CSE373: Data Structures & Algorithms
Lecture 14: Hash Collisions

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Spring 2015

Announcements

- Project 3 is due today at 11pm
- Project 4 is out tonight, due Wed. May 13 at 11pm
  - You can work with a partner for project 4
  - Partner selection due next Wednesday
- Midterm next Wednesday in class

Midterm, in-class Wednesday May 6th

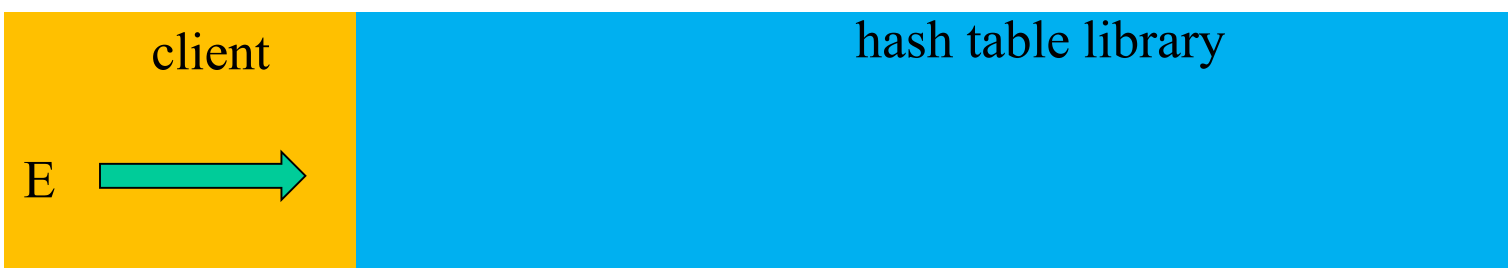
- In class, closed notes, closed book.

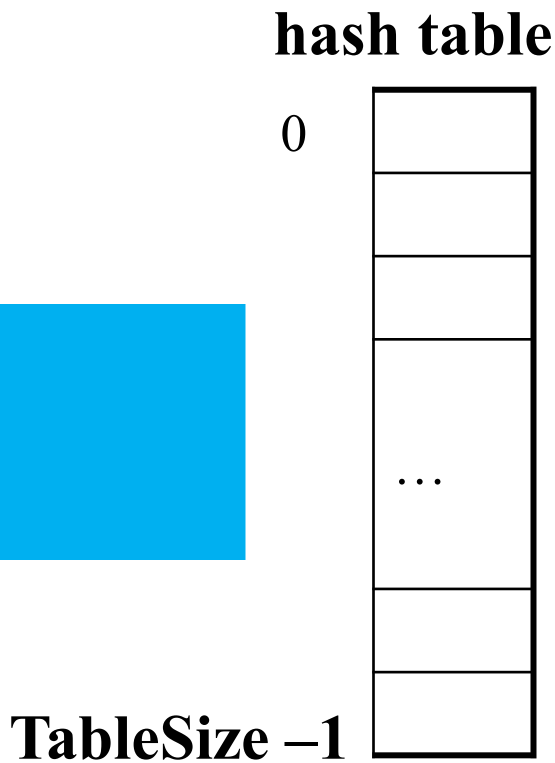
- Covers everything up to and including hashing.
  – Stacks, queues
  – Induction
  – Asymptotic analysis and Big-Oh
  – Dictionaries, BSTs, AVL Trees
  – Binary heaps and Priority Queues
  – Disjoint sets and Union-Find
  – Hash Tables and Collisions

- Information, sample past exams and solutions posted online.

Hash Tables: Review

- Aim for constant-time (i.e., *O*(1)) **`find`**, **`insert`**, and **`delete`**
  - "On average" under some reasonable assumptions
- A hash table is an array of some fixed size
  - But growable as we'll see





Collision resolution

Collision:

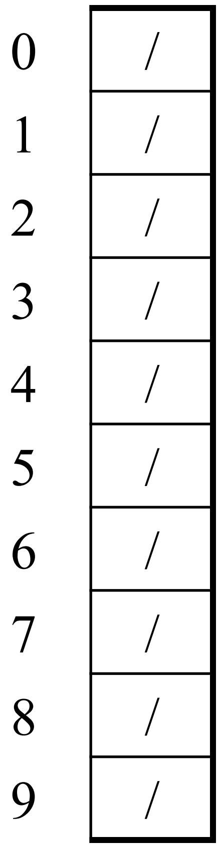
When two keys map to the same location in the hash table

We try to avoid it, but number-of-keys exceeds table size

So hash tables should support collision resolution

– Ideas?

