CSE373: Data Structures & Algorithms

Lecture 19: Spanning Trees

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Winter 2016

Announcements

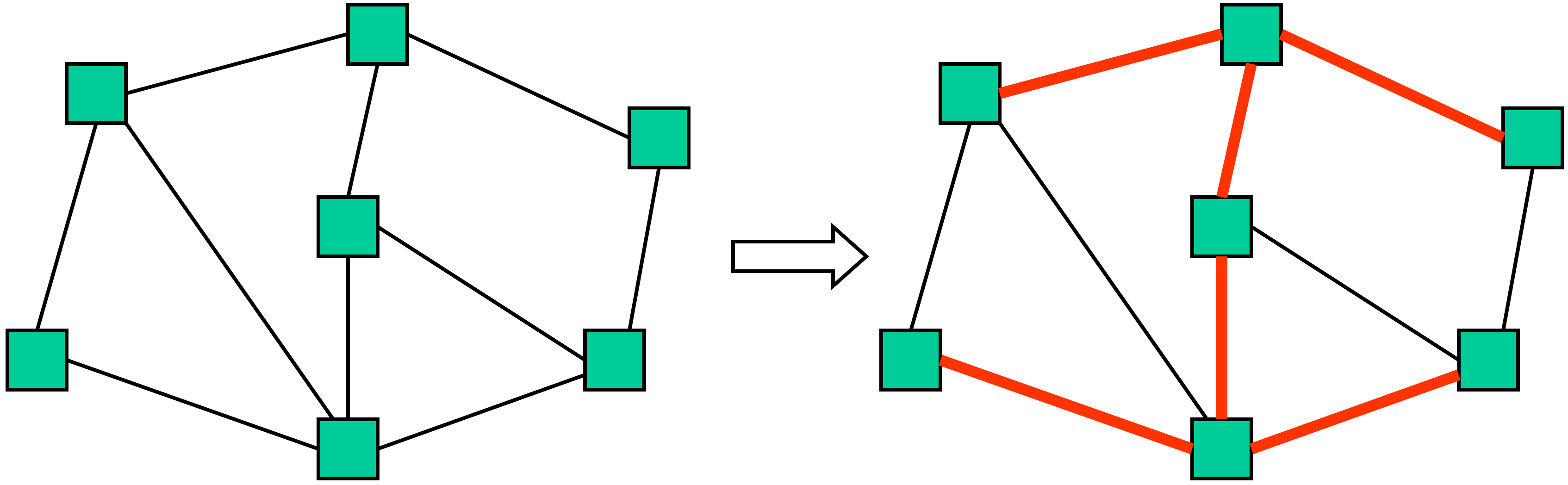
- HW 4 due Wed, May 18

Done with Dijkstra

- You will implement Dijkstra's algorithm in homework 5.
- Onward….. Spanning trees!

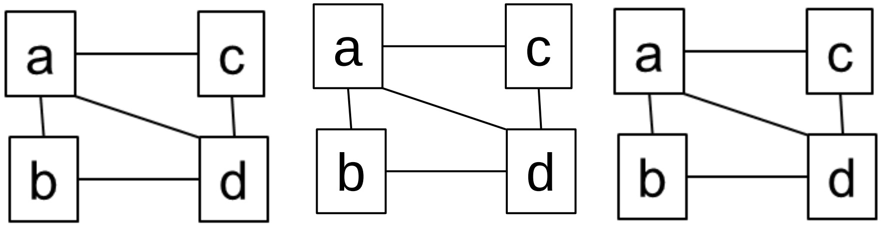
Spanning Trees

- A simple problem: Given a *connected* undirected graph **G**=(**V**,**E**), find a minimal subset of edges such that **G** is still connected
  - A graph **G2**=(**V**,**E2**) such that **G2** is connected and removing any edge from **E2** makes **G2** disconnected



Observations

|V|=4
|V|-1=3



1. Any solution to this problem is a tree
   - Recall a tree does not need a root; just means acyclic
   - For any cycle, could remove an edge and still be connected

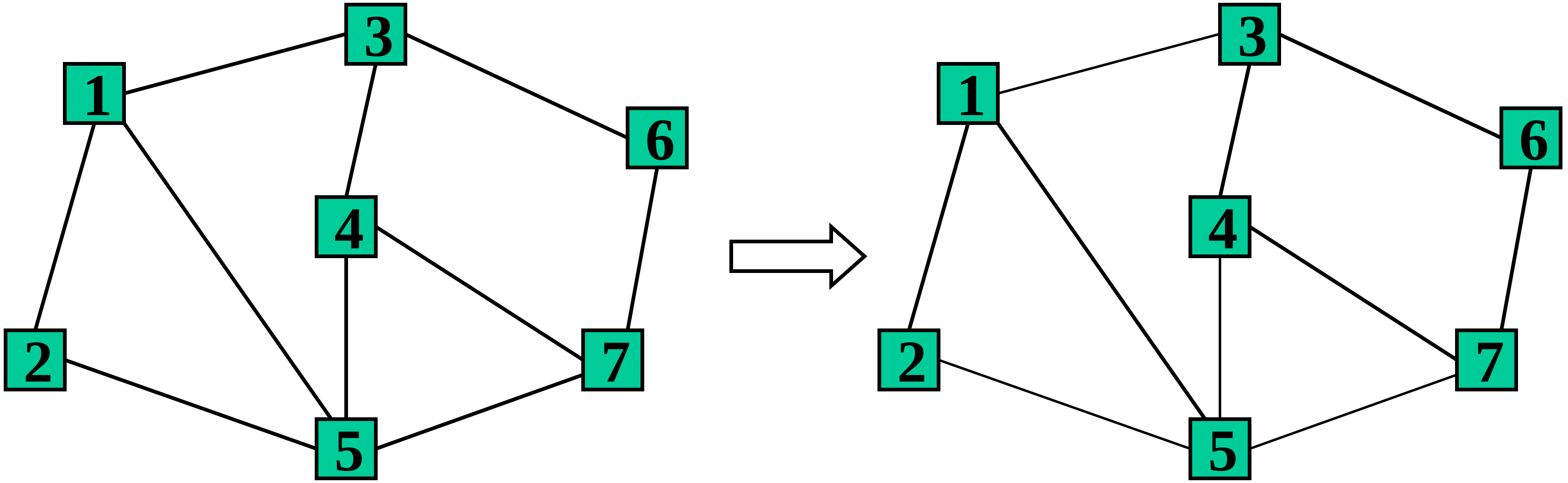
2. Solution not unique unless original graph was already a tree

3. Problem ill-defined if original graph not connected
   - So **|E| ≥ |V|-1**

4. A tree with **|V|** nodes has **|V|-1** edges
   - So every solution to the spanning tree problem has **|V|-1** edges

Spanning Trees

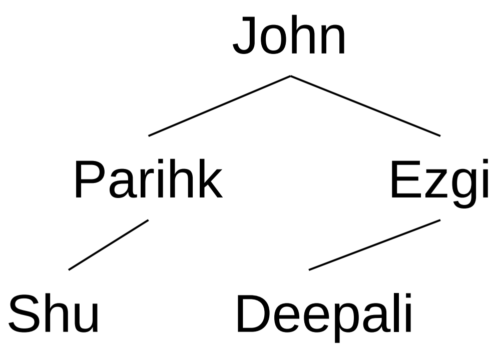
- Can we find another spanning tree in the bigger one?
- Pick a start node and think like a tree.

