CSE373: Data Structures & Algorithms

Lecture 10: Disjoint Sets and the Union-Find ADT

Linda Shapiro

Spring 2016

Announcements

- Get started on HW03
  - Keyword search in binary search trees

Where we are

Last lecture:

- Priority queues and binary heaps

Today:

- Disjoint sets
- The union-find ADT for disjoint sets

Disjoint sets

- A set is a collection of elements (no-repeats)

- In computer science, two sets are said to be disjoint if they have no element in common.
  - $\mathbf{S_1 \cap S_2 = \varnothing}$

- For example, {1, 2, 3} and {4, 5, 6} are disjoint sets.
- For example, {x, y, z} and {t, u, x} are not disjoint.

Partitions

A partition *P* of a set *S* is a set of sets {*S1*,*S2*,…,*Sn*} such that every element of *S* is in **exactly one** *Si*

Formally:

- $S_1 \cup S_2 \cup \ldots \cup S_k = S$
- $i \neq j$ implies $S_i \cap S_j = \emptyset$ (sets are disjoint with each other)

Example:

- Let *S* be {a,b,c,d,e}
- One partition: {a}, {d,e}, {b,c}
- Another partition: {a,b,c}, ∅, {d}, {e}
- A third: {a,b,c,d,e}
- Not a partition: {a,b,d}, {c,d,e} *…. element d appears twice*
- Not a partition of *S*: {a,b}, {e,c} *…. missing element d*

Binary relations

- *S x S* is the set of all pairs of elements of *S* (Cartesian product)
  - Example: If *S* = {a,b,c}
    then *S x S* = {(a,a),(a,b),(a,c),(b,a),(b,b),(b,c), (c,a),(c,b),(c,c)}
- A binary relation *R* on a set *S* is any subset of *S x S*
  - i.e. a collection of ordered pairs of elements of S.
  - Write *R*(x,y) to mean (x,y) is "in the relation"
  - (Unary, ternary, quaternary, … relations defined similarly)
- Examples for *S* = people-in-this-room
  - Sitting-next-to-each-other relation
  - First-sitting-right-of-second relation
  - Went-to-same-high-school relation
  - First-is-younger-than-second relation

Properties of binary relations

- A relation *R* over set *S* is reflexive means *R*(a,a) for *all* a in *S*
  - e.g. The relation “<=“ on the set of integers {1, 2, 3} is

    {<1, 1>, <1, 2>, <1, 3>, <2, 2>, <2, 3>, <3, 3>}

    It is reflexive because <1, 1>, <2, 2>, <3, 3> are in this relation.

- A relation R on a set S is symmetric if and only if for any a and b in S, whenever <a, b> is in R , <b, a> is in R .
  - e.g. The relation “=“ on the set of integers {1, 2, 3} is

    {<1, 1> , <2, 2> <3, 3> } and it is symmetric.
  - The relation "being acquainted with" on a set of people is symmetric.

- A binary relation *R* over set *S* is transitive means:

  If *R*(a,b) and *R*(b,c) then *R*(a,c) for *all* a,b,c in *S*
  - e.g. The relation “<=“ on the set of integers {1, 2, 3} is transitive, because for <1, 2> and <2, 3> in “<=“, <1, 3> is also in “<=“ (and similarly for the others)

Equivalence relations

- A binary relation *R* is an equivalence relation if *R* is reflexive, symmetric, *and* transitive

- Examples
  - Same gender
  - Connected roads in the world
  - "Is equal to" on the set of real numbers
  - "Has the same birthday as" on the set of all people
  - …

Punch-line

- Equivalence relations give rise to partitions.

