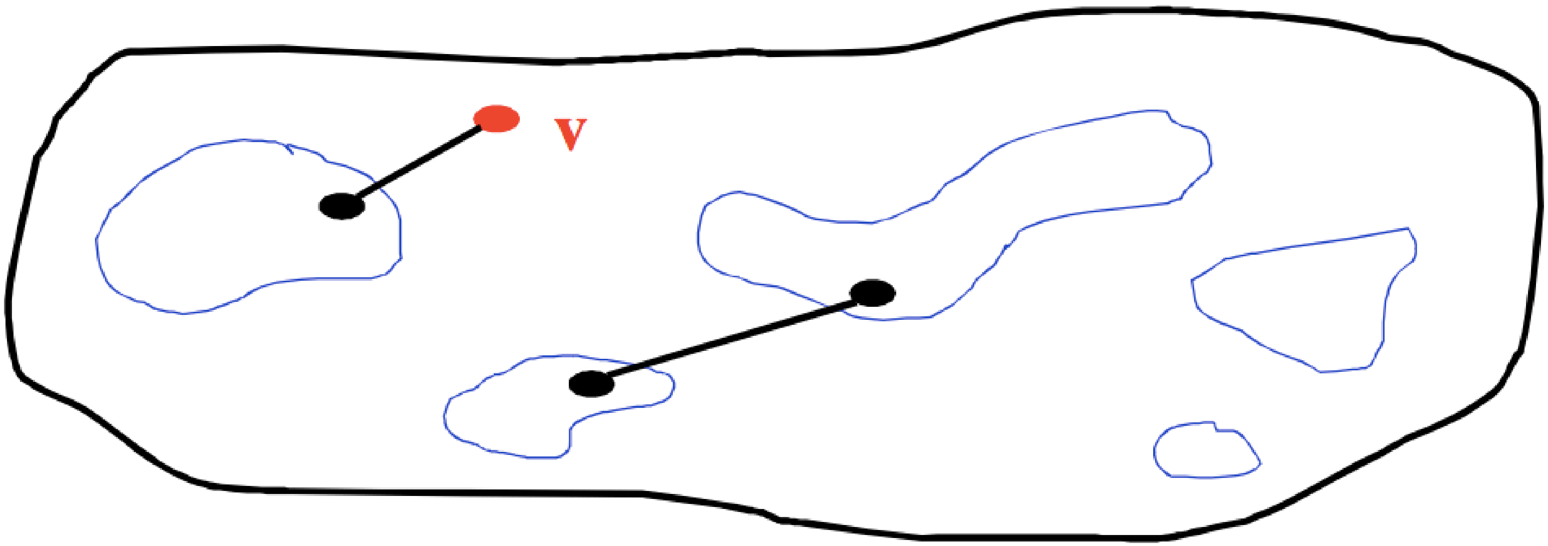
Kruskal's Algorithm

Idea: Grow a forest out of edges that do not create a cycle. Pick an edge with the smallest weight.

An *edge-based* greedy algorithm

Builds MST by greedily adding edges

G=(V,E)



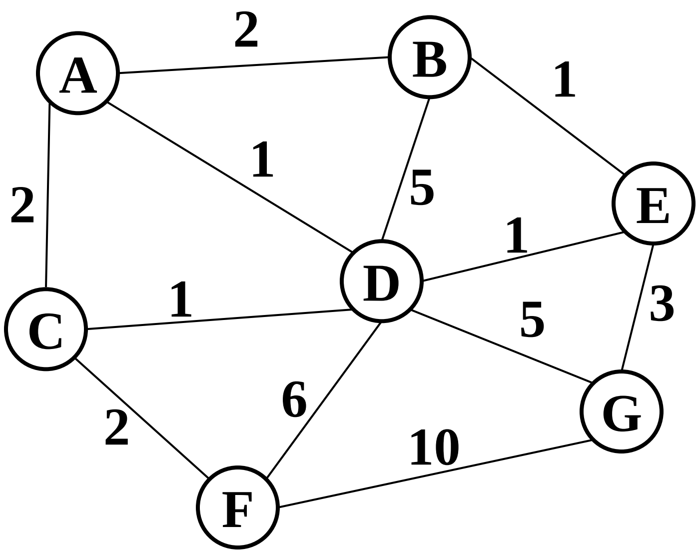
Kruskal's Algorithm Pseudocode

1. Sort **edges** by weight (min-heap)
2. Each **node** in its own set (up-trees)
3. While output size **< |V|-1**
   - Consider next smallest edge **`(u,v)`**
   - if **`find(u)`** and **`find(v)`** indicate **`u`** and **`v`** are in different sets
     - output **`(u,v)`**
     - **`union(find(u),find(v))`**

Recall invariant:

`u` and **`v`** in same set if and only if connected in output-so-far

Kruskal's Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

5: (D,G), (B,D)

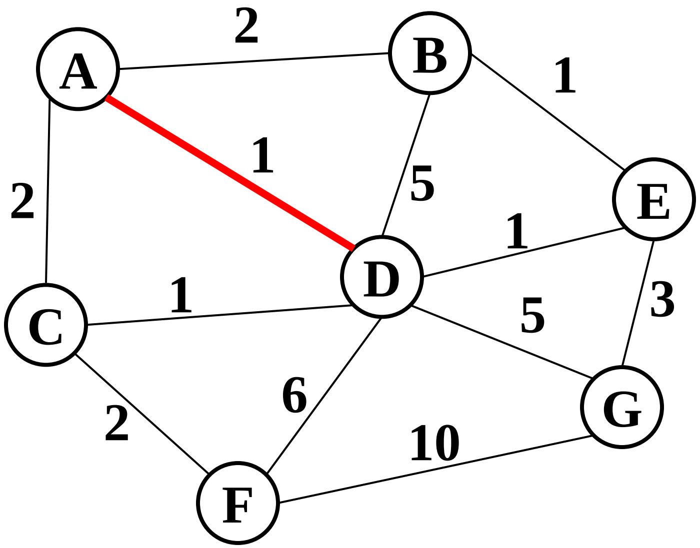
6: (D,F)

10: (F,G)

Output:

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

5: (D,G), (B,D)