Separate Chaining

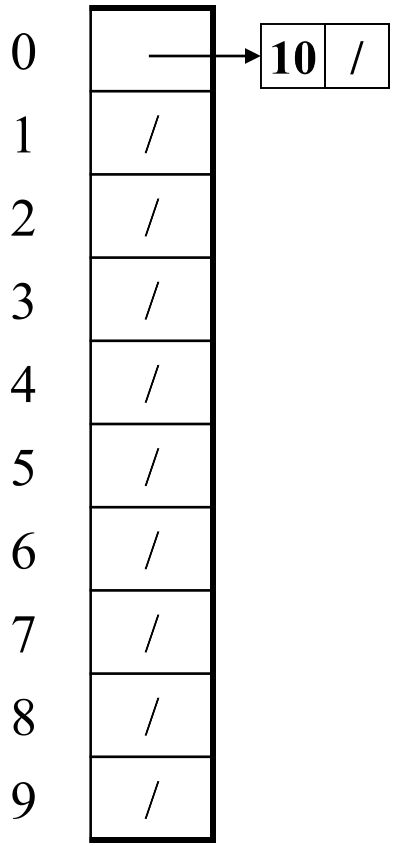


Chaining:
  All keys that map to the same table location are kept in a list (a.k.a. a "chain" or "bucket")

As easy as it sounds

Example:
  insert 10, 22, 107, 12, 42
  with mod hashing
  and **`TableSize`** = 10

Separate Chaining



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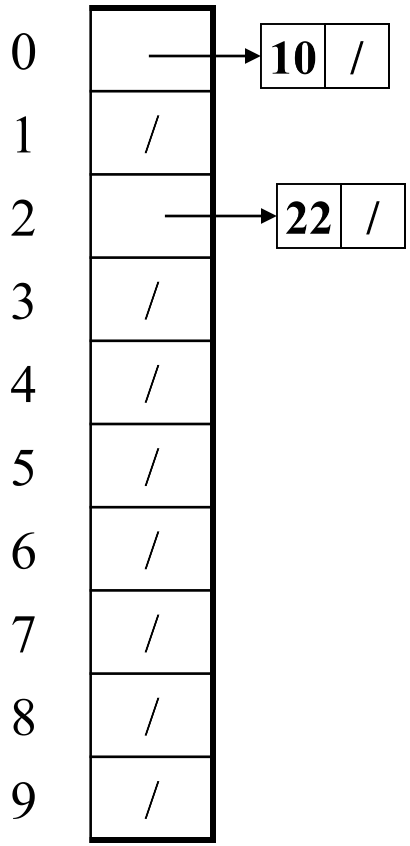

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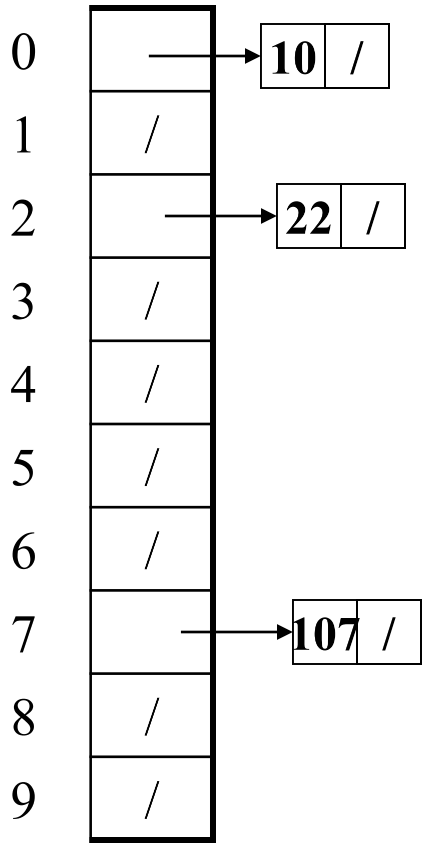

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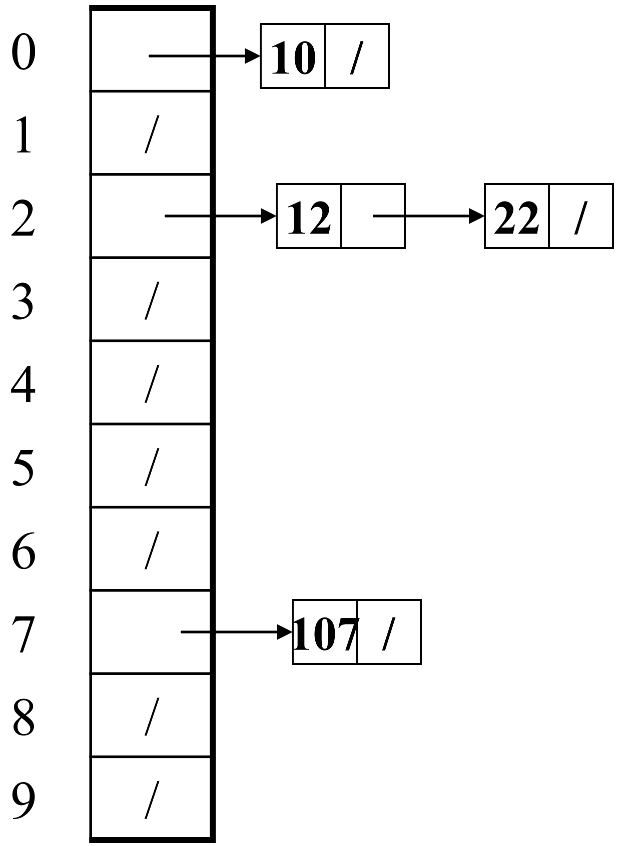

