CSE373: Data Structures & Algorithms
Lecture 11: Implementing Union-Find

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Announcements

- Homework 3 due in ONE week…Wednesday April 29th!
- TA sessions
- Catie will be back on Monday.

The plan

Last lecture:

- Disjoint sets
- The union-find ADT for disjoint sets

Today's lecture:

- Basic implementation of the union-find ADT with "up trees"
- Optimizations that make the implementation much faster

Union-Find ADT

- Given an unchanging set *S*, `create` an initial partition of a set
  - Typically each item in its own subset: {a}, {b}, {c}, …
  - Give each subset a "name" by choosing a *representative element*

- Operation `find` takes an element of *S* and returns the representative element of the subset it is in

- Operation `union` takes two subsets and (permanently) makes one larger subset
  - A different partition with one fewer set
  - Affects result of subsequent `find` operations
  - Choice of representative element up to implementation

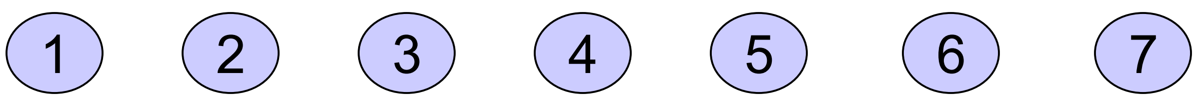
Implementation – our goal

- Start with an initial partition of n subsets
  - Often 1-element sets, e.g., {1}, {2}, {3}, …, {n}

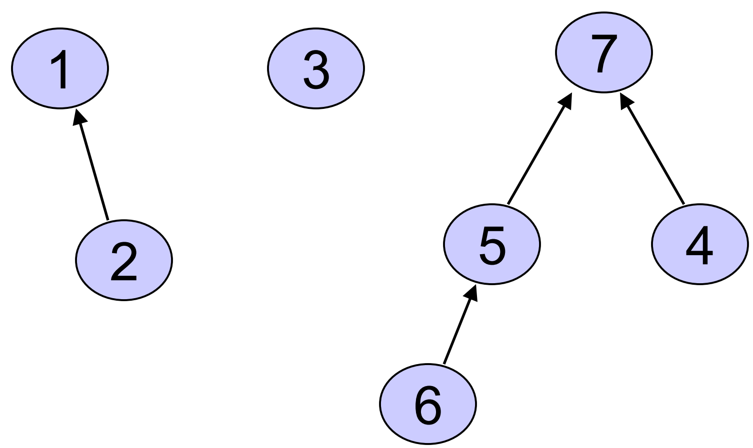
- May have m **`find`** operations
- May have up to n-1 **`union`** operations in any order
  - After n-1 **`union`** operations, every **`find`** returns same 1 set

Up-tree data structure

- Tree with:
  - No limit on branching factor
  - References from children to parent
- Start with *forest* of 1-node trees



- Possible forest after several unions:
  - Will use roots for set names

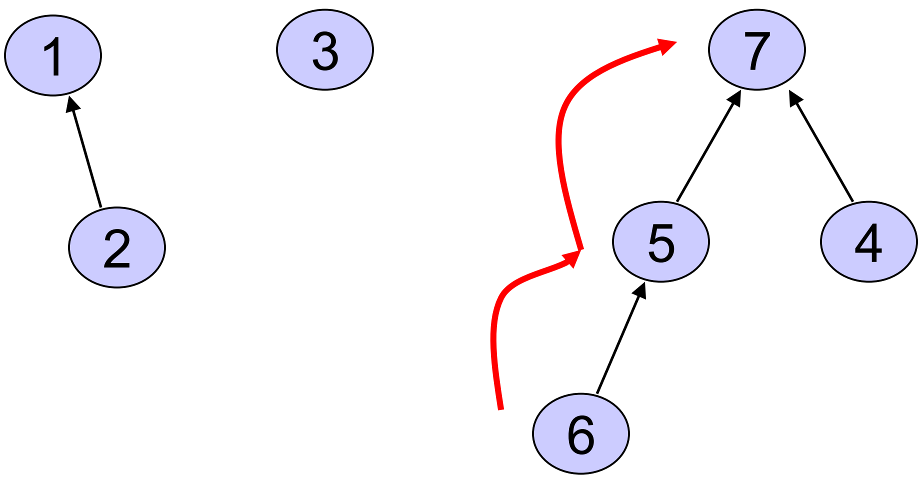


Find

`find`(**`x`**):

- *Assume* we have *O*(1) access to each node
  - Will use an array where index **`i`** holds node **`i`**
- Start at **`x`** and follow parent pointers to root
- Return the root