- Every partition induces an equivalence relation
- Every equivalence relation induces a partition

- Suppose $P=\{S1,S2,\ldots,Sn\}$ is a partition
  - Define $R(x,y)$ to mean x and y are in the same Si
    - R is an equivalence relation

- Suppose R is an equivalence relation over S
  - Consider a set of sets S1,S2,…,Sn where

    (1) x and y are in the same Si if and only if R(x,y)

    (2) Every x is in some Si
    - This set of sets is a partition

Example

- Let *S* be {a,b,c,d,e}
- One partition: {a,b,c}, {d}, {e}
- The corresponding equivalence relation:

  (a,a), (b,b), (c,c), (a,b), (b,a), (a,c), (c,a), (b,c), (c,b), (d,d), (e,e)

The Union-Find ADT

- The union-find ADT (or "Disjoint Sets" or "Dynamic Equivalence Relation") keeps track of a set of elements partitioned into a number of disjoint subsets.

- Many uses (which is why an ADT taught in CSE 373):
  - Road/network/graph connectivity (will see this again)
    - “connected components” e.g., in social network
  - Partition an image by connected-pixels-of-similar-color
  - Type inference in programming languages

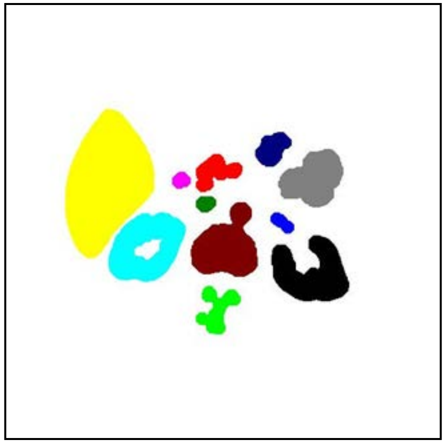
- Not as common as dictionaries, queues, and stacks, but valuable because implementations are very fast, so when applicable can provide big improvements

Connected Components of an Image

gray tone image

binary image

cleaned up



components

Union-Find Operations

- Given an unchanging set *S*, **`create`** an initial partition of a set
  - Typically each item in its own subset: {a}, {b}, {c}, …
  - Give each subset a "name" by choosing a *representative element*

- Operation **`find`** takes an element of *S* and returns the representative element of the subset it is in

- Operation **`union`** takes two subsets and (permanently) makes one larger subset
  - A different partition with one fewer set
  - Affects result of subsequent **`find`** operations
  - Choice of representative element up to implementation

Example

- Let *S* = {1,2,3,4,5,6,7,8,9}
- Let initial partition be (will highlight representative elements red)

  {1}, {2}, {3}, {4}, {5}, {6}, {7}, {8}, {9}
- **`union`**(2,5):

  {1}, {2, 5}, {3}, {4}, {6}, {7}, {8}, {9}
- **`find`**(4) = 4, **`find`**(2) = 2, **`find`**(5) = 2
- **`union`**(4,6), **`union`**(2,7)

  {1}, {2, 5, 7}, {3}, {4, 6}, {8}, {9}
- **`find`**(4) = 6, **`find`**(2) = 2, **`find`**(5) = 2
- **`union`**(2,6)

  {1}, {2, 4, 5, 6, 7}, {3}, {8}, {9}

No other operations

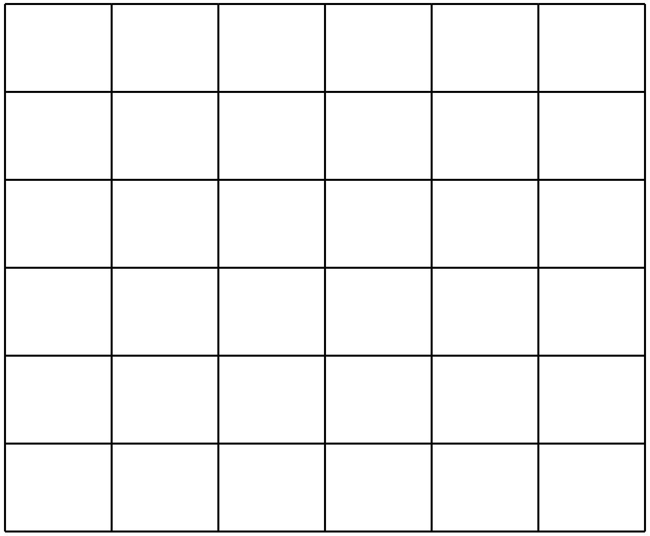
- All that can “happen” is sets get unioned
  - No “un-union” or “create new set” or …