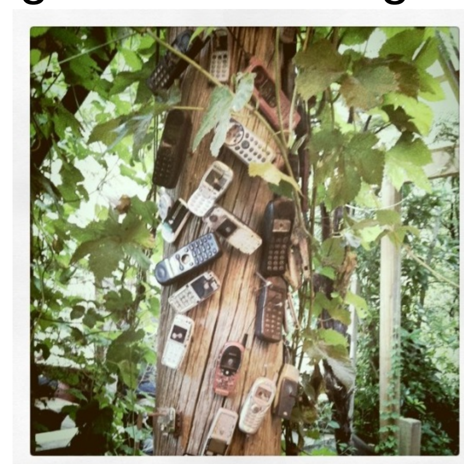


Motivation

A spanning tree connects all the nodes with as few edges as possible

- Example: A "phone tree" so everybody gets the message and no unnecessary calls get made





In most compelling uses, we have a *weighted* undirected graph and we want a tree of least total cost

- Example: Electrical wiring for a house or clock wires on a chip

Two Approaches

Different algorithmic approaches to the (unweighted) spanning-tree problem:

1. Do a graph traversal (e.g., depth-first search, but any traversal will do), keeping track of edges that form a tree

2. Iterate through edges; add to output any edge that does not create a cycle

Spanning tree via DFS

```
spanning_tree(Graph G) {
  for each node i
      i.marked = false
  for some node i: f(i)
}
f(Node i) {
  i.marked = true
  for each j adjacent to i:
    if(!j.marked) {
      add(i,j) to output
      f(j) // DFS
    }
}
```

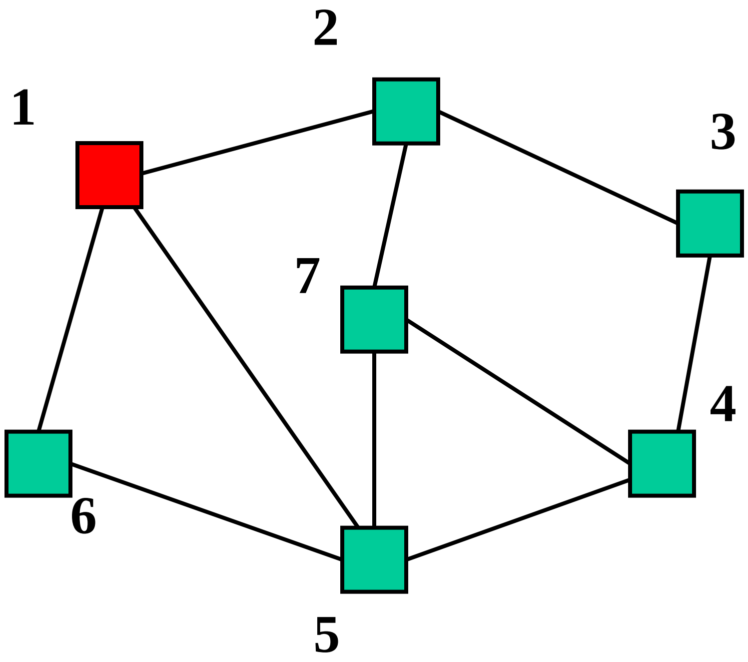
Correctness: DFS reaches each node. We add one edge to connect it to the already visited nodes. Order affects result, not correctness.

Time: $O(|E|)$

Example

Stack

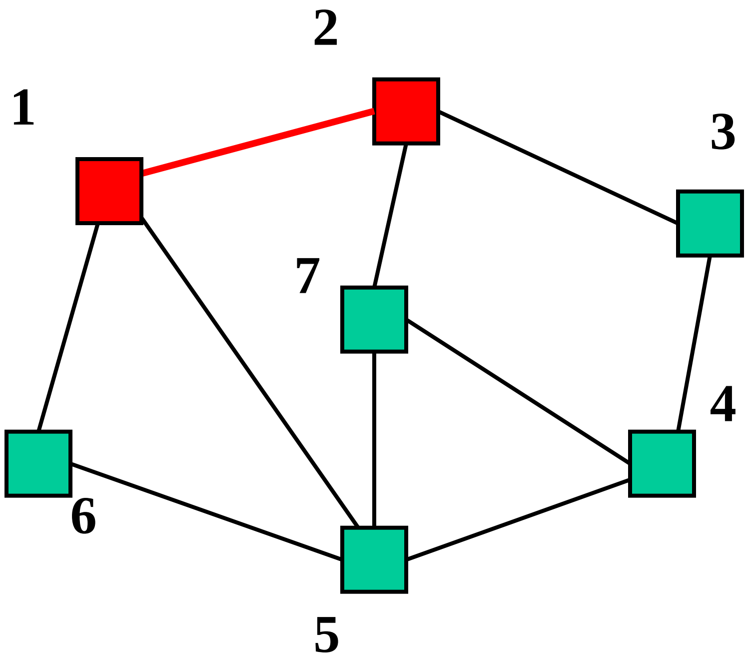
f(1)



Output:

Example

Stack
f(1)
f(2)



Output: (1,2)

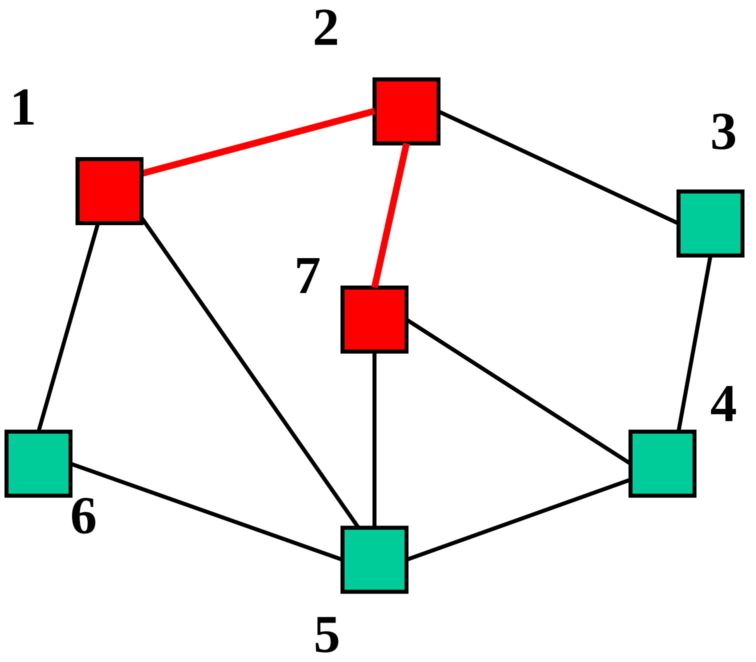
Example

Stack

f(1)

f(2)

f(7)



Output: (1,2), (2,7)

Example

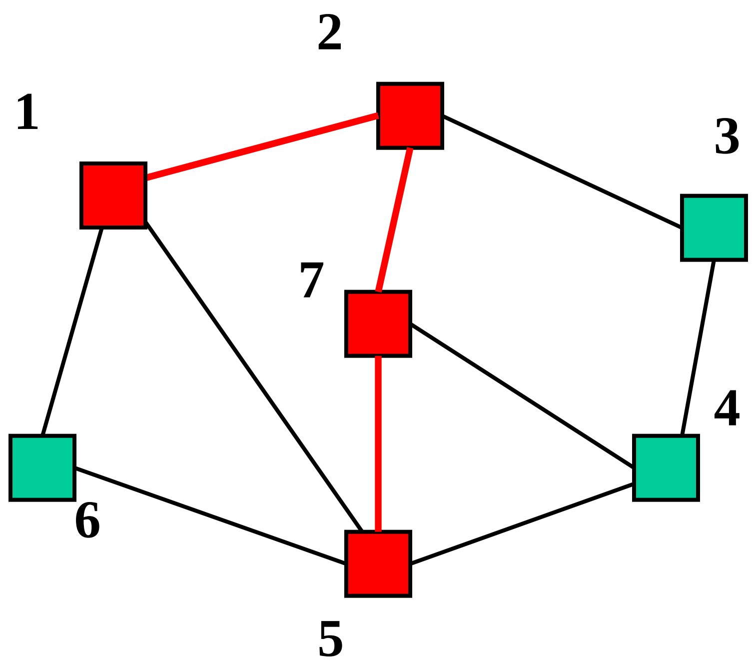
Stack

f(1)

f(2)

f(7)