`find`(6) = 7

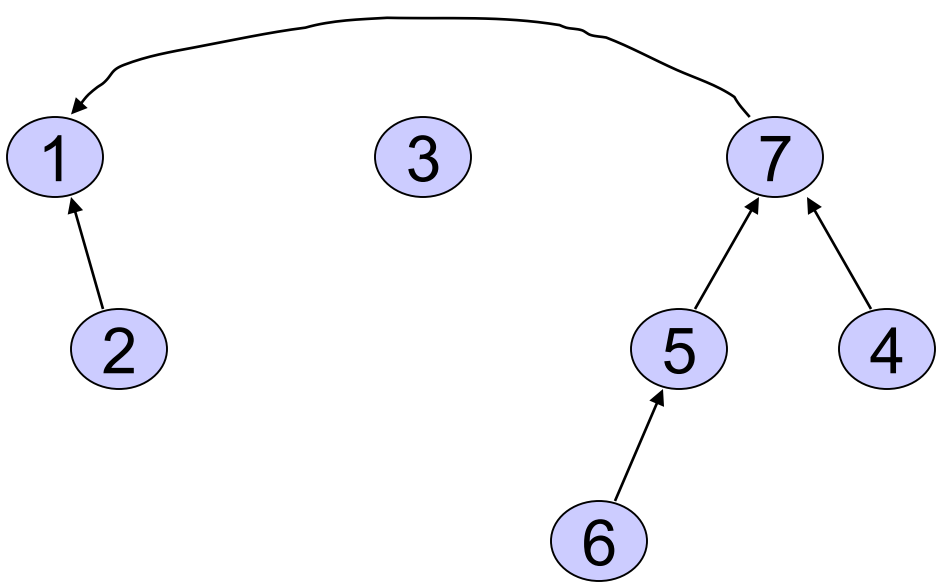


Union

`union`(**`x`**,**`y`**):

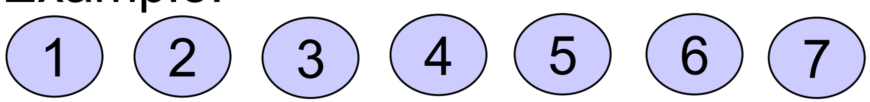
- Assume **`x`** and **`y`** are roots
  - Else **`find`** the roots of their trees
- Assume distinct trees (else do nothing)
- Change root of one to have parent be the root of the other
  - Notice no limit on branching factor

`union`(1,7)



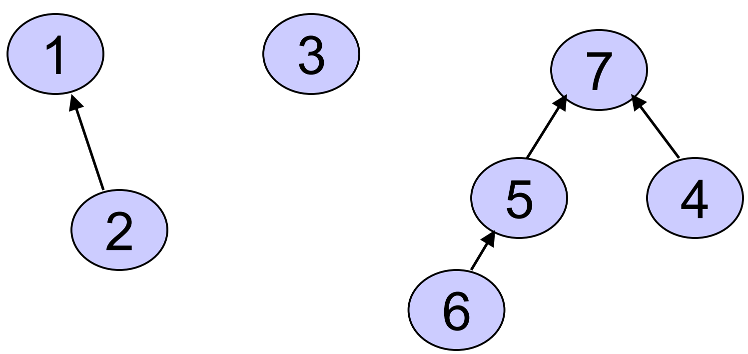
Simple implementation

- If set elements are contiguous numbers (e.g., 1,2,…,*n*), use an array of length *n* called **`up`**
  - Starting at index 1 on slides
  - Put in array index of parent, with 0 (or -1, etc.) for a root
- Example:



	1	2	3	4	5	6	7
up	0	0	0	0	0	0	0

- Example:



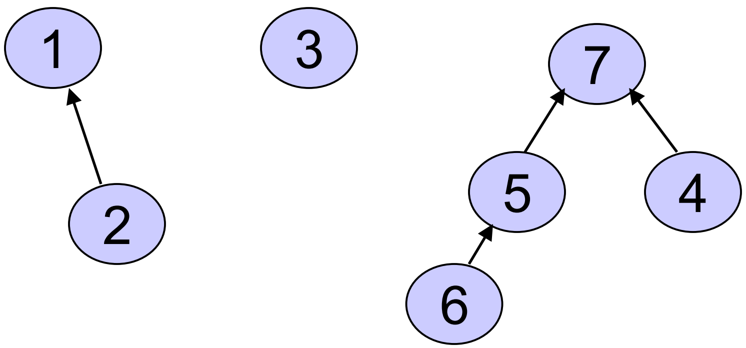
	1	2	3	4	5	6	7
up	0	1	0	7	7	5	0

- If set elements are not contiguous numbers, could have a separate dictionary to map elements (keys) to numbers (values)