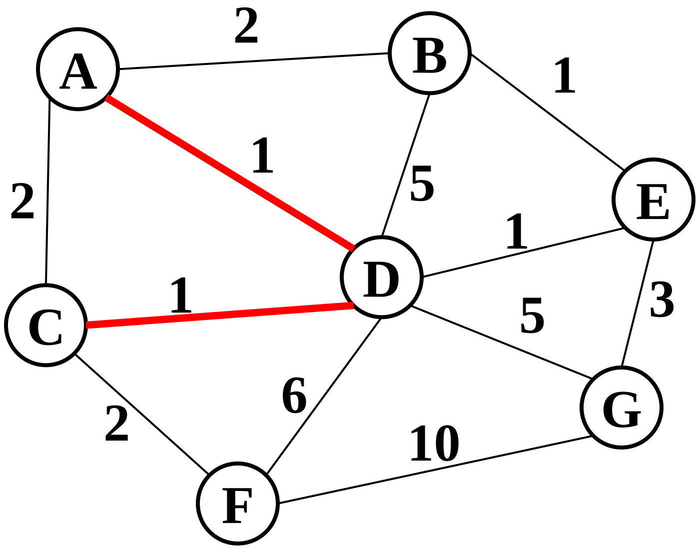
6: (D,F)

10: (F,G)

Output: (A,D)

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

5: (D,G), (B,D)

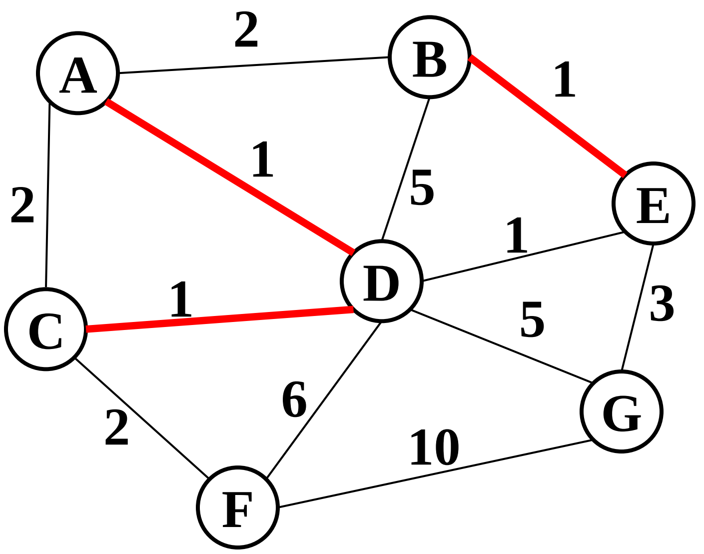
6: (D,F)

10: (F,G)

Output: (A,D), (C,D)

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

5: (D,G), (B,D)

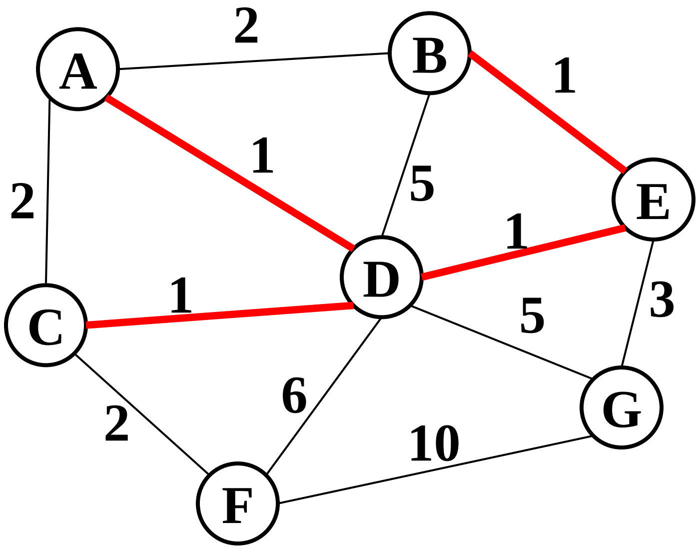
6: (D,F)

10: (F,G)

Output: (A,D), (C,D), (B,E)

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Example



Edges in sorted order:

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2: (A,B), (C,F), (A,C)

3: (E,G)

5: (D,G), (B,D)

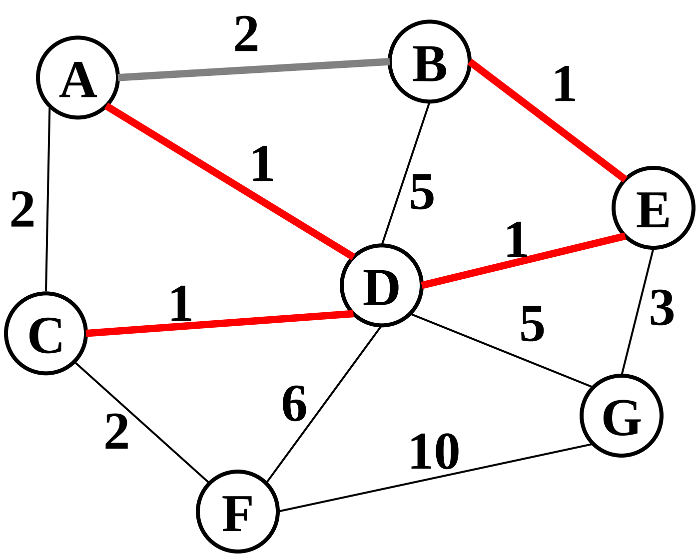
6: (D,F)

10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E)

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

5: (D,G), (B,D)