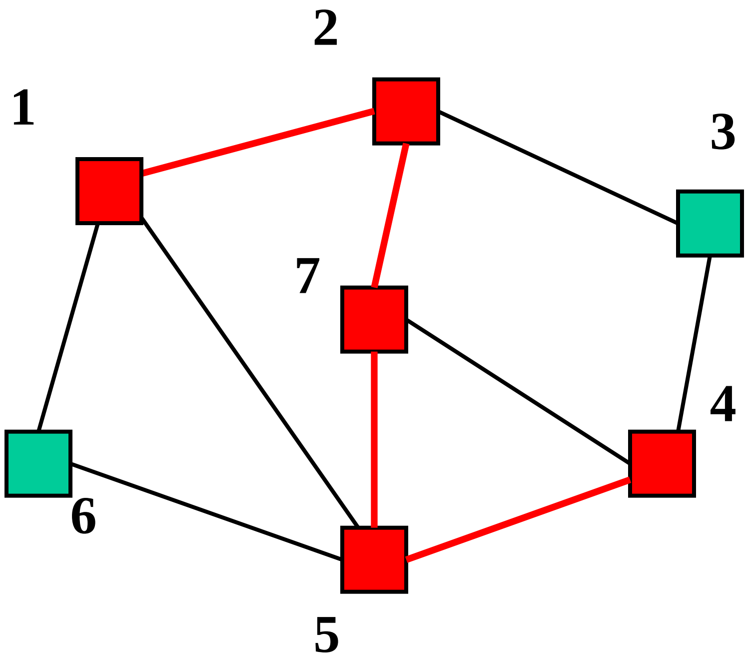
f(5)



Output: (1,2), (2,7), (7,5)

Example

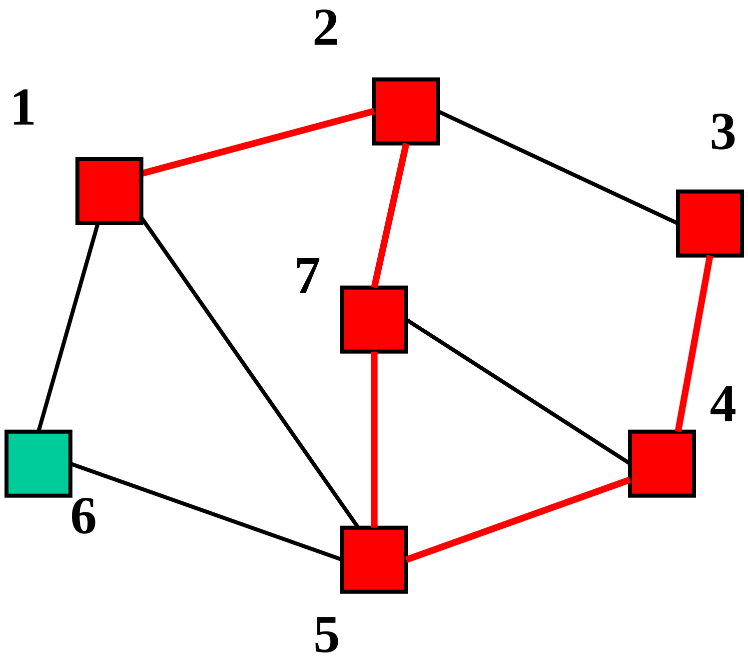
Stack
f(1)
f(2)
f(7)
f(5)
f(4)



Output: (1,2), (2,7), (7,5), (5,4)

Example

Stack
f(1)
f(2)
f(7)
f(5)
f(4)
f(3)



Output: (1,2), (2,7), (7,5), (5,4),(4,3)

Example

Stack

f(1)

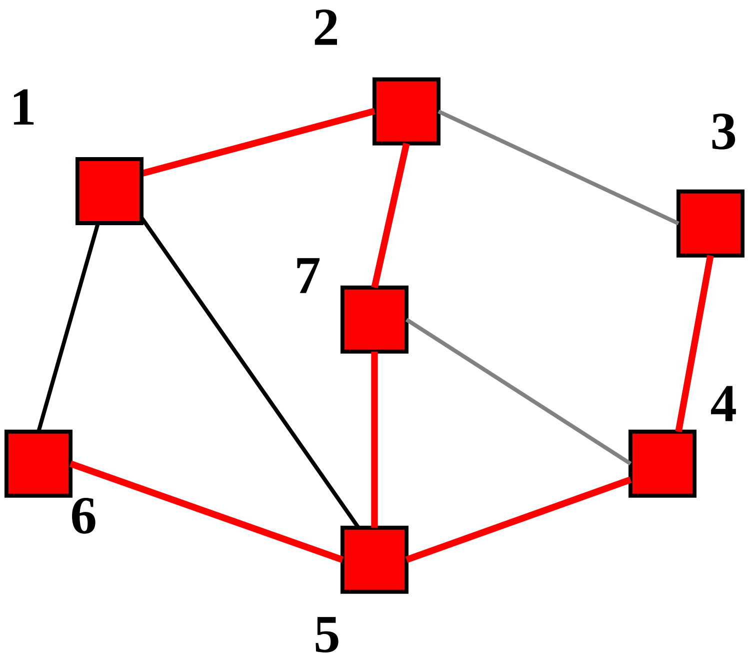
f(2)

f(7)

f(5)

f(4) f(6)

f(3)



Output: (1,2), (2,7), (7,5), (5,4), (4,3), (5,6)

Example

Stack

f(1)

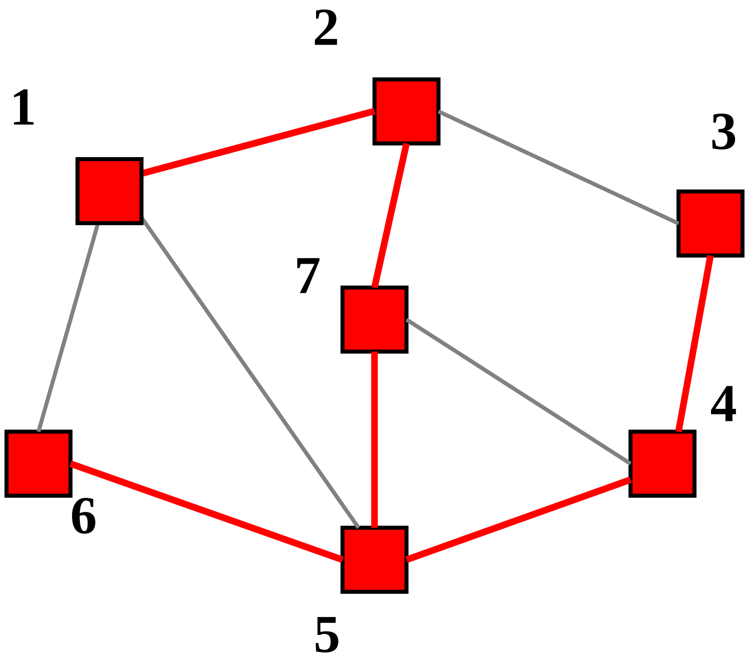
f(2)

f(7)

f(5)

f(4) f(6)

f(3)