Implement operations

```
// assumes x in range 1,n
int find(int x) {
  while(up[x] != 0) {
     x = up[x];
  }
  return x;
}
```

```
// assumes x,y are roots
void union(int x, int y){
  up[y] = x;
}
```



	1	2	3	4	5	6	7
up	0	1	0	7	7	5	0

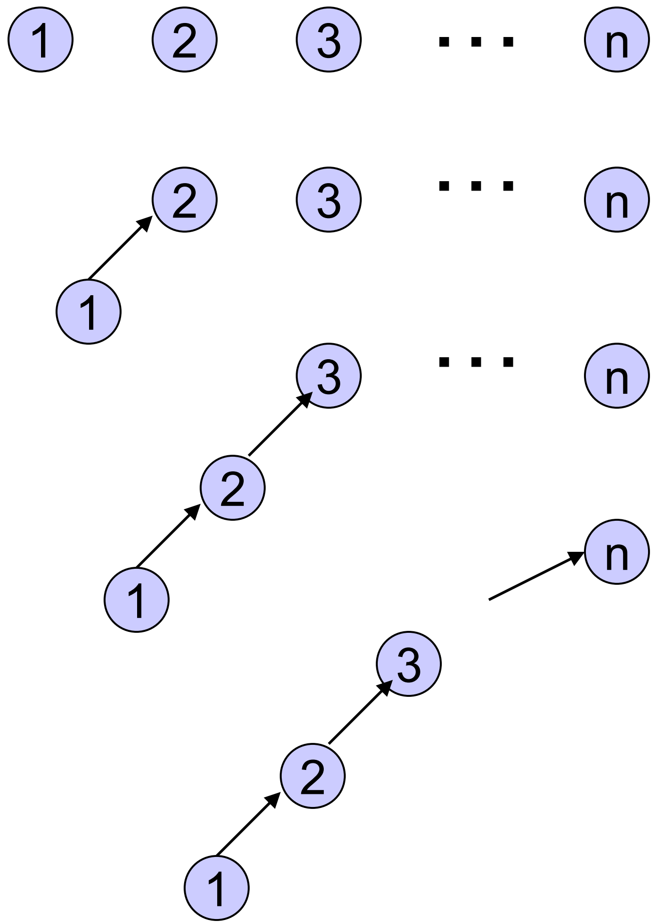
- Worst-case run-time for **`union`**? ***O*(1)**
- Worst-case run-time for **`find`**? ***O*(n)**
- Worst-case run-time for *m* **`finds`** and *n*-1 **`unions`**? ***O*(m*n)**

Two key optimizations

1. Improve **`union`** so it stays *O(1)* but makes **`find`** $O(\log n)$
   - So *m* **`find`**s and *n*-1 **`union`**s is $O(m \log n + n)$
   - *Union-by-size:* connect smaller tree to larger tree

2. Improve **`find`** so it becomes even faster
   - Make *m* **`find`**s and *n*-1 **`union`**s ***almost*** $O(m + n)$
   - *Path-compression:* connect directly to root during finds

The bad case to avoid



union(2,1)

union(3,2)

:

.

union(n,n-1)

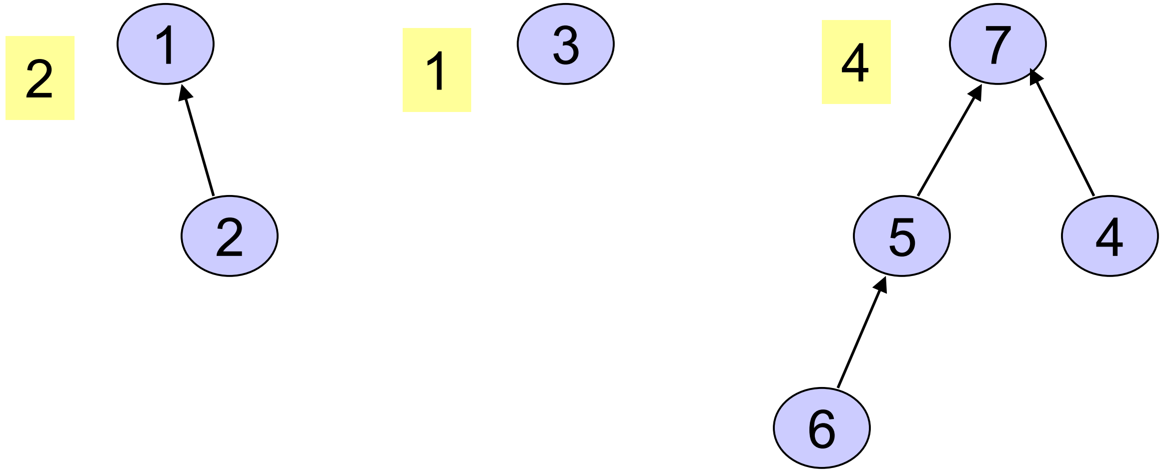
find(1) = n steps!!