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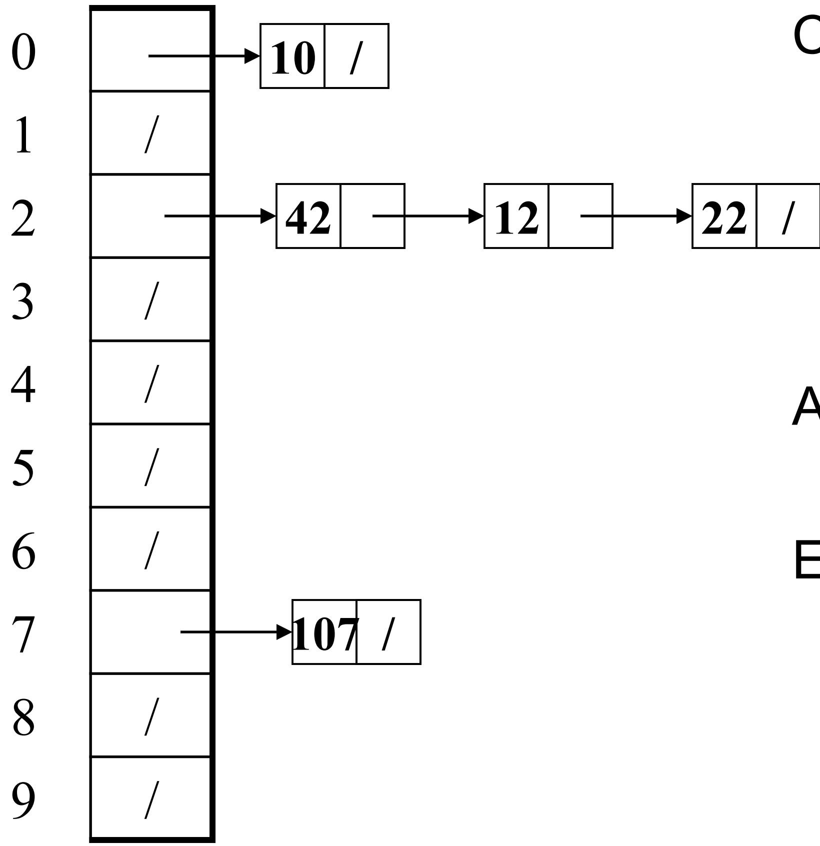

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As easy as it sounds

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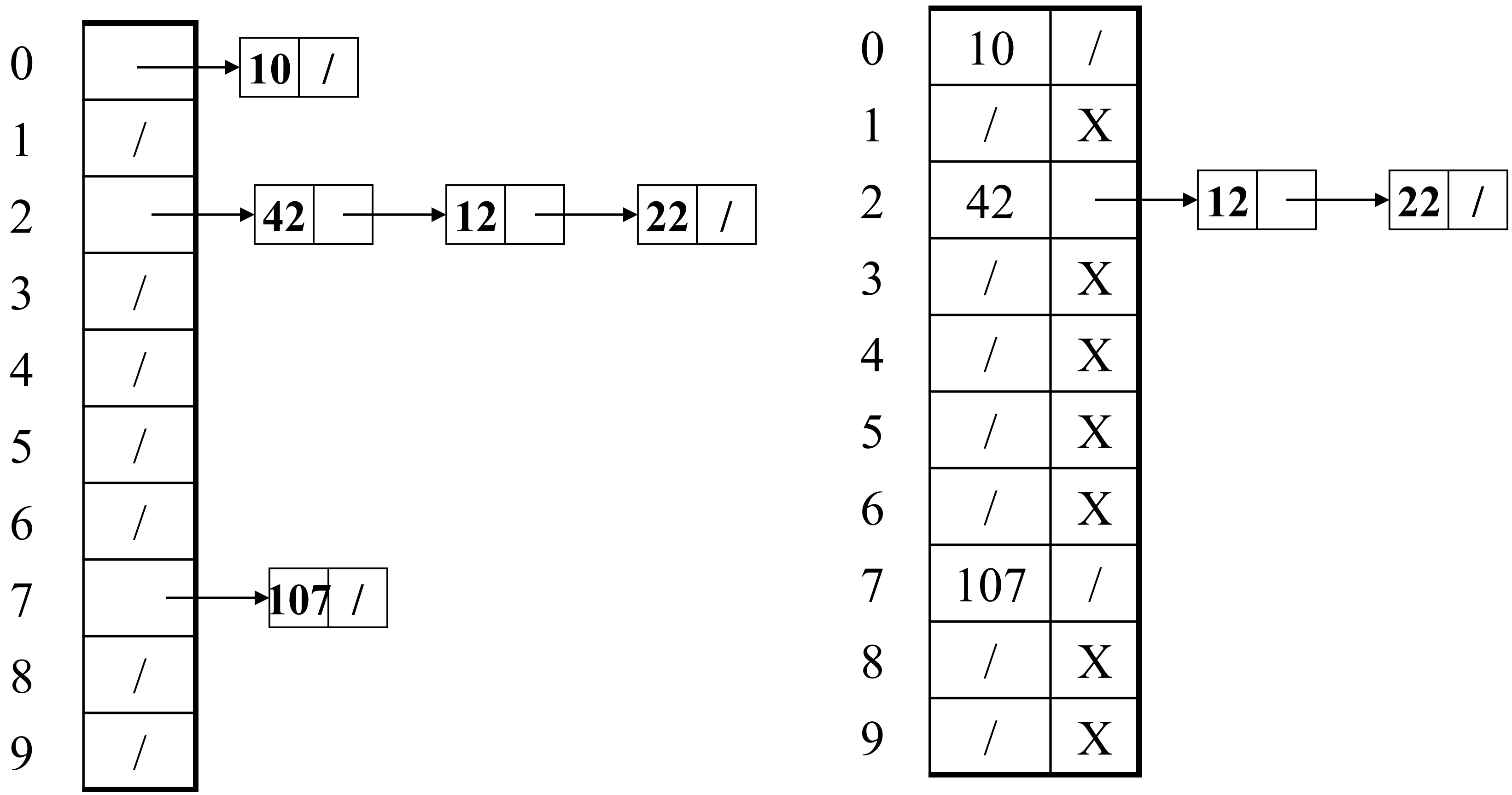
with mod hashing
and **`TableSize`** = 10