Output: (1,2), (2,7), (7,5), (5,4), (4,3), (5,6)

Second Approach

Iterate through edges; output any edge that does not create a cycle

Correctness (hand-wavy):

- Goal is to build an acyclic connected graph
- When we add an edge, it adds a vertex to the tree
  - Else it would have created a cycle
- The graph is connected, so we reach all vertices

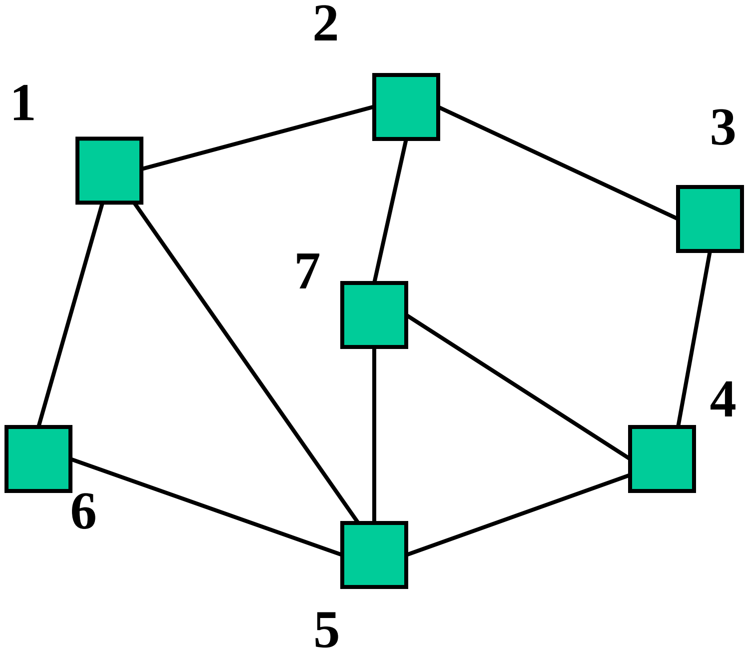
Efficiency:

- Depends on how quickly you can detect cycles
- Reconsider after the example

Example

Edges in some arbitrary order:

(1,2), (3,4), (5,6), (5,7),(1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

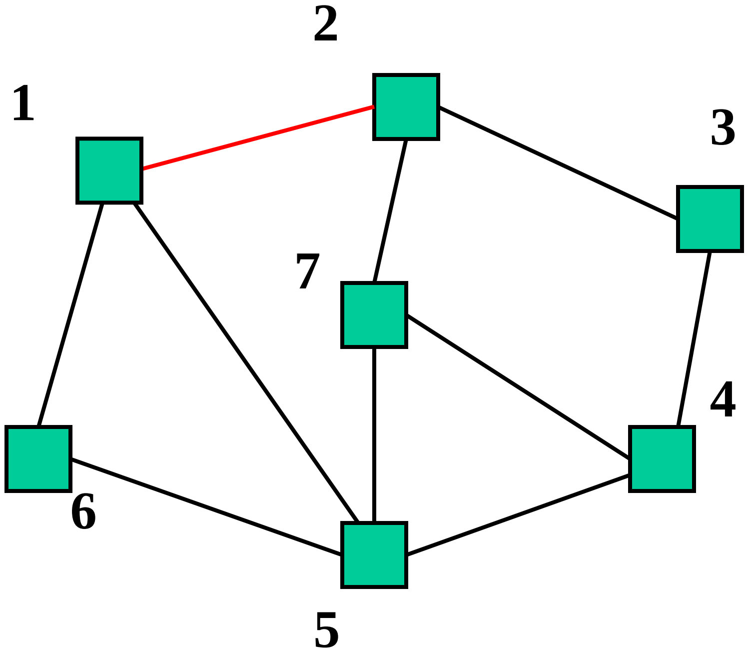


Output:

Example

Edges in some arbitrary order:

(1,2), (3,4), (5,6), (5,7),(1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

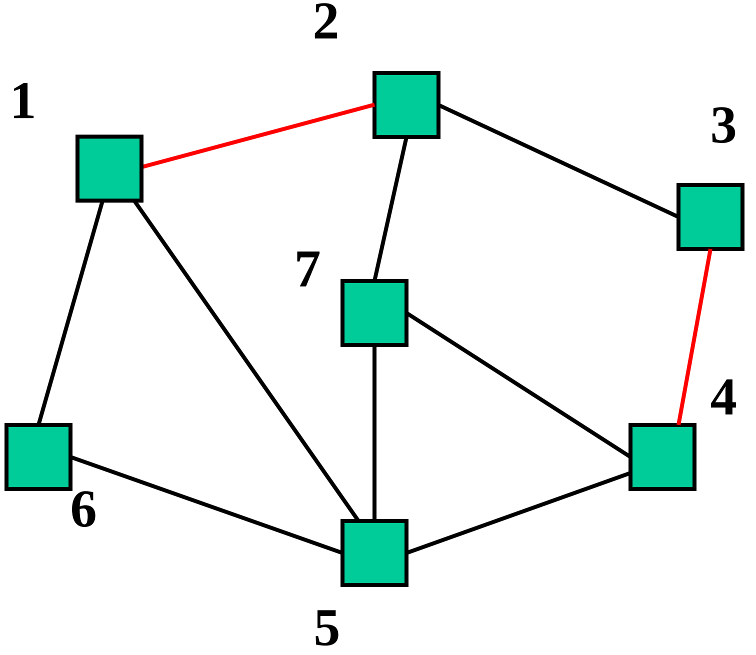


Output: (1,2)

Example

Edges in some arbitrary order:

(1,2), (3,4), (5,6), (5,7),(1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

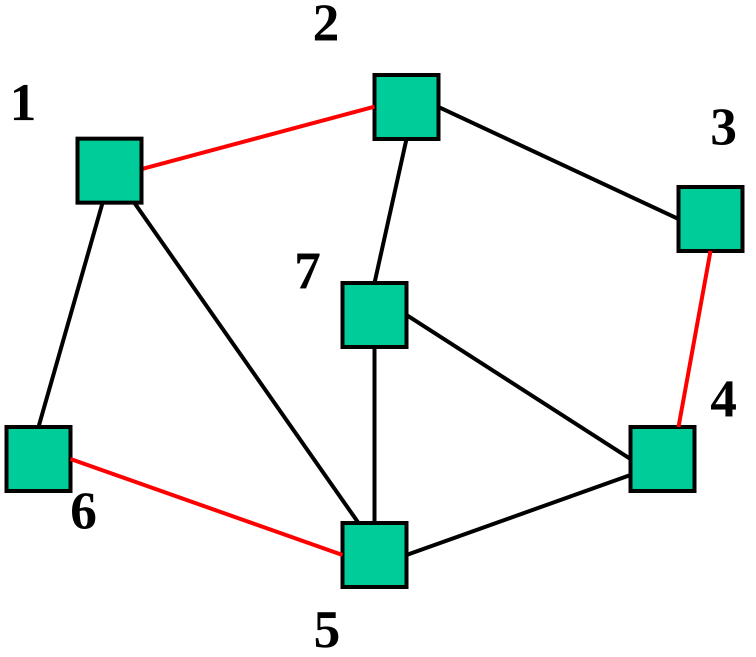


Output: (1,2), (3,4)