Union-by-size

Union-by-size:

- Always point the *smaller* (total # of nodes) tree to the root of the larger tree

`union`(1,7)



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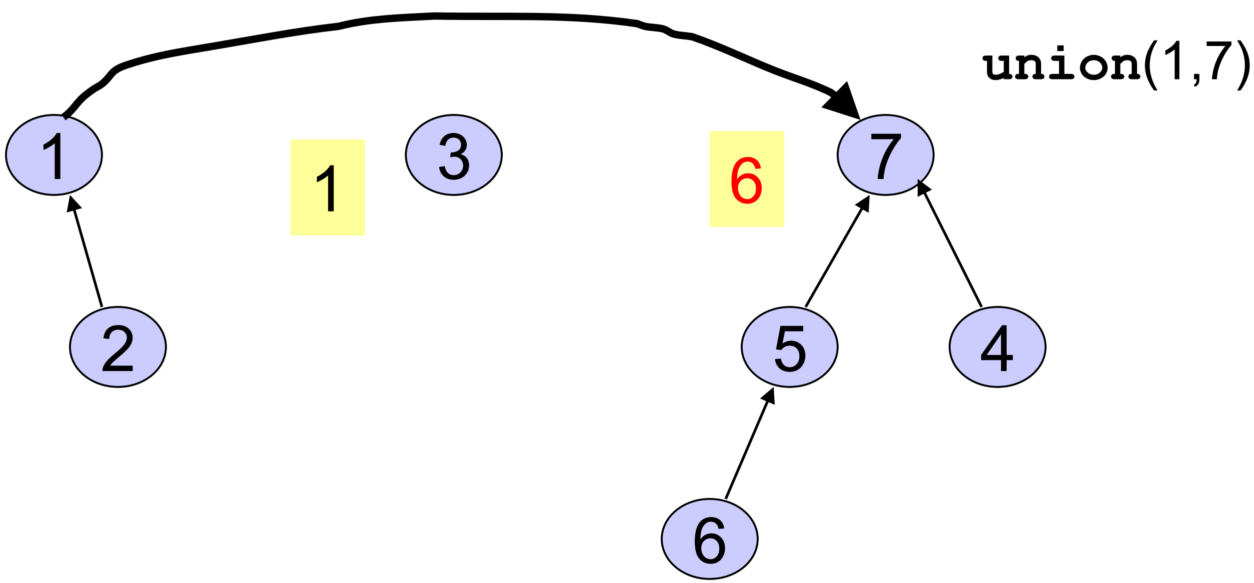
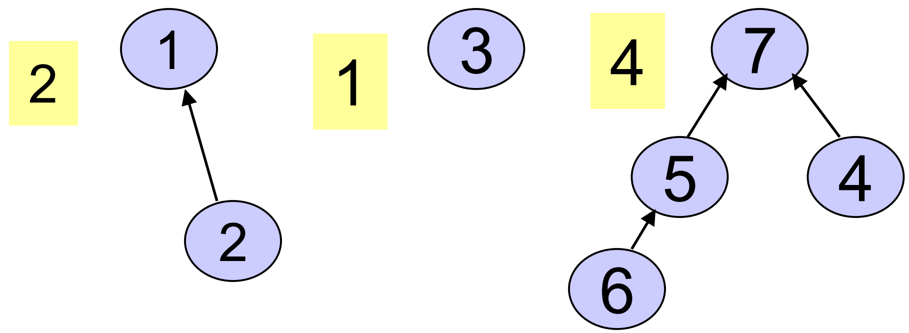


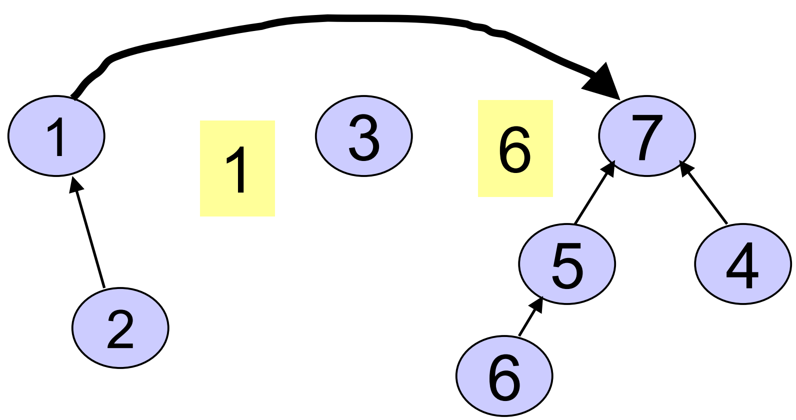
Array implementation

Keep the size (number of nodes in a second array)