- As always: trade-offs
  - Implementations will exploit this small ADT

- Surprisingly useful ADT
  - But not as common as dictionaries or priority queues

Example application: maze-building

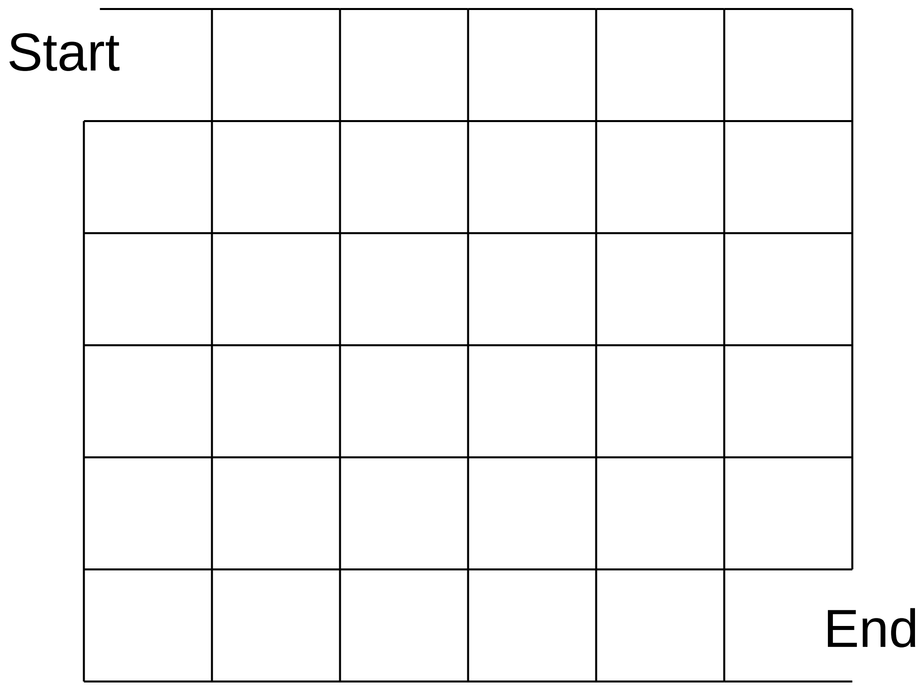
- Build a random maze by erasing edges



  - Possible to get from anywhere to anywhere
    - Including “start” to “finish”
  - No loops possible without backtracking
    - After a “bad turn” have to “undo”