Thoughts on chaining

- Worst-case time for **`find`**?
  - Linear
  - But only with really bad luck or bad hash function
  - So not worth avoiding (e.g., with balanced trees at each bucket)

- Beyond asymptotic complexity, some "data-structure engineering" may be warranted
  - Linked list vs. array vs. chunked list (lists should be short!)
  - Move-to-front
  - Maybe leave room for 1 element (or 2?) in the table itself, to optimize constant factors for the common case
    - A time-space trade-off…

Time vs. space (constant factors only here)



More rigorous chaining analysis

Definition: The load factor, λ, of a hash table is

$$\lambda = \frac{\text{N}}{\text{TableSize}}$$ **← number of elements**

Under chaining, the average number of elements per bucket is ___

More rigorous chaining analysis

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Under chaining, the average number of elements per bucket is λ

So if some inserts are followed by *random* finds, then on average:

- Each unsuccessful **`find`** compares against ____ items

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So if some inserts are followed by *random* finds, then on average:

- Each unsuccessful **`find`** compares against λ items
- Each successful **`find`** compares against _____ items

More rigorous chaining analysis

Definition: The load factor, λ, of a hash table is

$$\lambda = \frac{\mathrm{N}}{\mathrm{TableSize}} \qquad \leftarrow \textbf{number of elements}$$

Under chaining, the average number of elements per bucket is λ

So if some inserts are followed by *random* finds, then on average:

- Each unsuccessful **`find`** compares against λ items
- Each successful **`find`** compares against $\lambda / 2$ items

So we like to keep λ fairly low (e.g., 1 or 1.5 or 2) for chaining

Alternative: Use empty space in the table

- Another simple idea: If `h(key)` is already full,
  - try `(h(key) + 1) % TableSize`. If full,
  - try `(h(key) + 2) % TableSize`. If full,
  - try `(h(key) + 3) % TableSize`. If full…

- Example: insert 38, 19, 8, 109, 10

0	/
1	/
2	/
3	/
4	/
5	/
6	/
7	/
8	38
9	/

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1	/
2	/
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4	/
5	/
6	/
7	/
8	38
9	19

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- Example: insert 38, 19, 8, 109, 10

0	8
1	/
2	/
3	/
4	/
5	/
6	/
7	/
8	38
9	19

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0	8
1	109
2	10
3	/
4	/
5	/
6	/
7	/
8	38
9	19

Probing hash tables

Trying the next spot is called probing (also called open addressing)

- We just did linear probing
  - **i**th probe was **`(h(key) + i) % TableSize`**
- In general have some probe function **`f`** and use **`h(key) + f(i) % TableSize`**

Open addressing does poorly with high load factor λ

- So want larger tables
- Too many probes means no more $O(1)$

Other operations

`insert` finds an open table position using a probe function

What about `find`?

- Must use same probe function to "retrace the trail" for the data
- Unsuccessful search when reach empty position

What about `delete`?