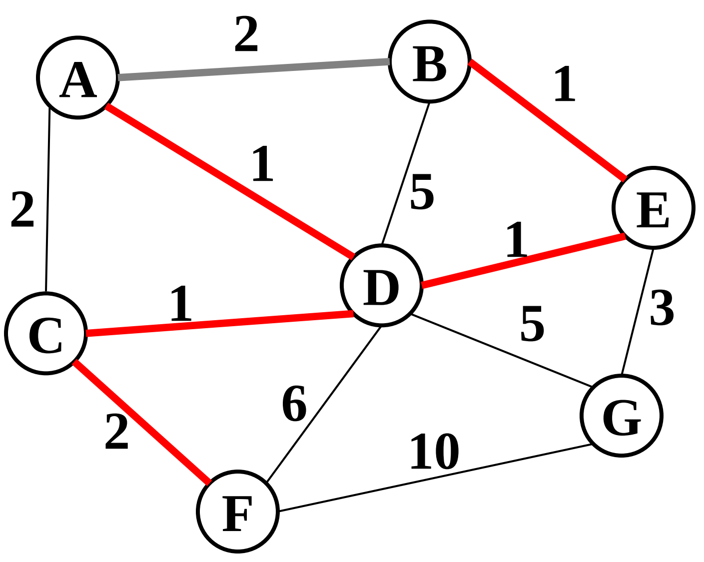
6: (D,F)

10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E)

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

5: (D,G), (B,D)

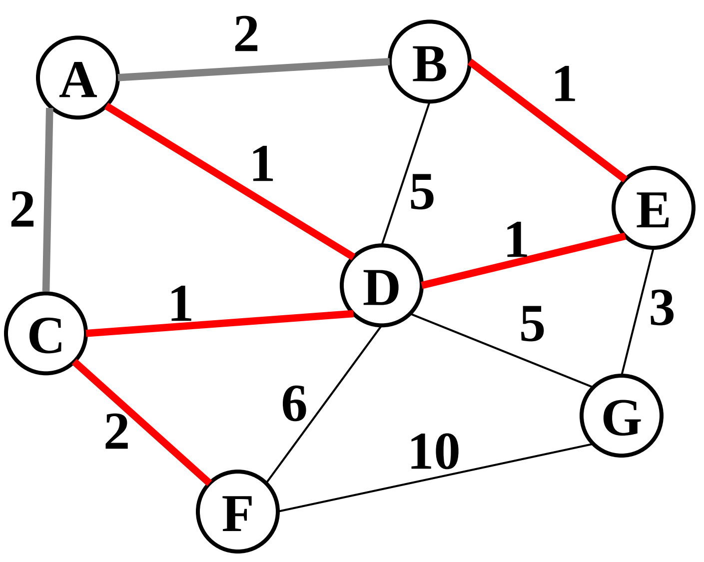
6: (D,F)

10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E), (C,F)

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

5: (D,G), (B,D)

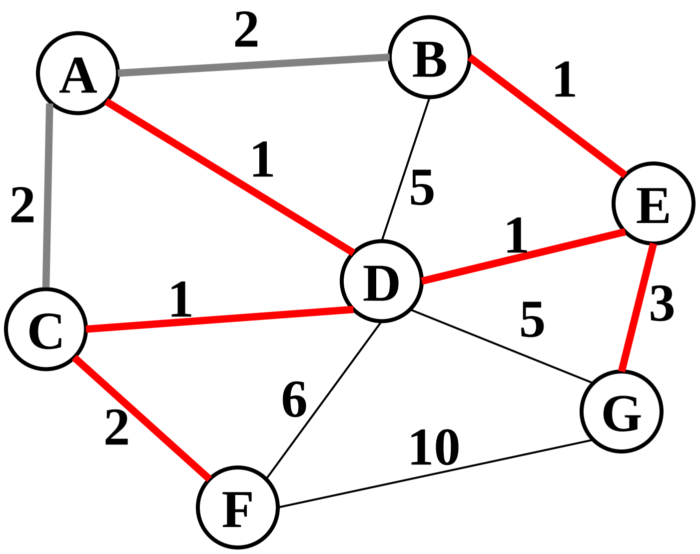
6: (D,F)

10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E), (C,F)

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Example



Edges in sorted order:

1: (A,D), (C,D), (B,E), (D,E)

2: (A,B), (C,F), (A,C)

3: (E,G)

5: (D,G), (B,D)

6: (D,F)

10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E), (C,F), (E,G)

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Algorithm Analysis

Idea: Grow a forest out of edges that do not grow a cycle, just like for the spanning tree problem.

– But now consider the edges in order by weight

So:

– Sort edges: *O*(**|E|`log` |E|**) (next course topic)

– Iterate through edges using union-find for cycle detection almost *O*(**|E|**)

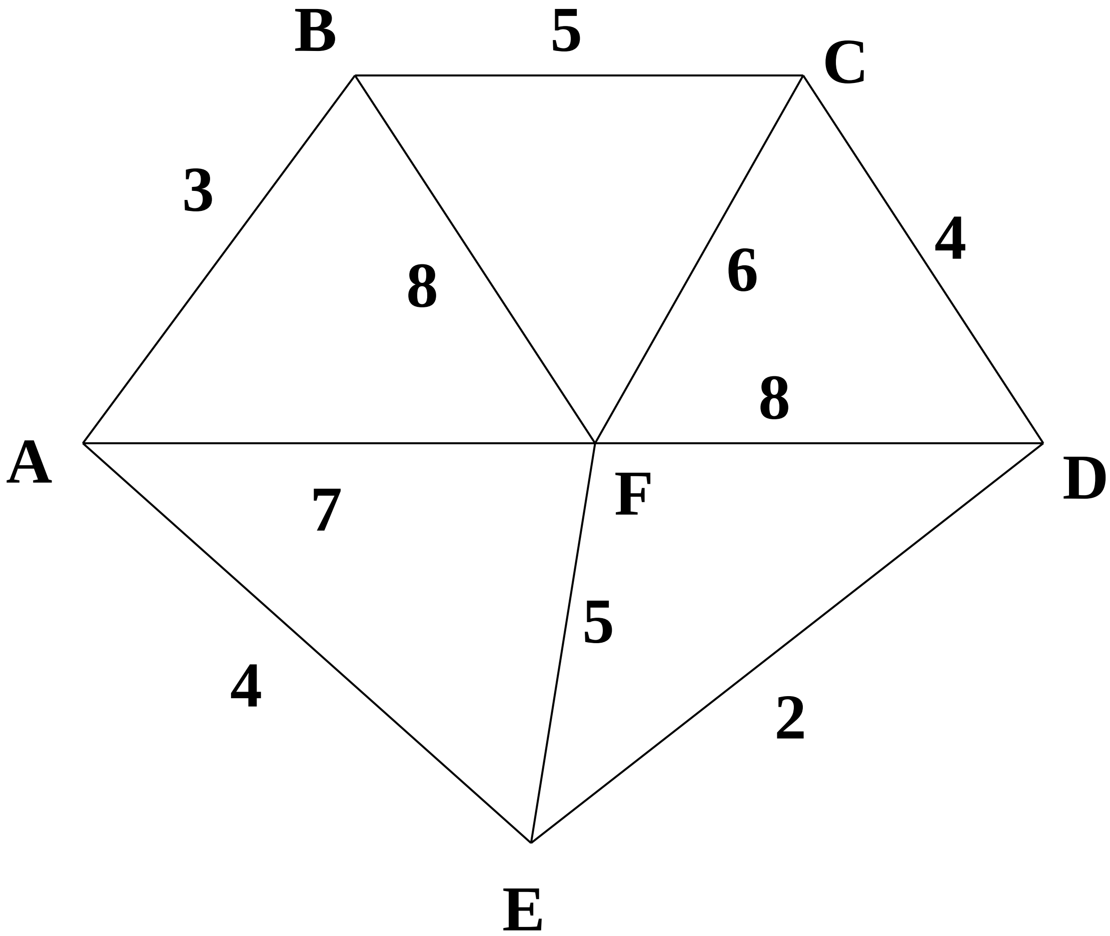
Somewhat better:

– Floyd's algorithm to build min-heap with edges *O*(**|E|**)

– Iterate through edges using union-find for cycle detection and **`deleteMin`** to get next edge *O*(**|E| `log` |E|**)

– Not better *worst-case* asymptotically, but often stop long before considering all edges.

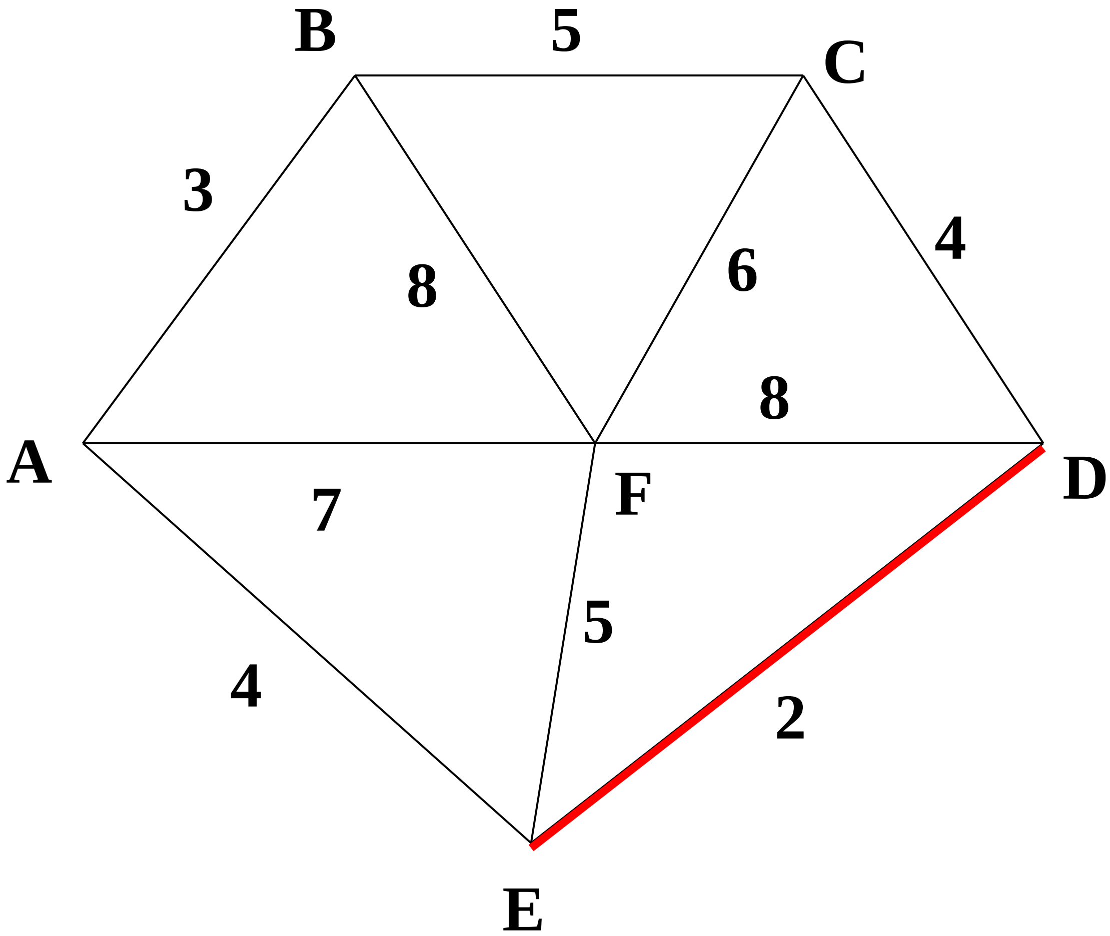
Kruskal's Algorithm



List the edges in order of size:

ED 2
AB 3
AE 4
CD 4
BC 5
EF 5
CF 6
AF 7
BF 8
CF 8

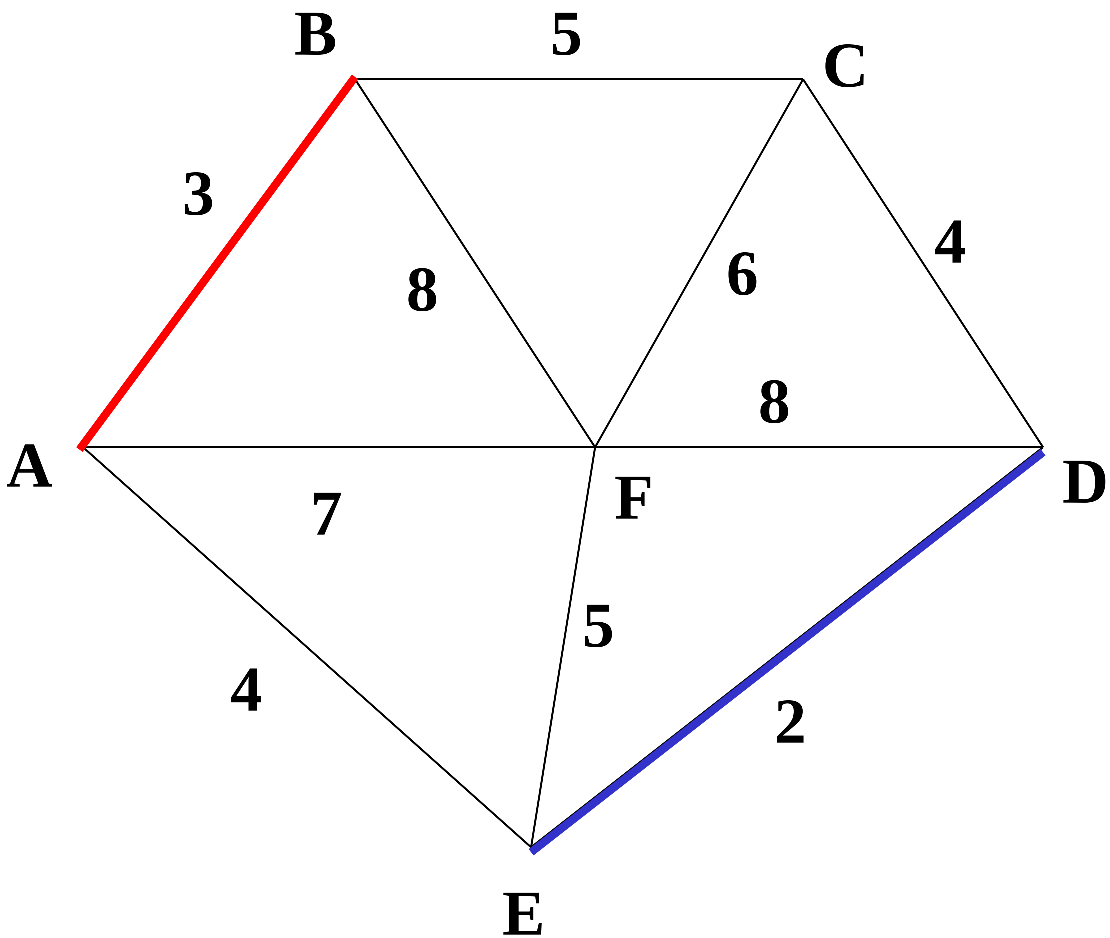
Kruskal's Algorithm



Select the edge
with min cost

ED 2

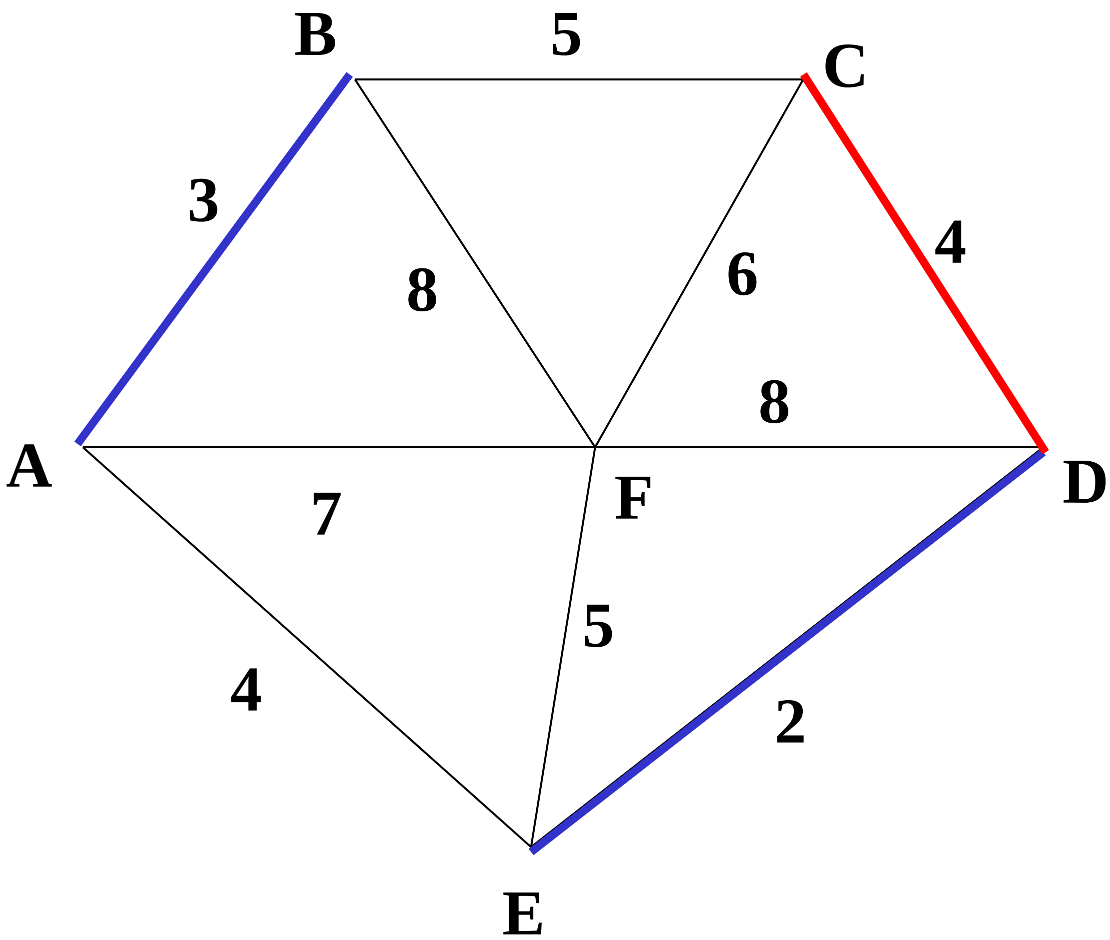
Kruskal's Algorithm