- Or have one array of objects with two fields



	1	2	3	4	5	6	7
up	0	1	0	7	7	5	0
weight	2		1				4

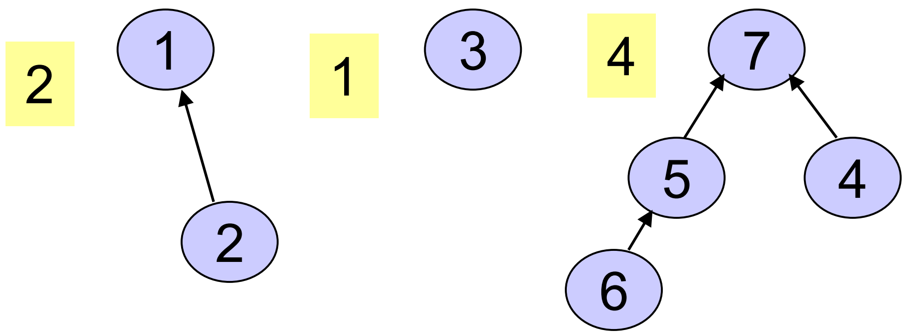


	1	2	3	4	5	6	7
up	7	1	0	7	7	5	0
weight			1				6

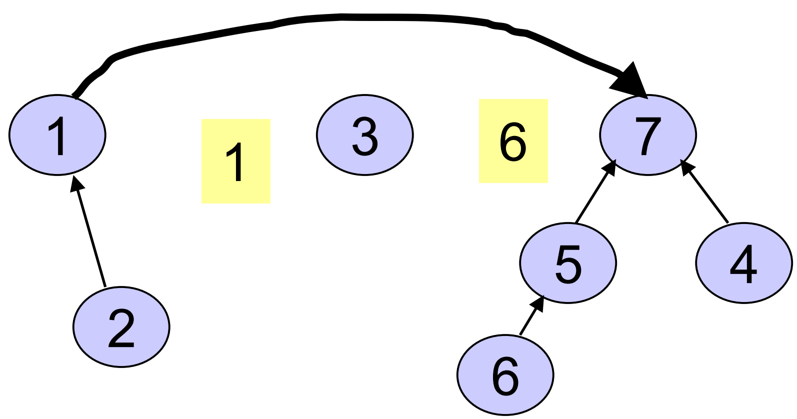
Nifty trick

Actually we do not need a second array…

– Instead of storing 0 for a root, store negation of size
– So up value < 0 means a root

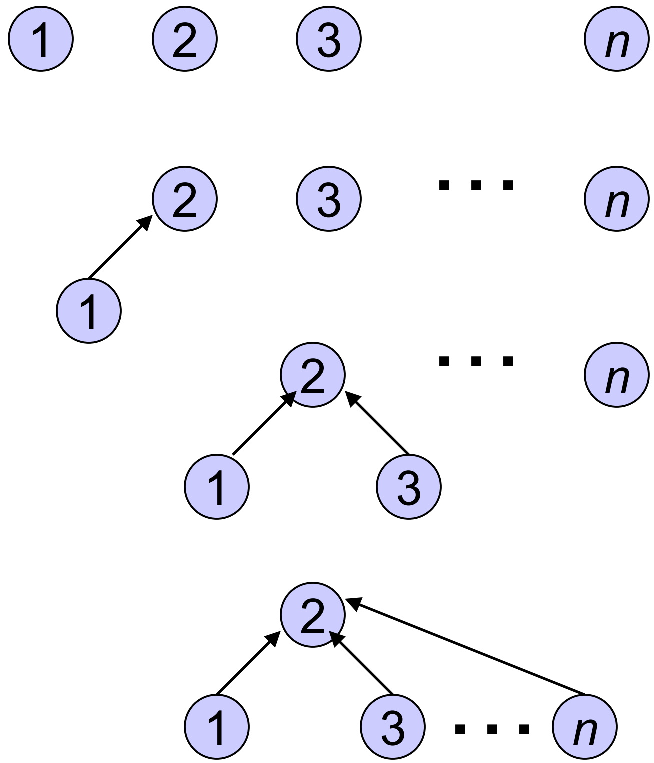


	1	2	3	4	5	6	7
up	-2	1	-1	7	7	5	-4



	1	2	3	4	5	6	7
up	7	1	-1	7	7	5	-6

The Bad case? Now a Great case…



`union`(2,1)

`union`(3,2)

:

`union`(n,n-1)

`find`(1) *constant here*

General analysis

- Showing one worst-case example is now good is *not* a proof that the worst-case has improved