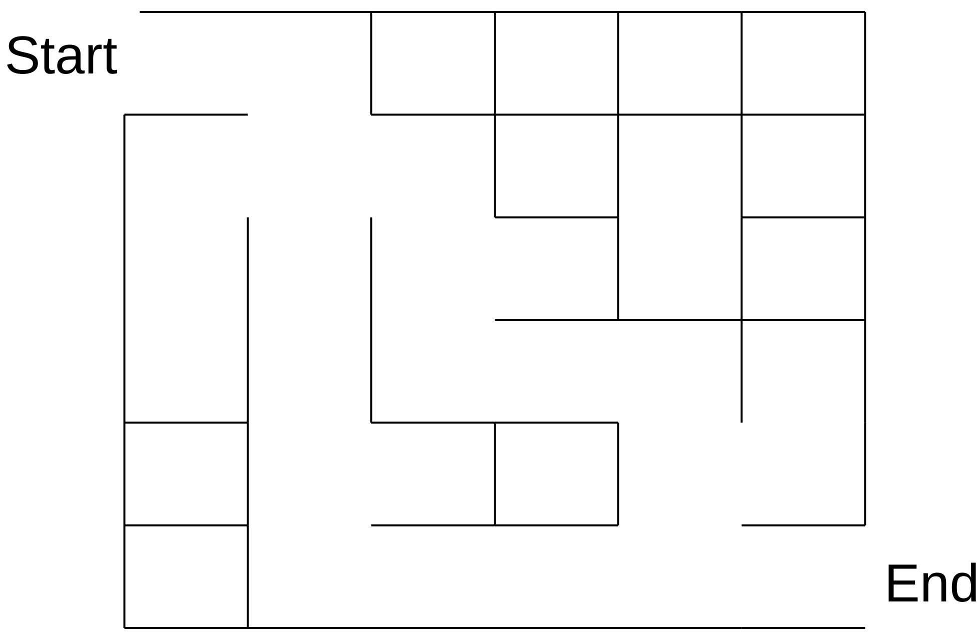
Maze building

Pick start edge and end edge



Repeatedly pick random edges to delete

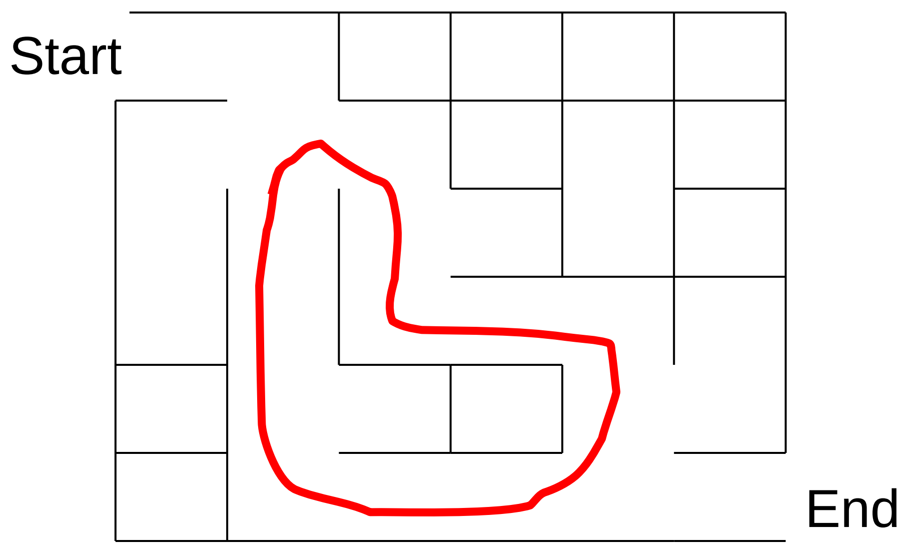
One approach: just keep deleting random edges until you can get from start to finish



Problems with this approach

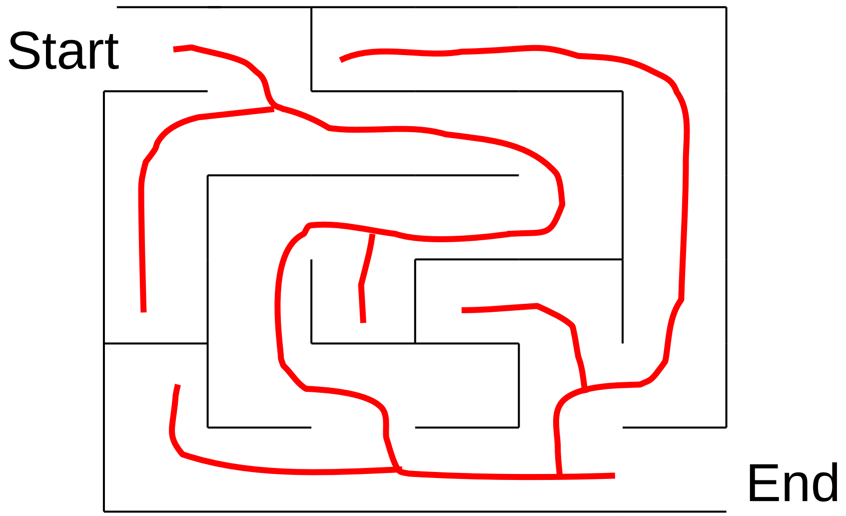
1. How can you tell when there is a path from start to finish?
   – We do not really have an algorithm yet