Select the next minimum cost edge that does not create a cycle

ED 2
AB 3

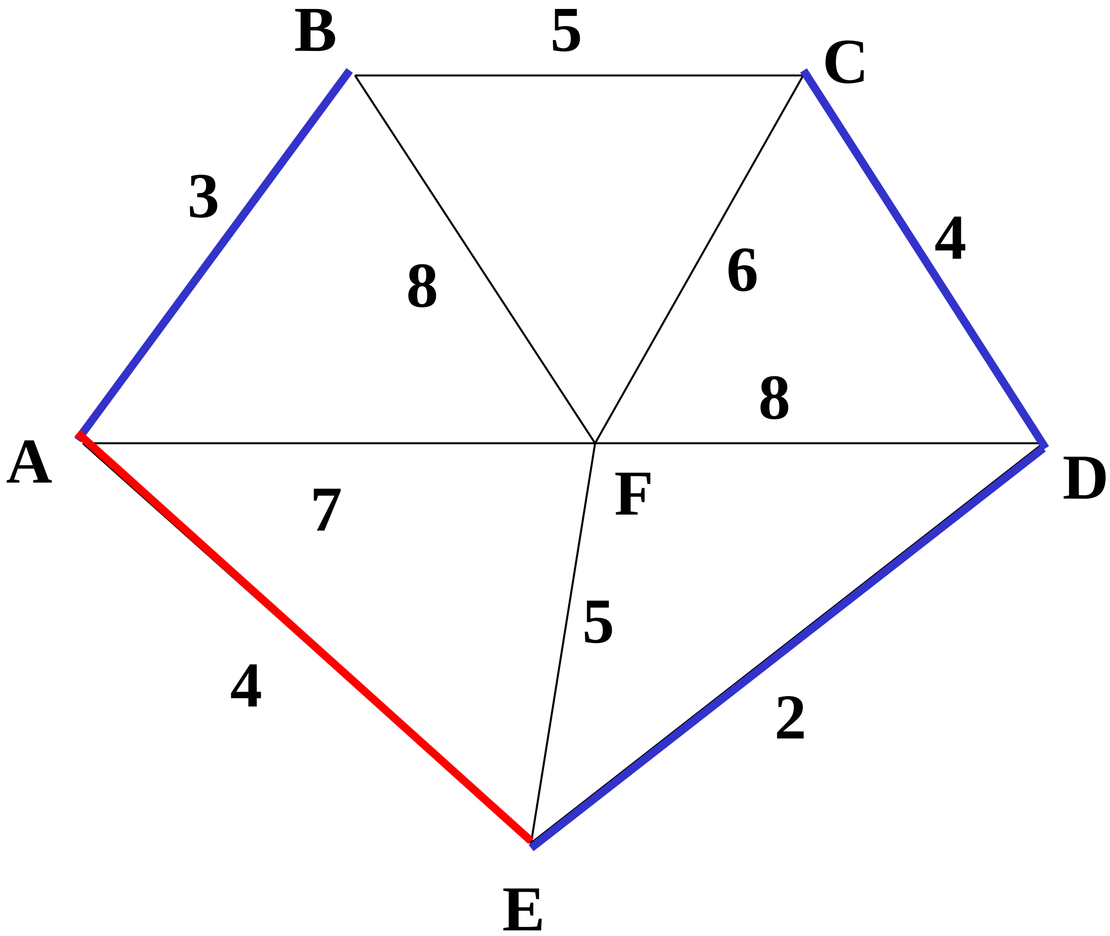
Kruskal's Algorithm



Select the next minimum cost edge that does not create a cycle

ED 2
AB 3
CD 4 (or AE 4)