- So let's prove:
  - **`union`** is still $O(1)$ – this is "obvious"
  - **`find`** is now $O(\log n)$

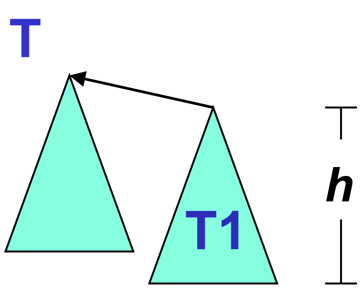
- Claim: If we use union-by-size, an up-tree of height h has at least 2^h nodes
  - Proof by induction on h…

Exponential number of nodes

P(h)= With union-by-size, up-tree of height h has at least 2^h nodes

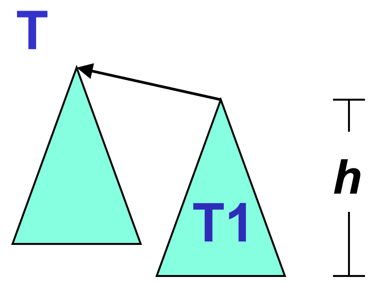
Proof by induction on h…

- Base case: h = 0: The up-tree has 1 node and 2^0= 1
- Inductive case: Assume P(h) and show P(h+1)
  - A height h+1 tree T has at least one height h child T1
  - T1 has at least 2^h nodes by induction (assumption)
  - And T has *at least* as many nodes not in T1 than in T1
    - Else union-by-size would have had T point to T1, not T1 point to T (!!)
  - So total number of nodes is *at least* $2^h + 2^h = 2^{h+1}$



The key idea

Intuition behind the proof: No one child can have more than half the nodes

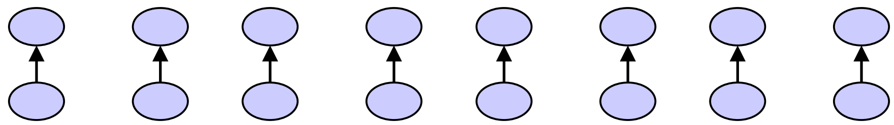


So, as usual, if number of nodes is exponential in height, then height is logarithmic in number of nodes