2. We could have *cycles*, which a “good” maze avoids
   – Want one solution and no cycles



Revised approach

- Consider edges in random order (i.e. pick an edge)
- Only delete an edge if it introduces no cycles (how? TBD)
- When done, we will have a way to get from any place to any other place (including from start to end points)



Cells and edges

- Let's number each cell
  - 36 total for 6 x 6
- An (internal) edge (x,y) is the line between cells x and y
  - 60 total for 6x6: (1,2), (2,3), …, (1,7), (2,8), …

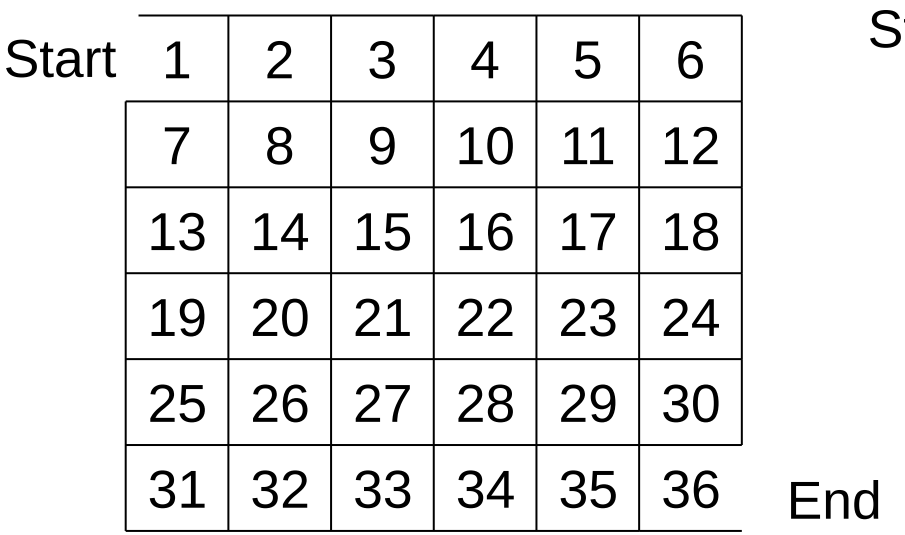
Start

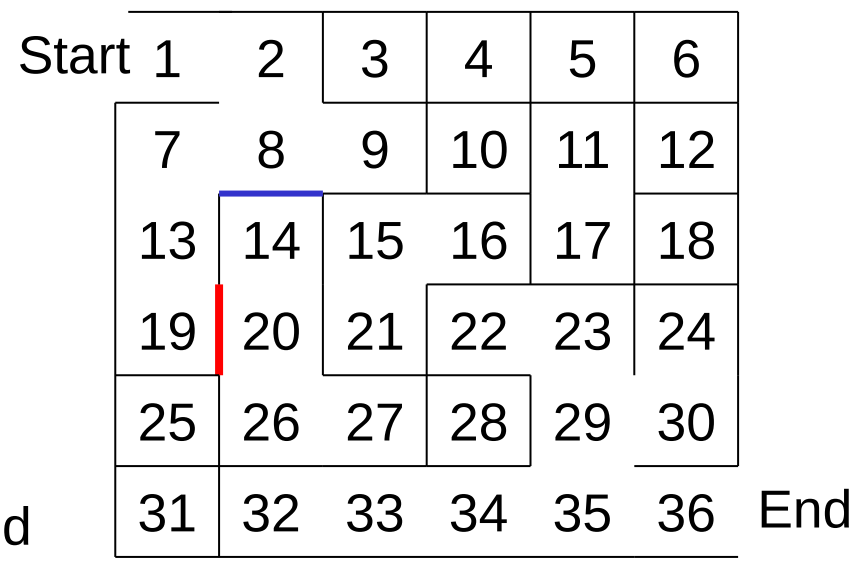
1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