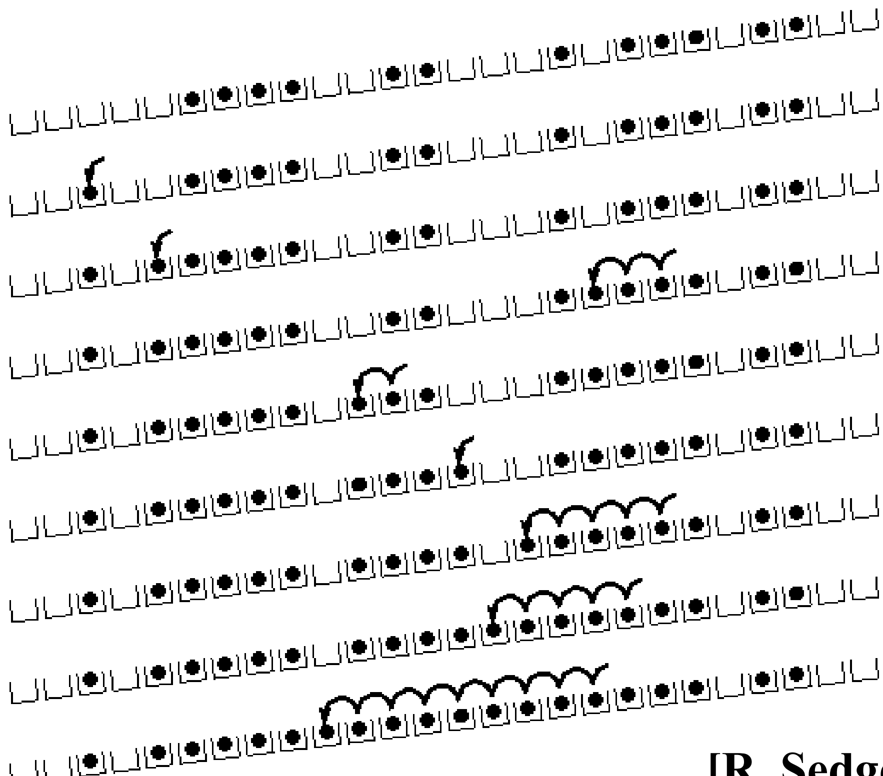
- ***Must*** use "lazy" deletion. Why?
  - Marker indicates "no data here, but don't stop probing"
- Note: `delete` with chaining is plain-old list-remove

(Primary) Clustering

It turns out linear probing is a *bad idea*, even though the probe function is quick to compute (which is a good thing)

Tends to produce *clusters*, which lead to long probing sequences

- Called primary clustering
- Saw this starting in our example



[R. Sedgewick]

Analysis of Linear Probing

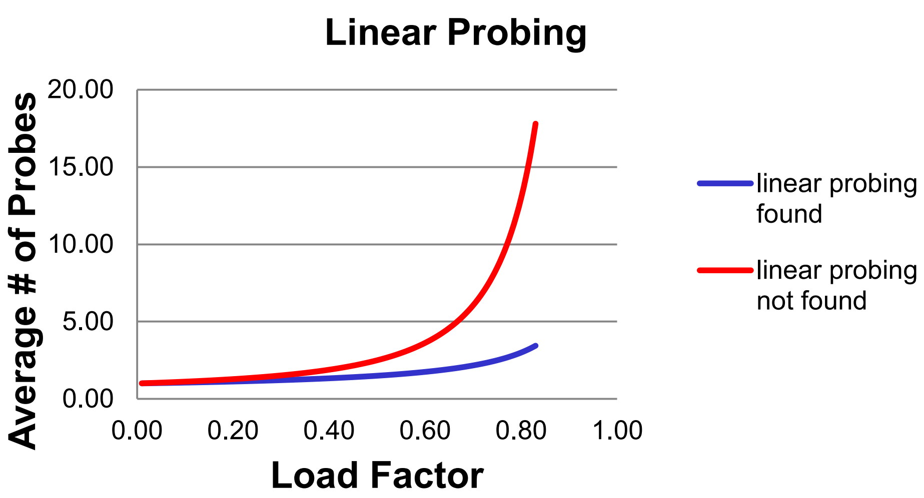
- Trivial fact: For any $\lambda < 1$, linear probing will find an empty slot
  - It is "safe" in this sense: no infinite loop unless table is full