Example

Edges in some arbitrary order:

(1,2), (3,4), (5,6), (5,7),(1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

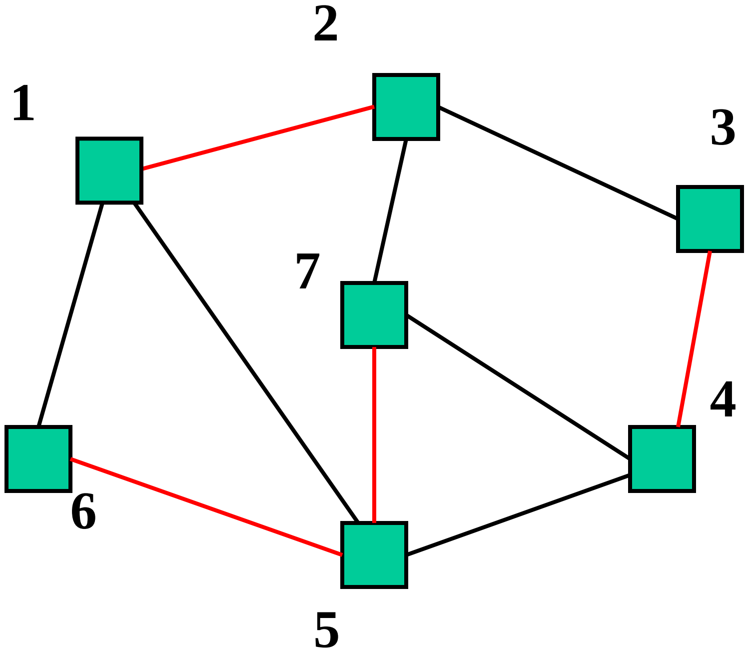


Output: (1,2), (3,4), (5,6),

Example

Edges in some arbitrary order:

(1,2), (3,4), (5,6), (5,7),(1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

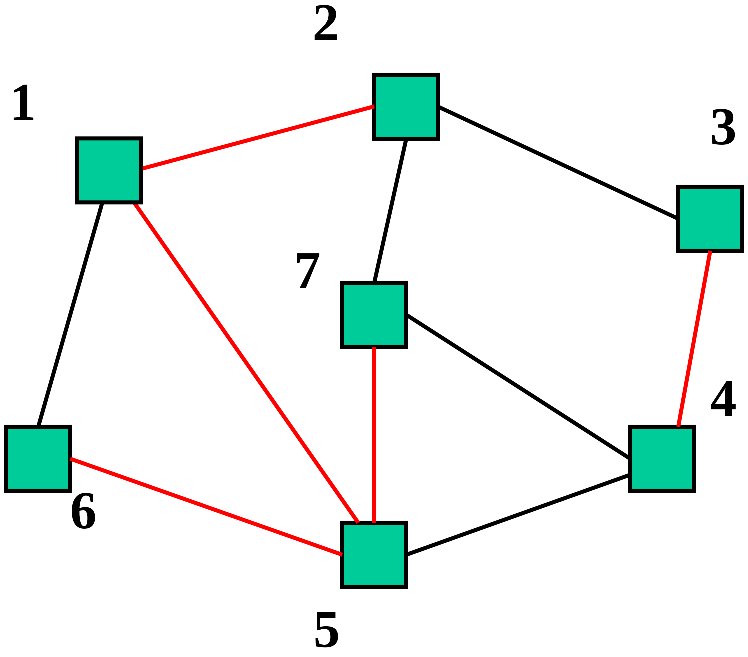


Output: (1,2), (3,4), (5,6), (5,7)

Example

Edges in some arbitrary order:

(1,2), (3,4), (5,6), (5,7), (1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

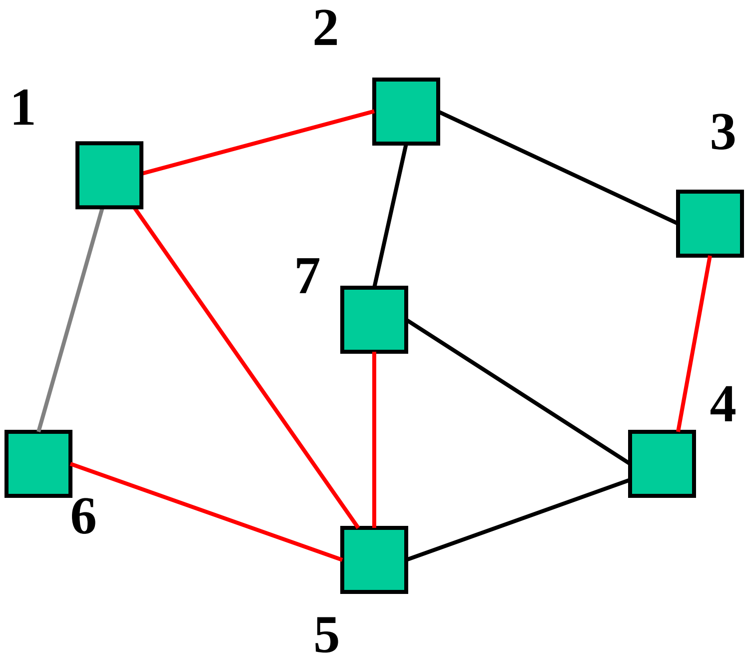


Output: (1,2), (3,4), (5,6), (5,7), (1,5)

Example

Edges in some arbitrary order:

(1,2), (3,4), (5,6), (5,7), (1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

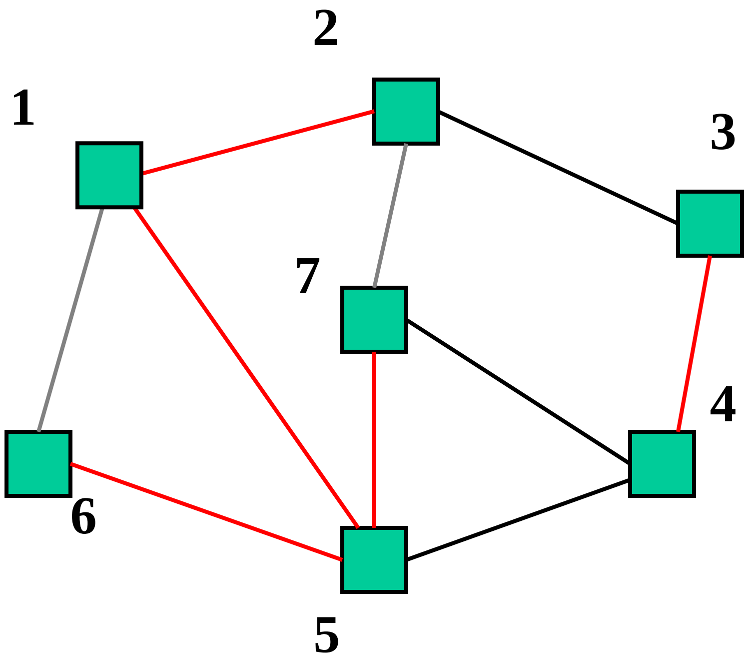


Output: (1,2), (3,4), (5,6), (5,7), (1,5)

Example

Edges in some arbitrary order:

(1,2), (3,4), (5,6), (5,7), (1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