End

The trick

- Partition the cells into disjoint sets
  - Two cells in same set if they are “connected”
  - Initially every cell is in its own subset
- If removing an edge would connect two different subsets:
  - then remove the edge and **union** the subsets
  - else leave the edge because removing it makes a cycle





The algorithm

- P = **disjoint sets** of connected cells
  initially each cell in its own 1-element set
- E = **set** of edges not yet processed, initially all (internal) edges
- M = **set** of edges kept in maze (initially empty)

while P has more than one set {

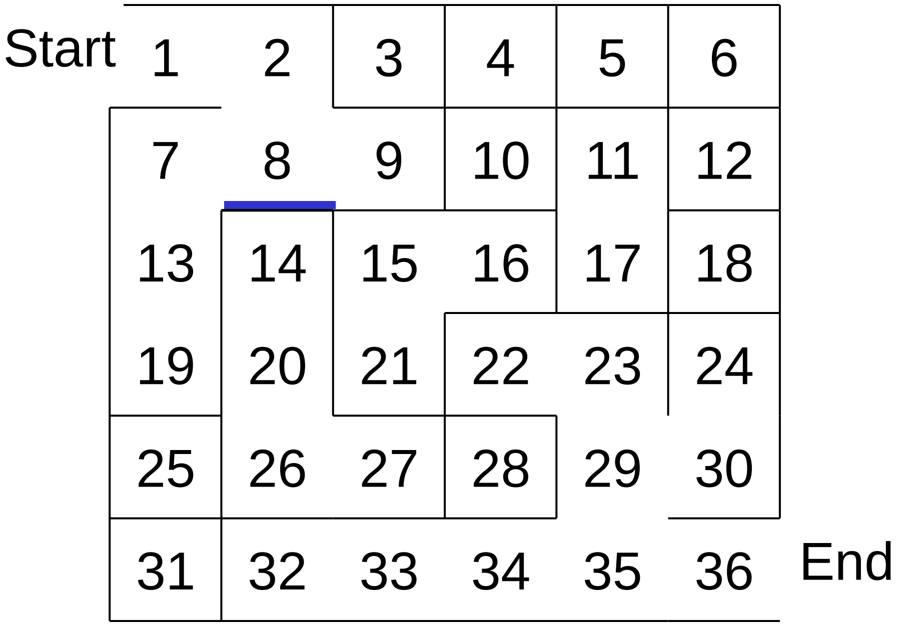
- Pick a random edge (x,y) to remove from E
- u = **`find`**(x)
- v = **`find`**(y)
- if u==v
  add (x,y) to M // same subset, do not remove edge, do not create cycle
  else
  `union`(u,v) // connect subsets, do not put edge in M

}

Add remaining members of E to M, then output M as the maze

Example at some step

Pick edge (8,14)



P

{1,2,7,8,9,13,19}

{3}

{4}

{5}

{6}

{10}

{11,17}

{12}

{14,20,26,27}

{15,16,21}

{18}

{25}

{28}

{31}

{22,23,24,29,30,32
 33,34,35,36}

Example

P
{1,2,7,8,9,13,19}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{14,20,26,27}
{15,16,21}
{18}
{25}
{28}
{31}
{22,23,24,29,30,32,33,34,35,36}