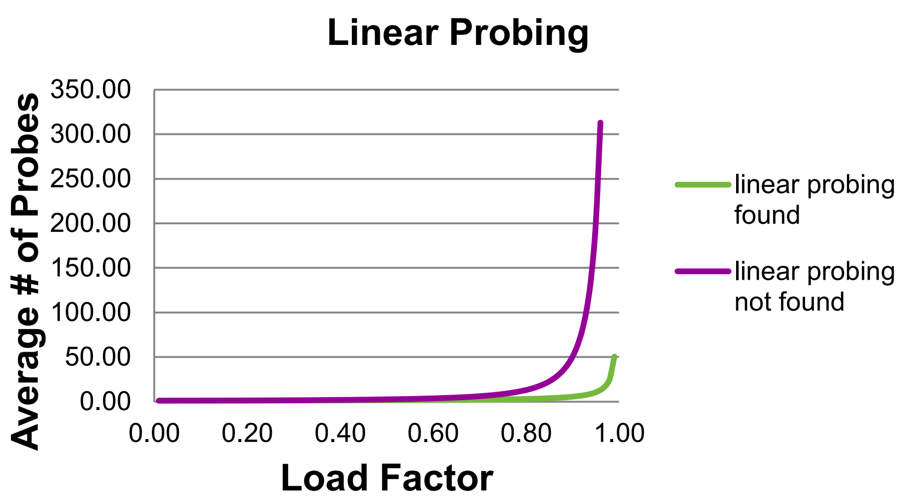
- Non-trivial facts we won't prove:
  Average # of probes given λ (in the limit as **`TableSize`** $\rightarrow \infty$)
  - Unsuccessful search: $$\frac{1}{2}\left(1+\frac{1}{(1-\lambda)^2}\right)$$
  - Successful search: $$\frac{1}{2}\left(1+\frac{1}{(1-\lambda)}\right)$$

- This is pretty bad: need to leave sufficient empty space in the table to get decent performance (see chart)

In a chart

- Linear-probing performance degrades rapidly as table gets full
  - (Formula assumes “large table” but point remains)





- By comparison, chaining performance is linear in λ and has no trouble with λ***>1***

Quadratic probing

- We can avoid primary clustering by changing the probe function

  `(h(key) + f(i)) % TableSize`

- A common technique is quadratic probing:

  `f(i) =` $\texttt{i}^2$

  – So probe sequence is:
    - 0^{th} probe: `h(key) % TableSize`
    - 1^{st} probe: `(h(key) + 1) % TableSize`
    - 2^{nd} probe: `(h(key) + 4) % TableSize`
    - 3^{rd} probe: `(h(key) + 9) % TableSize`
    - …
    - i^{th} probe: `(h(key) +` $\texttt{i}^2$`) % TableSize`

- Intuition: Probes quickly “leave the neighborhood”

Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

TableSize=10
Insert:
89
18
49
58
79

Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	89

TableSize=10
Insert:
89
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Quadratic Probing Example

0	
1	
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9	89

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Quadratic Probing Example

0	49
1	
2	
3	
4	
5	
6	
7	
8	18
9	89

TableSize=10
Insert:
89
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Quadratic Probing Example

0	49
1	
2	58
3	
4	
5	
6	
7	
8	18
9	89

TableSize=10
Insert:
89
18
49
58
79

Quadratic Probing Example

0	49
1	
2	58
3	79
4	
5	
6	
7	
8	18
9	89

TableSize=10
Insert:
89
18
49
58
79

Another Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	

TableSize = 7

Insert:

76	**(76 % 7 = 6)**
40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

Another Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	76

TableSize = 7

Insert:

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1	
2	
3	
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TableSize = 7

Insert:

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40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

Another Quadratic Probing Example

0	48
1	
2	5
3	
4	
5	40
6	76

TableSize = 7

Insert:

76 (76 % 7 = 6)
40 (40 % 7 = 5)
48 (48 % 7 = 6)
5 (5 % 7 = 5)
55 (55 % 7 = 6)
47 (47 % 7 = 5)

Another Quadratic Probing Example

0	48
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2	5
3	55
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5	40
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TableSize = 7

Insert:

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TableSize = 7

Insert:

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47	**(47 % 7 = 5)**

Doh!: For all *n*, `((n*n) +5) % 7 is 0, 2, 5, or 6`

- Excel shows takes "at least" 50 probes and a pattern
- Proof (like induction) using $(n^2+5) \% 7 = ((n-7)^2+5) \% 7$
  - In fact, for all *c* and *k*, $(n^2+c) \% k = ((n-k)^2+c) \% k$

From Bad News to Good News

- Bad news:
  - Quadratic probing can cycle through the same full indices, never terminating despite table not being full
- Good news:
  - If **`TableSize`** is *prime* and $\lambda < ½$, then quadratic probing will find an empty slot in at most **`TableSize/2`** probes
  - So: If you keep $\lambda < ½$ and **`TableSize`** is *prime*, no need to detect cycles
  - Optional: Proof is posted in **`lecture14.txt`**
    - Also, slightly less detailed proof in textbook
    - Key fact: For prime $\mathbf{T}$ and $\mathbf{0 < i,j < T/2}$ where $\mathbf{i \neq j}$, $\mathbf{(k + i^2) \% T \neq (k + j^2) \% T}$ (i.e., no index repeat)

Clustering reconsidered

- Quadratic probing does not suffer from primary clustering: no problem with keys initially hashing to the same neighborhood