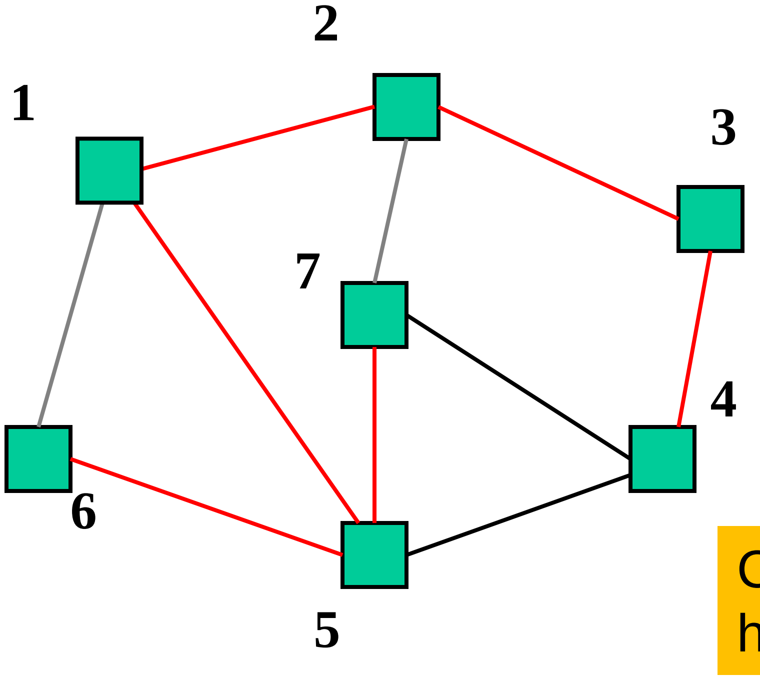


Output: (1,2), (3,4), (5,6), (5,7), (1,5)

Example

Edges in some arbitrary order:

(1,2), (3,4), (5,6), (5,7), (1,5), (1,6), (2,7), (2,3), (4,5), (4,7)



Can stop once we have **|V|-1** edges

Output: (1,2), (3,4), (5,6), (5,7), (1,5), (2,3)

Cycle Detection

- To decide if an edge could form a cycle is $O(|V|)$ because we may need to traverse all edges already in the output
- So overall algorithm would be $O(|V||E|)$
- But there is a faster way we know
- Use union-find!
  - Initially, each item is in its own 1-element set
  - Union sets when we add an edge that connects them
  - Stop when we have one set

Using Disjoint-Sets

Can use a disjoint-set implementation in our spanning-tree algorithm to detect cycles:

Invariant: **u** and **v** are connected in output-so-far

iff

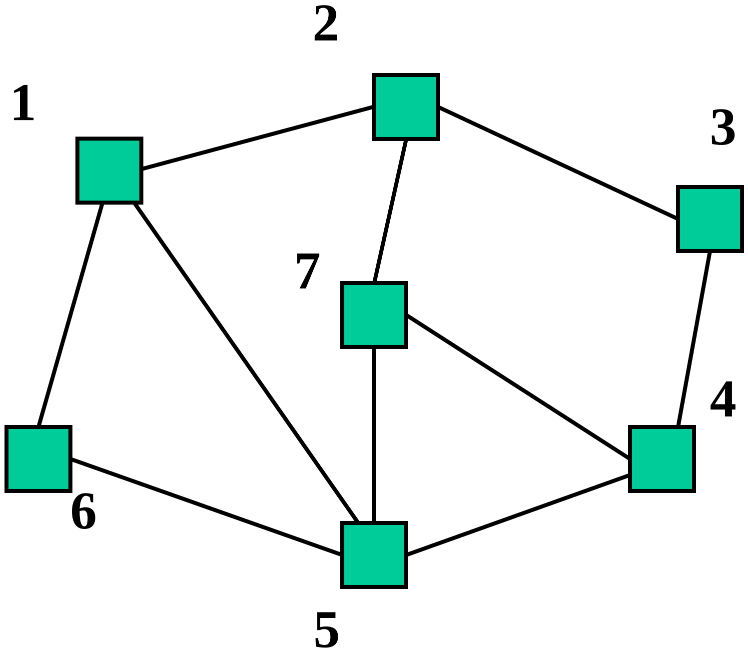
u and **v** in the same set

- Initially, each node is in its own set
- When processing edge **`(u,v)`**:
  - If **`find(u)`** equals **`find(v)`**, then do not add the edge
  - Else add the edge and **`union(find(u),find(v))`**
  - $O(|\mathbf{E}|)$ operations that are almost $O(1)$ amortized

Example

Edges (1,2), (3,4), (5,6), (5,7),(1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

Sets: {1} {2} {3} {4} {5} {6} {7}

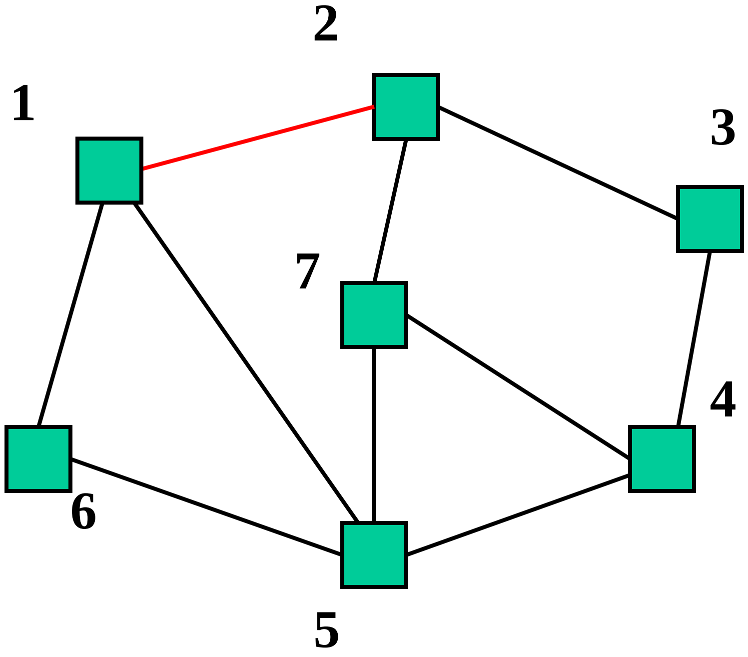


Output:

Example

Edges (1,2), (3,4), (5,6), (5,7),(1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

Sets: {1,2} {3} {4} {5} {6} {7}

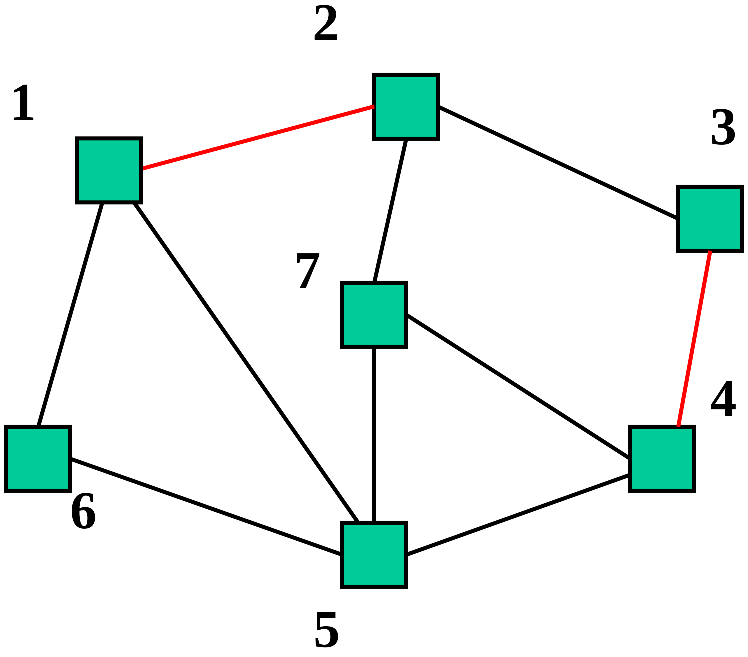


Output: (1,2)