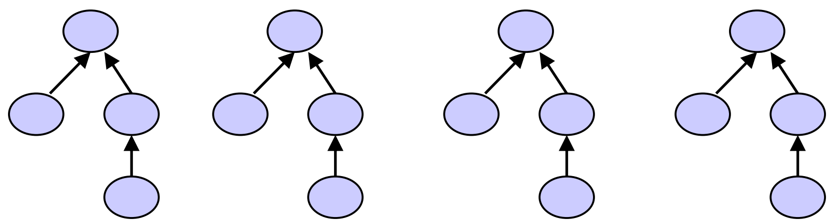
So **`find`** is $O($**`log`** $n)$

The new worst case

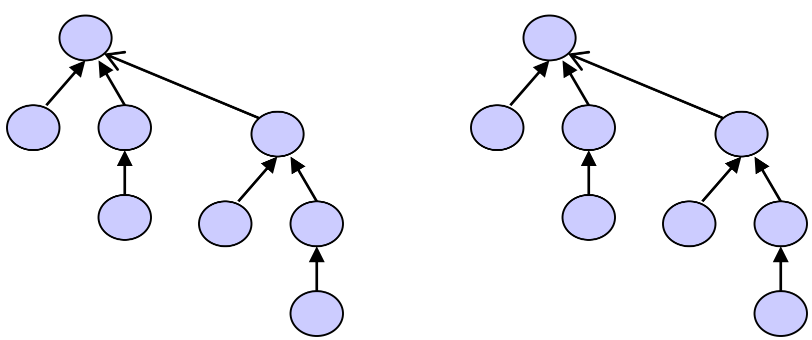
n/2 Unions-by-size



n/4 Unions-by-size

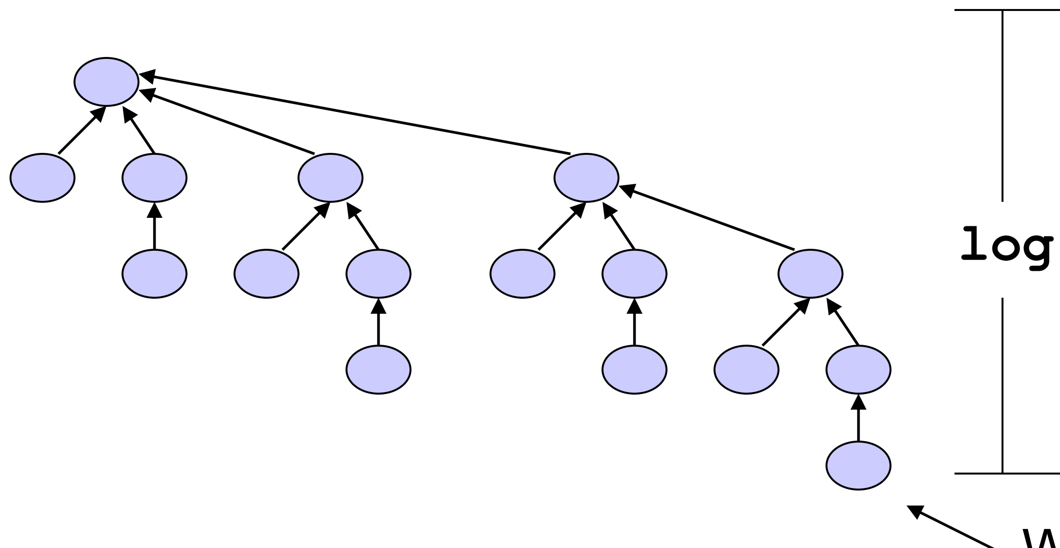


n/8 Unions-by-size



The new worst case (continued)

After n/2 + n/4 + …+ 1 Unions-by-size:



Height grows by 1 a total of **`log`** *n* times

Worst
find

What about union-by-height

We could store the height of each root rather than size

- Still guarantees logarithmic worst-case find
  - Proof left as an exercise if interested

- But does not work well with our next optimization
  - Maintaining height becomes inefficient, but maintaining size still easy

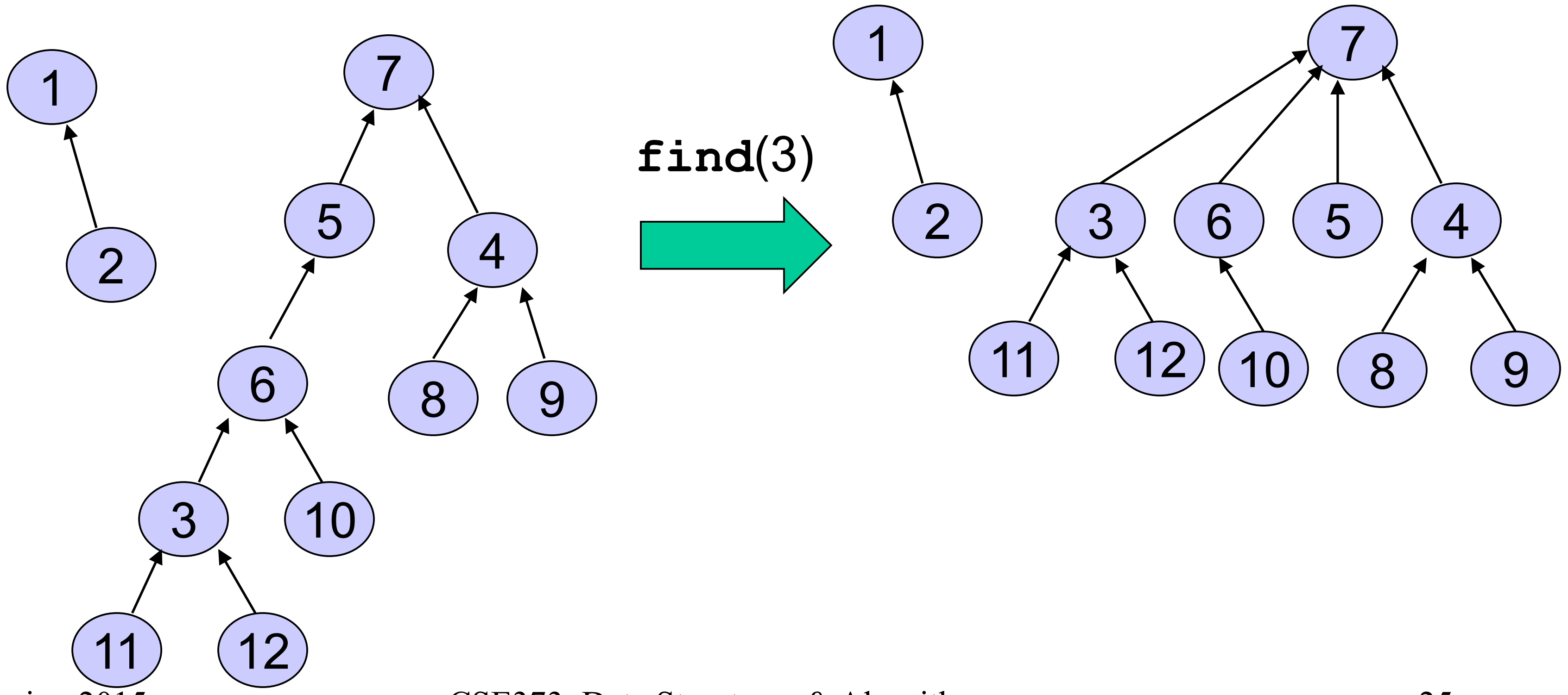
Two key optimizations

1. Improve **`union`** so it stays *O(1)* but makes **`find`** $O(\log n)$
   - So *m* **`finds`** and *n*-1 **`unions`** is $O(m \log n + n)$
   - *Union-by-size:* connect smaller tree to larger tree