- But it's no help if keys initially hash to the same index
  - Called secondary clustering

- Can avoid secondary clustering with a probe function that depends on the key: double hashing…

Double hashing

Idea:

- Given two good hash functions *h* and *g*, it is very unlikely that for some *key*, `h(key) == g(key)`
- So make the probe function `f(i) = i*g(key)`

Probe sequence:

- 0^{th} probe: `h(key) % TableSize`
- 1^{st} probe: `(h(key) + g(key)) % TableSize`
- 2^{nd} probe: `(h(key) + 2*g(key)) % TableSize`
- 3^{rd} probe: `(h(key) + 3*g(key)) % TableSize`
- …
- i^{th} probe: `(h(key) + i*g(key)) % TableSize`

Detail: Make sure `g(key)` cannot be `0`

Double-hashing analysis

- Intuition: Because each probe is "jumping" by **`g(key)`** each time, we "leave the neighborhood" *and* "go different places from other initial collisions"

- But we could still have a problem like in quadratic probing where we are not "safe" (infinite loop despite room in table)
  - It is known that this cannot happen in at least one case:
    - **`h(key) = key % p`**
    - **`g(key) = q - (key % q)`**
    - **`2 < q < p`**
    - **`p`** and **`q`** are prime

More double-hashing facts

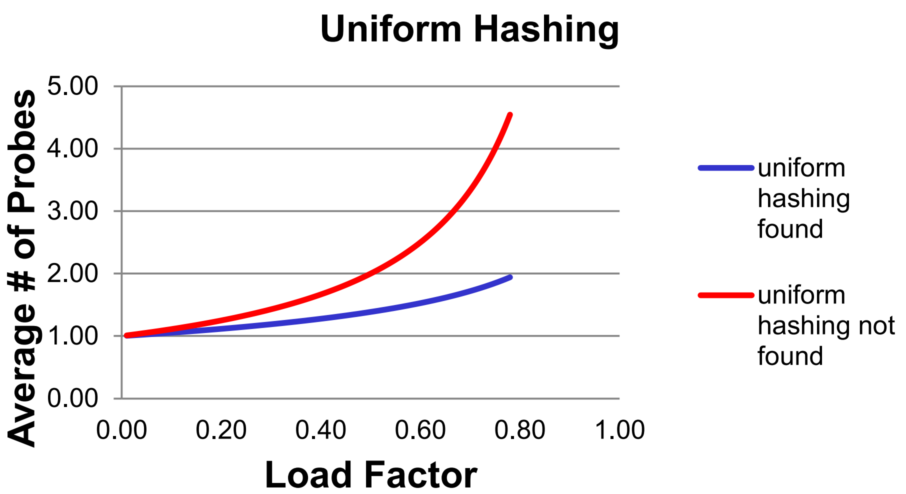
- Assume "uniform hashing"
  - Means probability of **`g(key1) % p == g(key2) % p`** is **`1/p`**

- Non-trivial facts we won't prove:

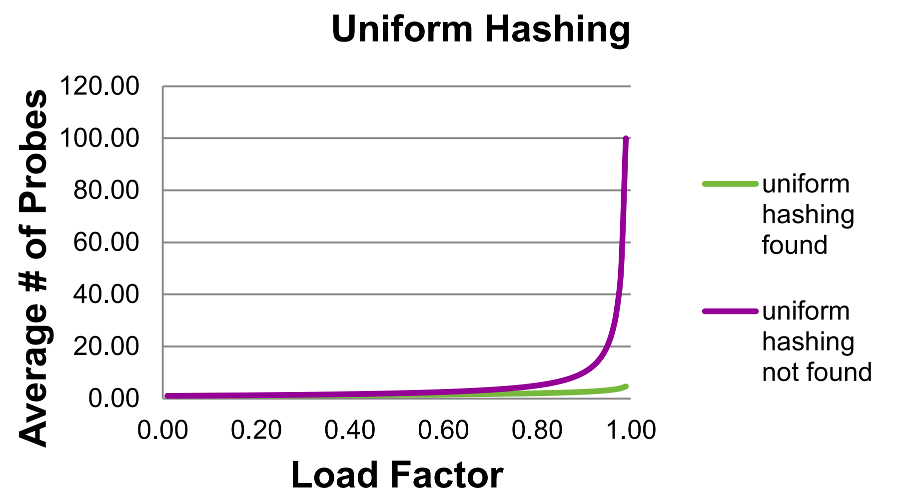
  Average # of probes given λ (in the limit as **`TableSize`** $\rightarrow \infty$)
  - Unsuccessful search (intuitive): $$\frac{1}{1-\lambda}$$
  - Successful search (less intuitive): $$\frac{1}{\lambda}\log_e\left(\frac{1}{1-\lambda}\right)$$

- Bottom line: unsuccessful bad (but not as bad as linear probing), but successful is not nearly as bad