Example

Edges (1,2), (3,4), (5,6), (5,7),(1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

Sets: {1,2} {3,4} {5} {6} {7}

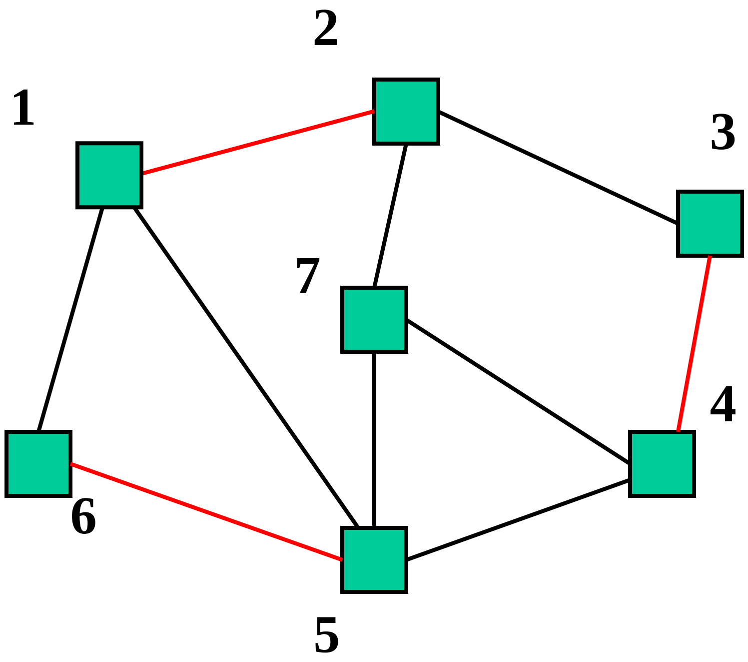


Output: (1,2) (3,4)

Example

Edges (1,2), (3,4), (5,6), (5,7),(1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

Sets: {1,2} {3,4} {5,6} {7}

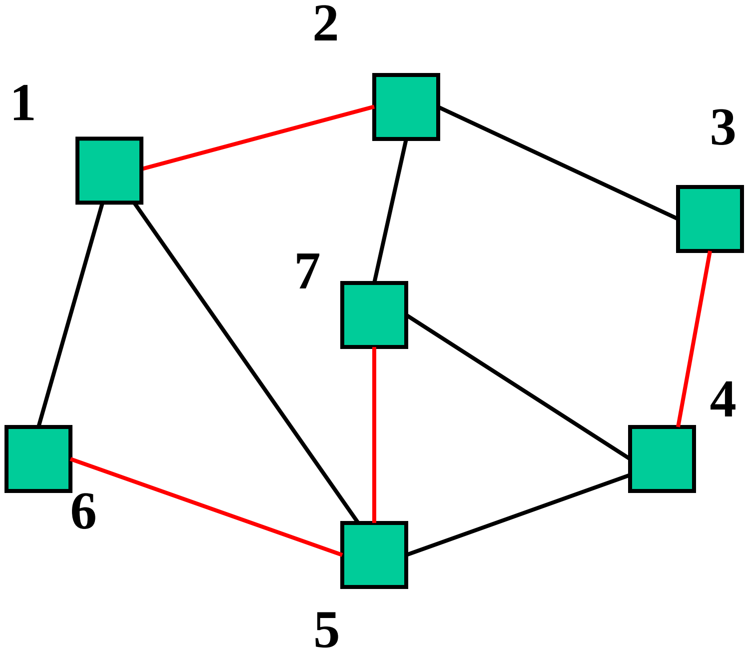


Output: (1,2) (3,4) (5,6)

Example

Edges (1,2), (3,4), (5,6), (5,7),(1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

Sets: {1,2} {3,4} {5,6,7}

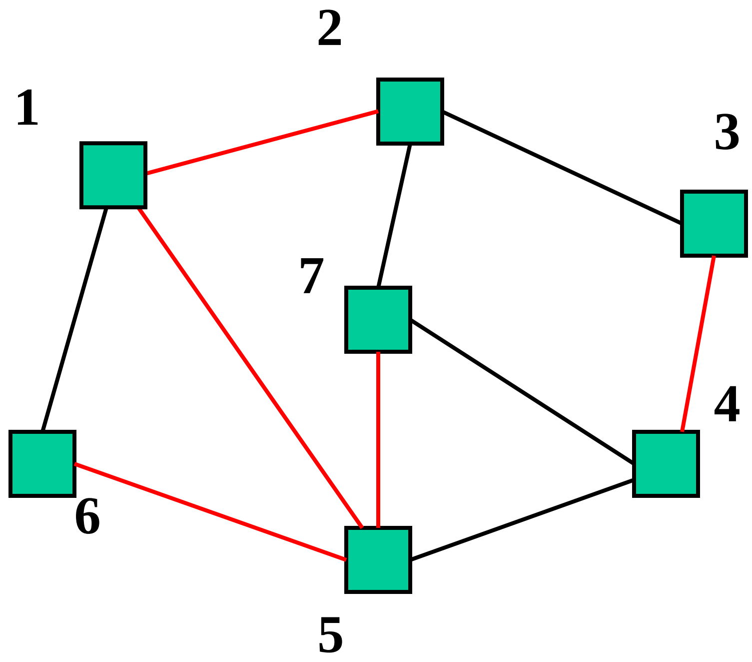


Output: (1,2) (3,4) (5,6) (5,7)

Example

Edges (1,2), (3,4), (5,6), (5,7),(1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

Sets: {3,4} {5,6,7,1,2}

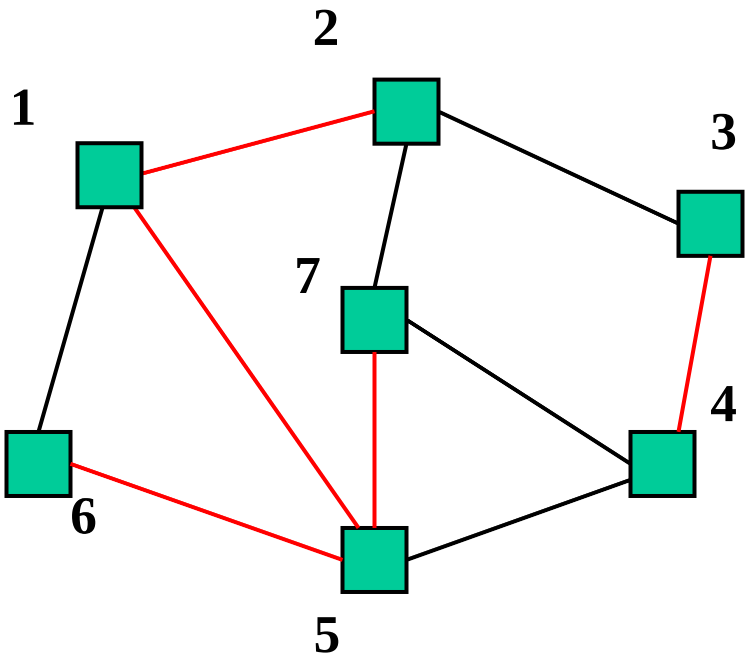


Output: (1,2) (3,4) (5,6) (5,7) (1,5)

Example

Edges (1,2), (3,4), (5,6), (5,7),(1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