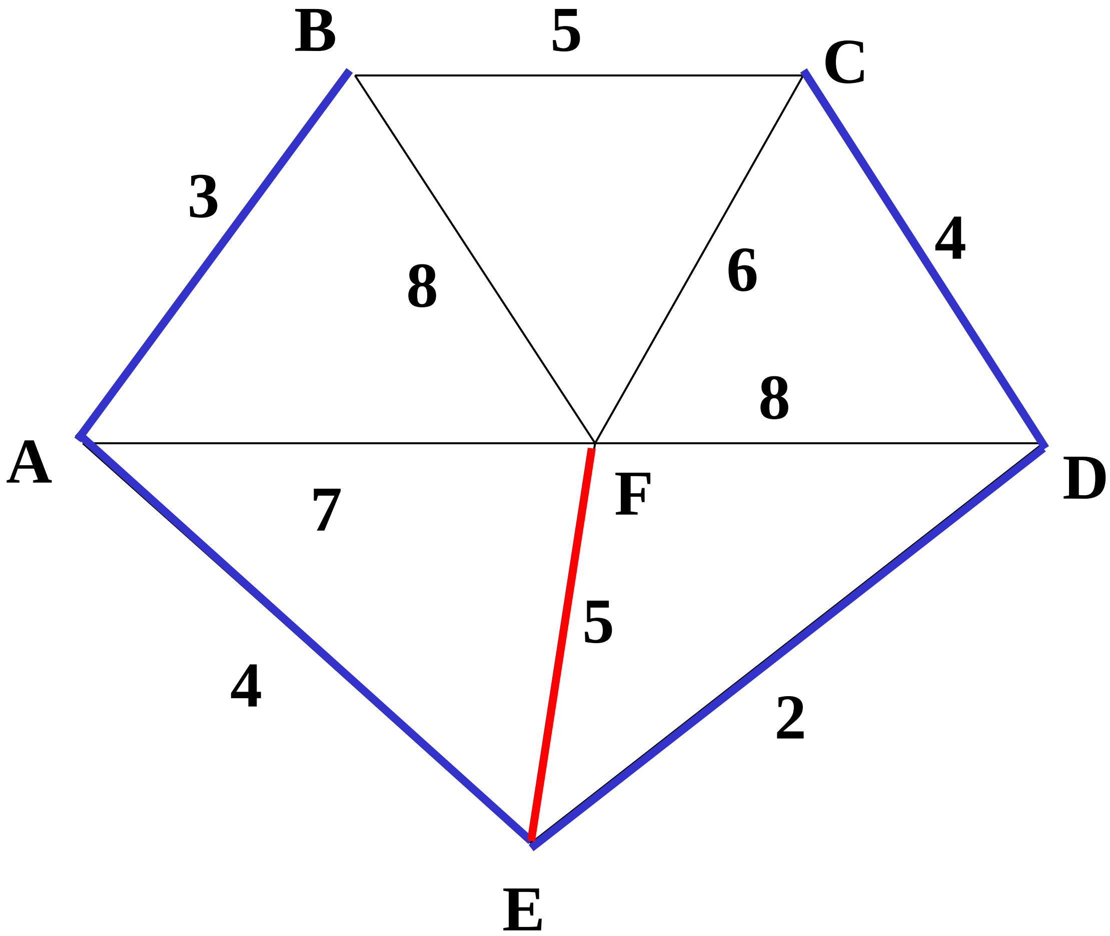
Kruskal’s Algorithm



Select the next minimum cost edge that does not create a cycle

ED 2
AB 3
CD 4
AE 4

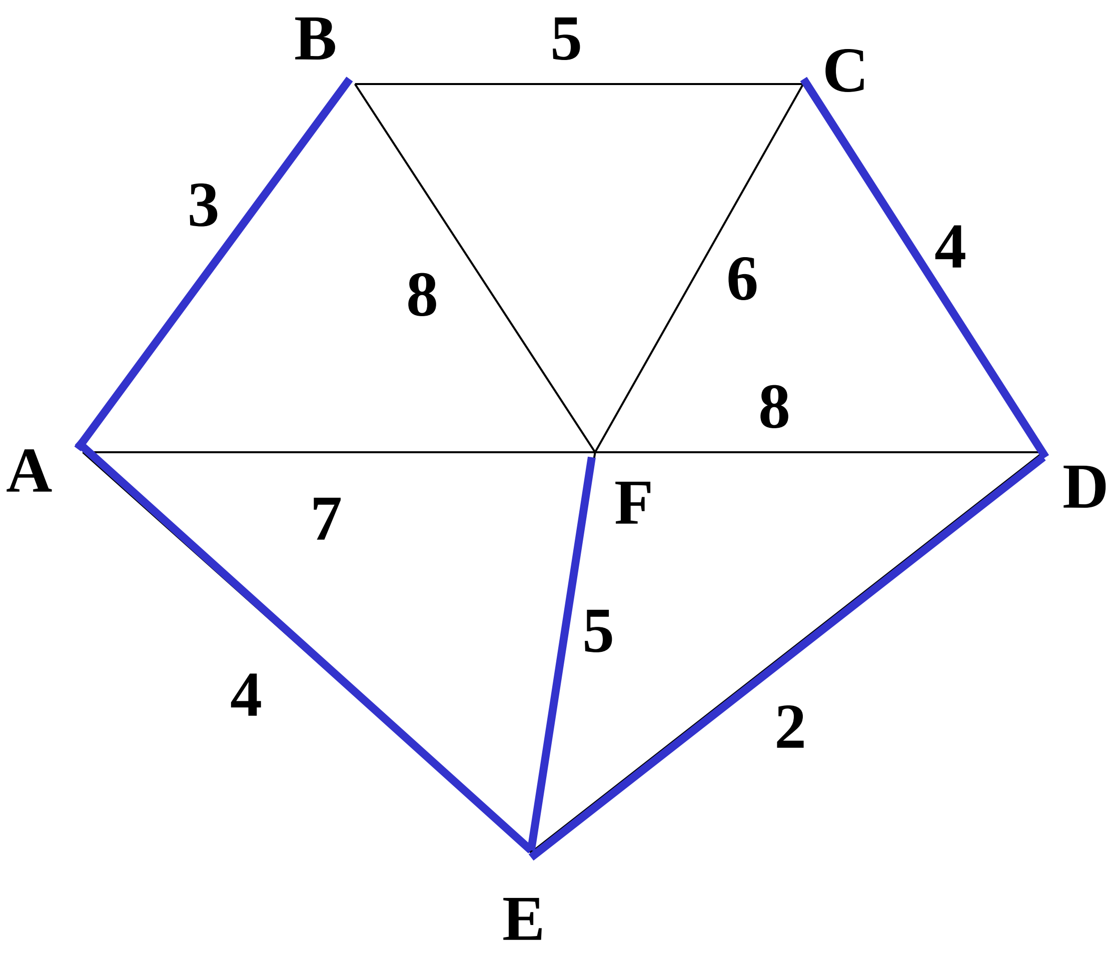
Kruskal's Algorithm



Select the next minimum cost edge that does not create a cycle

ED 2
AB 3
CD 4
AE 4
BC 5 – forms a cycle
EF 5

Kruskal's Algorithm



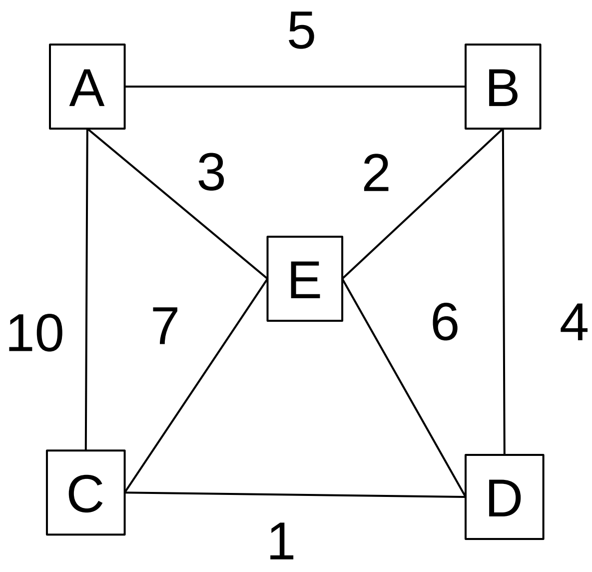
All vertices have been connected.

The solution is

ED 2
AB 3
CD 4
AE 4
EF 5

Total weight of tree: 18

Practice



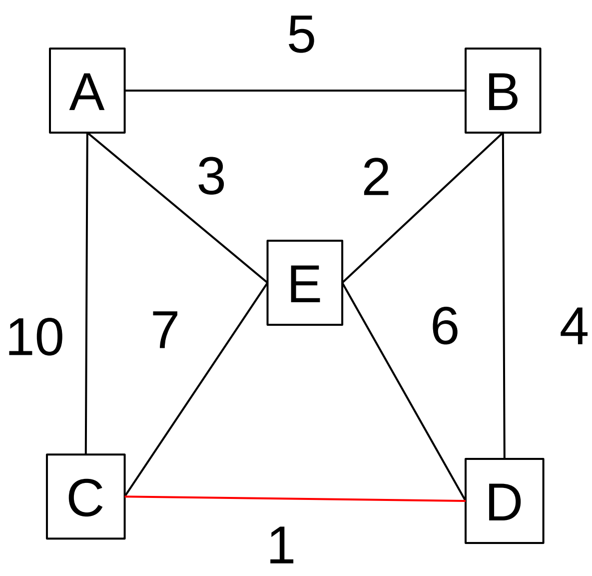
Order the edges.