2. Improve **`find`** so it becomes even faster
   - Make *m* **`finds`** and *n*-1 **`unions`** ***almost*** $O(m + n)$
   - *Path-compression:* connect directly to root during finds

Path compression

- Simple idea: As part of a **`find`**, change each encountered node's parent to point directly to root
  - Faster future **`find`**s for everything on the path (and their descendants)



Pseudocode

```
// performs path compression
int find(i) {
  // find root
  int r = i
  while(up[r] > 0)
     r = up[r]

  // compress path
  if i==r
    return r;
  int old_parent = up[i]
  while(old_parent != r) {
    up[i] = r
    i = old_parent;
    old_parent = up[i]
  }
  return r;
}
```

Example

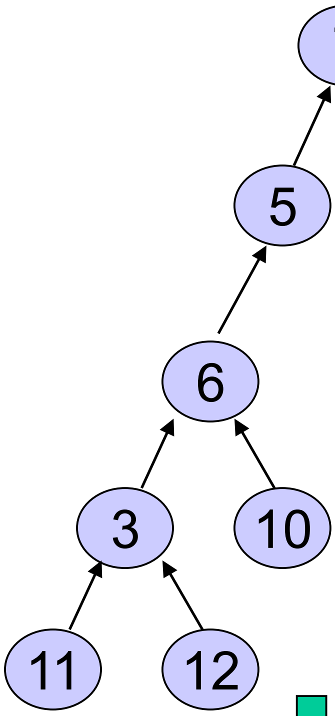
i=3
r=3

r=6
r=5
r=7

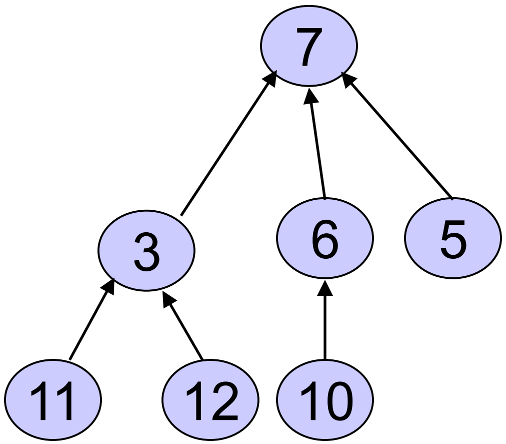
old_parent=6

up[3]=7
i=6
old_parent=5

up[6]=7
i=5
old_parent=7



find(3)



So, how fast is it?

A single worst-case **`find`** could be $O(\mathbf{log}\ n)$

- But only if we did a lot of worst-case unions beforehand
- And path compression will make future finds faster

Turns out the amortized worst-case bound is much better than $O(\mathbf{log}\ n)$

- We won't *prove* it – see text if curious
- But we will *understand* it:
  - How it is *almost* $O(1)$
  - Because total for m **`find`**s and n-1 **`union`**s is *almost* $O(m+n)$

A really slow-growing function

`log*` *x* is the minimum number of times you need to apply "**`log`** of **`log`** of **`log`** of" to go from *x* to a number <= 1

For just about every number we care about, **`log*`** *x* is less than or equal to *5* (!)

If $x <= 2^{65536}$ then **`log*`** $x <= 5$

- log* 2 = 1
- log* 4 = log* 2^2 = 2
- log* 16 = log* $2^{(2^2)}$ = 3 (log log log 16 = 1)
- log* 65536 = log* $2^{((2^2)^2)}$ = 4 (log log log log 65536 = 1)
- log* 2^{65536} = …………… = 5

Almost linear

- Turns out total time for *m* `finds` and *n*-1 `unions` is

  $O((m+n)*(\texttt{log*}\ (m+n))$

  – Remember, if $m+n < 2^{65536}$ then `log*` $(m+n) < 5$

  so effectively we have $O(m+n)$
- Because `log*` grows soooo slowly
  – For all practical purposes, amortized bound is constant, i.e., cost of find is constant, total cost for m finds is linear
  – We say "near linear" or "effectively linear"
- Need union-by-size and path-compression for this bound
  – Path-compression changes height but not weight, so they interact well
- As always, asymptotic analysis is separate from "coding it up"