Sets: {3,4} {5,6,7,1,2}

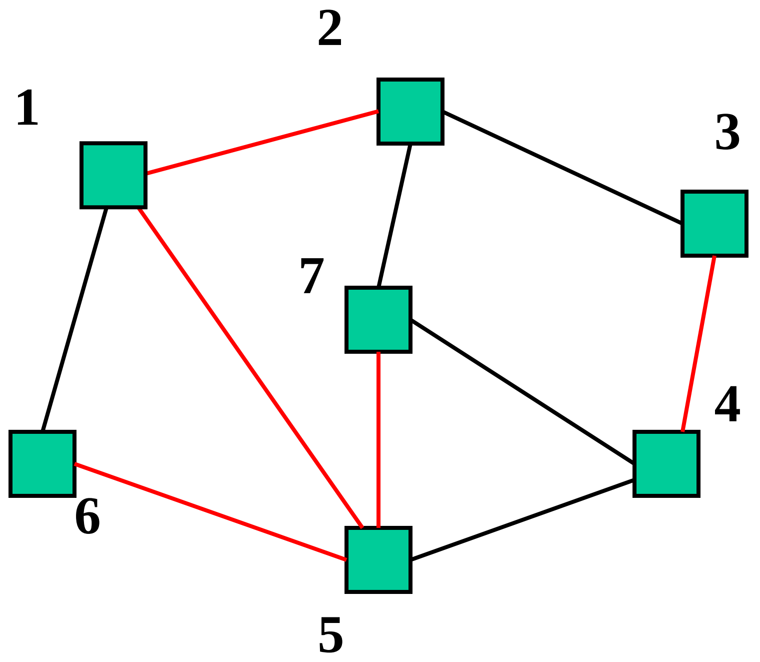


Output: (1,2) (3,4) (5,6) (5,7) (1,5)

Example

Edges (1,2), (3,4), (5,6), (5,7),(1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

Sets: {3,4} {5,6,7,1,2}

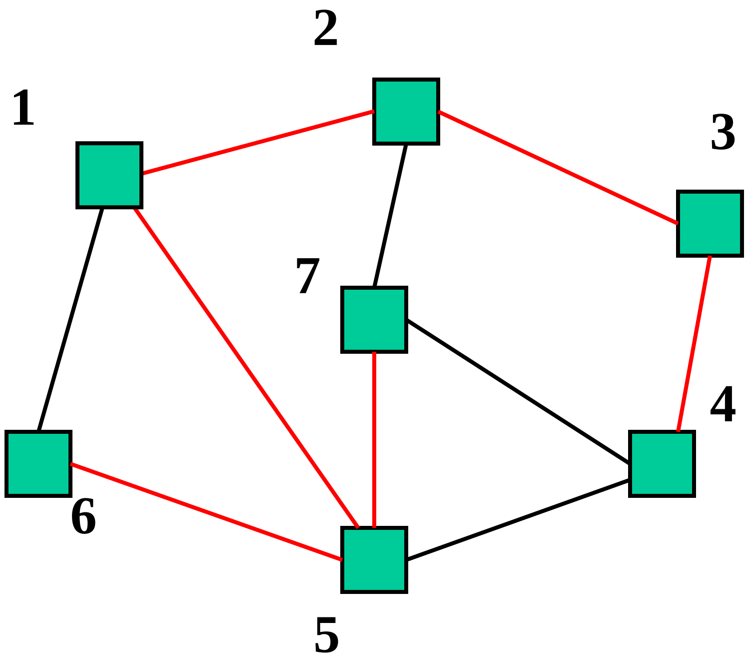


Output: (1,2) (3,4) (5,6) (5,7) (1,5)

Example

Edges (1,2), (3,4), (5,6), (5,7),(1,5), (1,6), (2,7), (2,3), (4,5), (4,7)

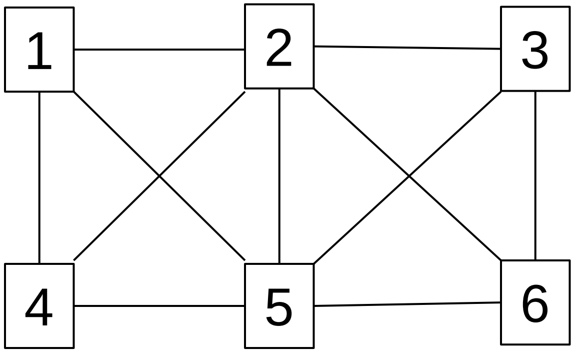
Sets: {3,4, 5,6,7,1,2}



Output: (1,2) (3,4) (5,6) (5,7) (1,5) (2,3)

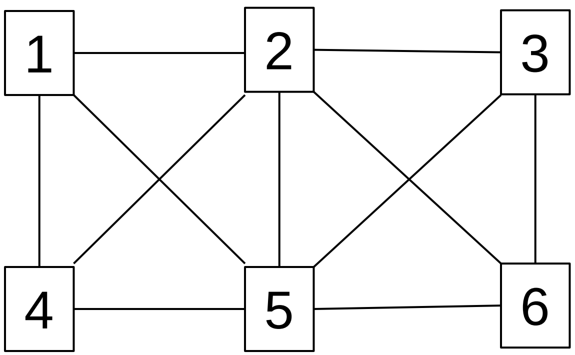
Practice Problem

Edges in arbitrary order:
(2,5) (2,3) (1,2) (1,4) (2,4) (3,6) (3,5) (1,5) (2,6) (4,5) (5,6)



Practice Problem

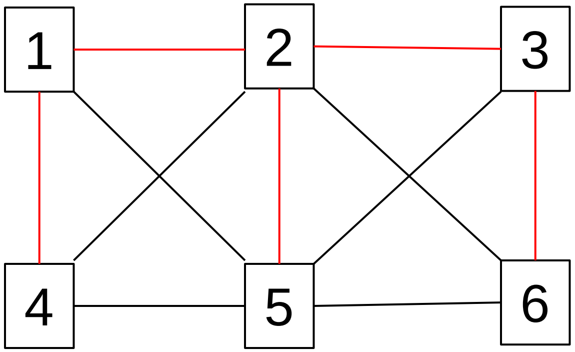
Edges in arbitrary order:
(2,5) (2,3) (1,2) (1,4) (2,4) (3,6) (3,5) (1,5) (2,6) (4,5) (5,6)



(2,5)	{2,5}
(2,3)	{2,3,5}
(1,2)	{1,2,3,5)
(1,4)	{1,2,3,4,5}
(2,4)	{1,2,3,4,5}
(3,6)	{1,2,3,4,5,6}

Practice Problem

Edges in arbitrary order:
(2,5) (2,3) (1,2) (1,4) (2,4) (3,6) (3,5) (1,5) (2,6) (4,5) (5,6)



(2,5)
(2,3)
(1,2)
(1,4)
(2,4)
(3,6)

Summary So Far

The spanning-tree problem

- Add nodes to partial tree approach is O(**|E|**)
- Add acyclic edges approach is *almost* O(**|E|**)
  - Using union-find "as a black box"

But really want to solve the minimum-spanning-tree problem

- Given a weighted undirected graph, give a spanning tree of minimum weight
- Same two approaches will work with minor modifications
- Both will be $O(|E| \log |V|)$

Minimum Spanning Tree Algorithms

Algorithm #1

Shortest-path is to Dijkstra's Algorithm

as

Minimum Spanning Tree is to Prim's Algorithm

(Both based on expanding cloud of known vertices, basically using a priority queue instead of a DFS stack)

Algorithm #2

Kruskal's Algorithm for Minimum Spanning Tree

is

Exactly our 2nd approach to spanning tree

but process edges in cost order