(C,D)
(E,B)
(A,E)
(B,D)
(A,B)
(E,D)
(E,C)
(A,C)

Practice

Order the edges.

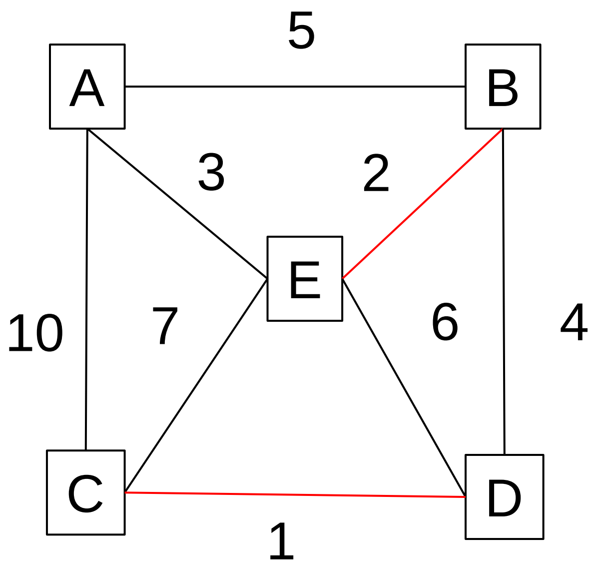
(C,D)
(E,B)
(A,E)
(B,D)
(A,B)
(E,D)
(E,C)
(A,C)



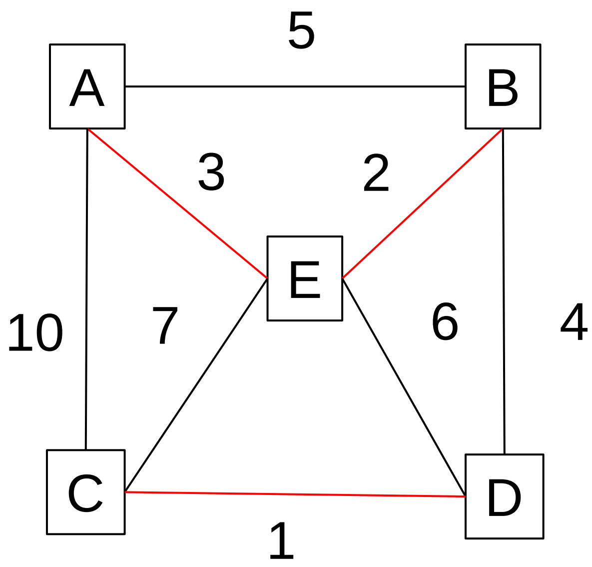
Practice

Order the edges.

(C,D) ☑
(E,B) ☑
(A,E)
(B,D)
(A,B)
(E,D)
(E,C)
(A,C)



Practice



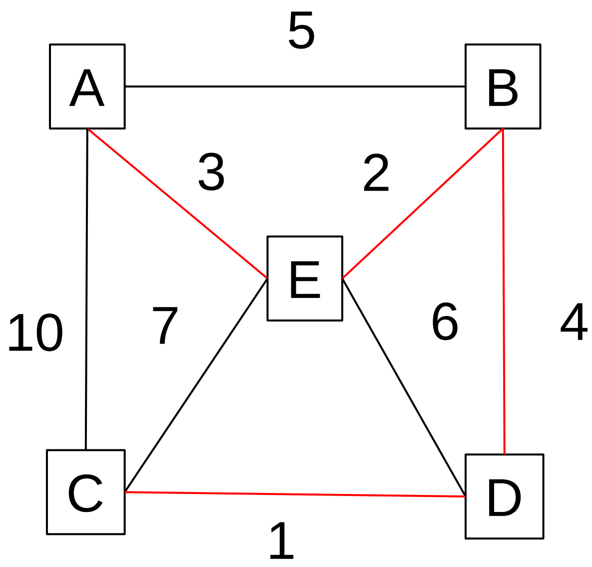
Order the edges.

(C,D) ☑
(E,B) ☑
(A,E) ☑
(B,D)
(A,B)
(E,D)
(E,C)
(A,C)

Practice

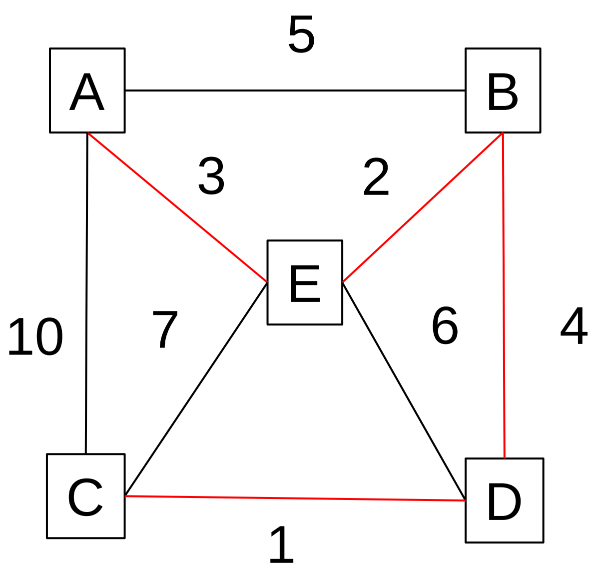
Order the edges.

(C,D) ☑
(E,B) ☑
(A,E) ☑
(B,D) ☑
(A,B)
(E,D)
(E,C)
(A,C)



Practice

Order the edges.



(C,D)
(E,B)
(A,E)
(B,D)
(A,B)
(E,D)
(E,C)
(A,C)

Done with graph algorithms!

Next lecture…

- Sorting