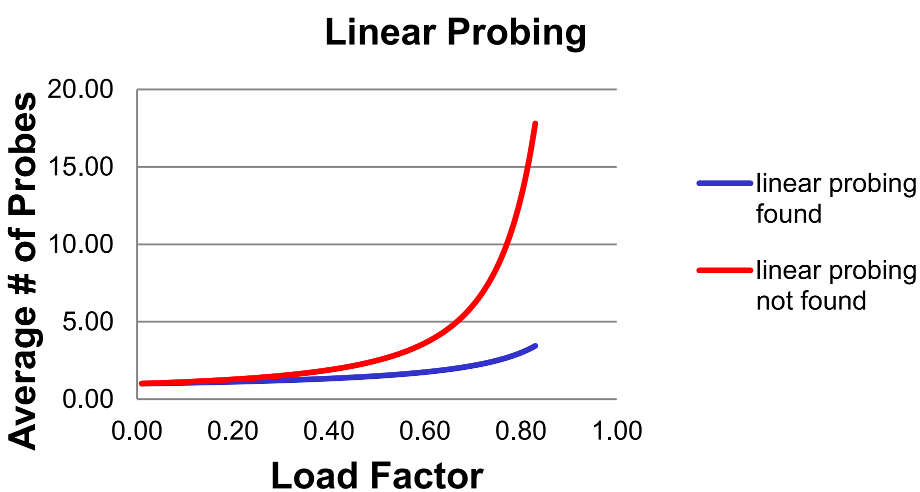
Charts



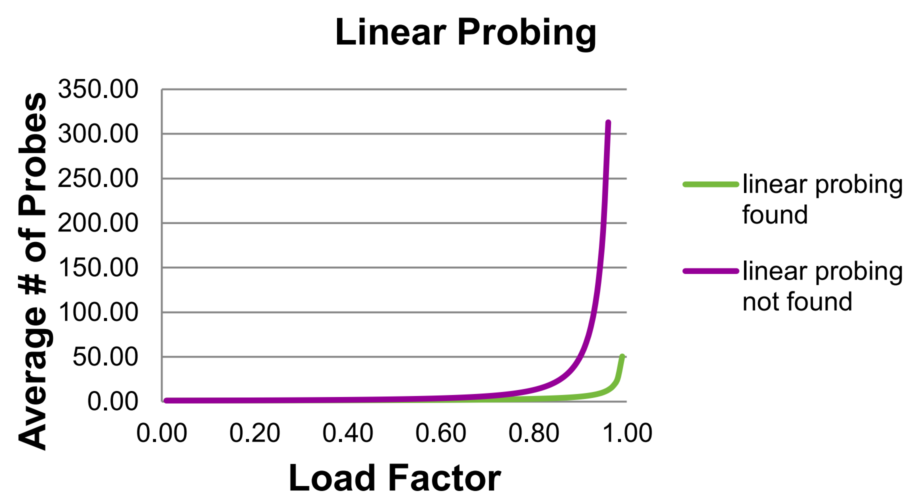
Uniform Hashing
Average # of Probes
5.00
4.00
3.00
2.00
1.00
0.00
0.00
0.20
0.40
0.60
0.80
1.00
Load Factor
uniform hashing found
uniform hashing not found



Uniform Hashing
Average # of Probes
120.00
100.00
80.00
60.00
40.00
20.00
0.00
0.00
0.20
0.40
0.60
0.80
1.00
Load Factor
uniform hashing found
uniform hashing not found



Linear Probing
Average # of Probes
20.00
15.00
10.00
5.00
0.00
0.00
0.20
0.40
0.60
0.80
1.00
Load Factor
linear probing found
linear probing not found



Linear Probing
Average # of Probes
350.00
300.00
250.00
200.00
150.00
100.00
50.00
0.00
0.00
0.20
0.40
0.60
0.80
1.00
Load Factor
linear probing found
linear probing not found

Rehashing

- As with array-based stacks/queues/lists, if table gets too full, create a bigger table and copy everything
- With chaining, we get to decide what “too full” means
  - Keep load factor reasonable (e.g., < 1)?
  - Consider average or max size of non-empty chains?
- For probing, half-full is a good rule of thumb
- New table size
  - Twice-as-big is a good idea, except that won’t be prime!
  - So go *about* twice-as-big
  - Can have a list of prime numbers in your code since you won’t grow more than 20-30 times

Hashtable Scenarios

- For each of the scenarios, answer the following questions:
  - Is a hashtable the best-suited data structure?
  - If so, what would be used at the keys? Values?
  - If not, what data structure would be best-suited?
  - What other assumptions, if any, about the scenario must you make to support your previous answers?

- Catalog of items (product id, name, price)
- Bookmarks in a web browser (favicon, URL, bookmark name)
- IT support requests (timestamp, ticket id, description)
- Character frequency analysis (character, # of appearances)
- Activation records for nested function calls (return addresses, local variables, etc.)