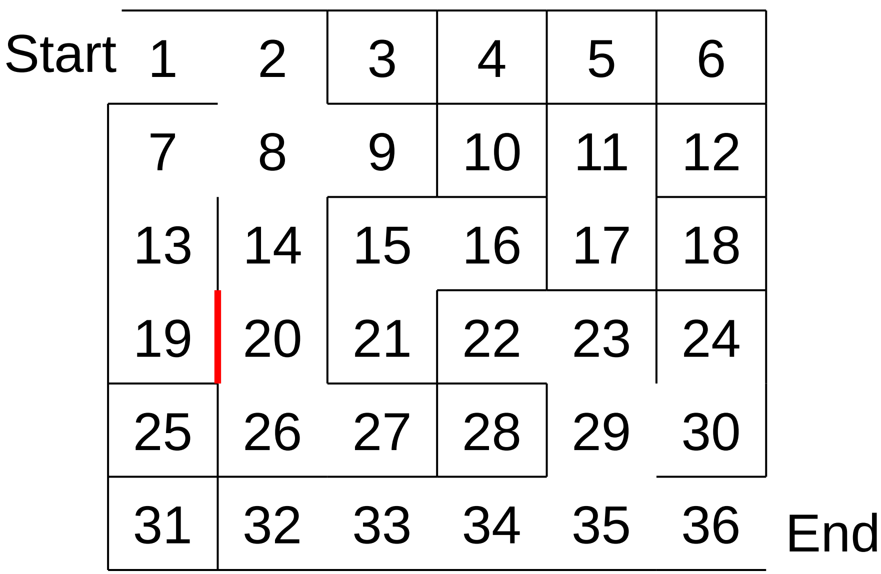
Find(8) = 7
Find(14) = 20

Union(7,20)

P
{1,2,7,8,9,13,19,14,20,26,27}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{15,16,21}
{18}
{25}
{28}
{31}
{22,23,24,29,30,32,33,34,35,36}

Example: Add edge to M step

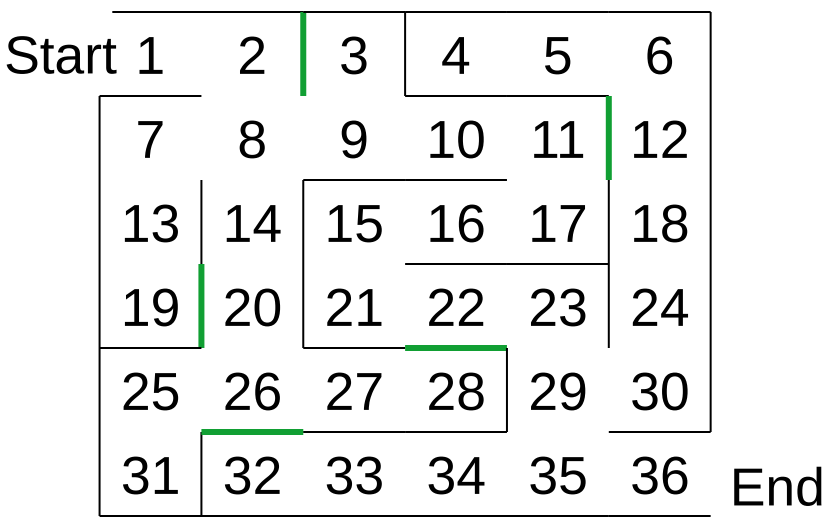
Pick edge (19,20)
Find (19) = 7
Find (20) = 7
Add (19,20) to M



P
{1,2,7,8,9,13,19,14,20,26,27}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{15,16,21}
{18}
{25}
{28}
{31}
{22,23,24,29,30,32
33,34,35,36}

At the end

- Stop when P has one set (i.e. all cells connected)
- Suppose green edges are already in M and black edges were not yet picked
  - Add all black edges to M



P
{1,2,3,4,5,6,7,… 36}

Done! ☺

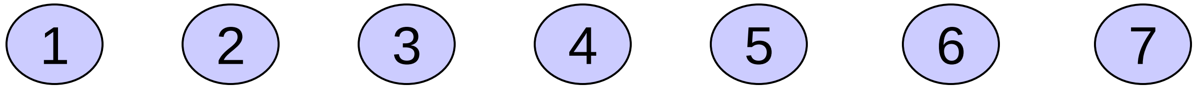
A data structure for the union-find ADT

- Start with an initial partition of *n* subsets
  - Often 1-element sets, e.g., {1}, {2}, {3}, …, {*n*}

- May have any number of **`find`** operations
- May have up to *n*-1 **`union`** operations in any order
  - After *n*-1 **`union`** operations, every **`find`** returns same 1 set

Teaser: the up-tree data structure

- Tree structure with:
  - No limit on branching factor
  - References from children to parent
- Start with *forest* of 1-node trees



- Possible forest after several unions:
  - Will use roots for set names