CSE373: Data Structures & Algorithms

Lecture 9: Binary Heaps, Continued

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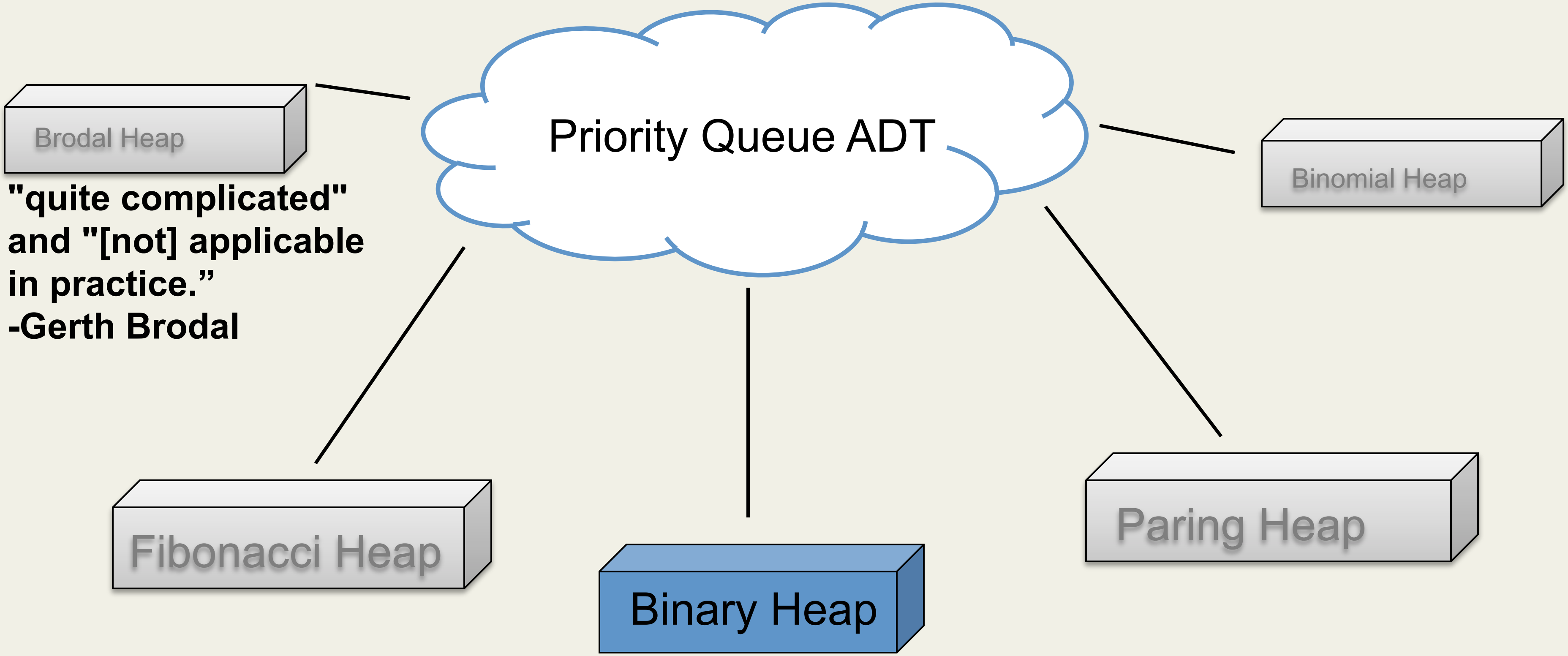
Fall 2015

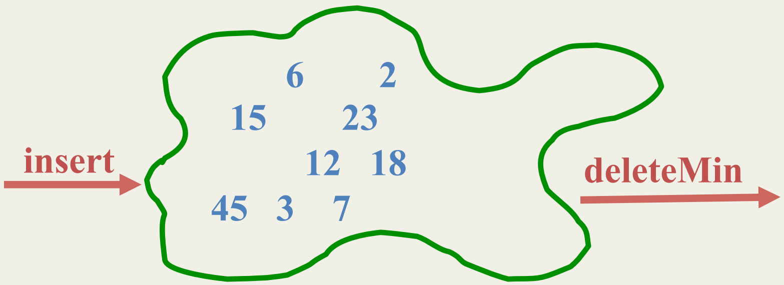


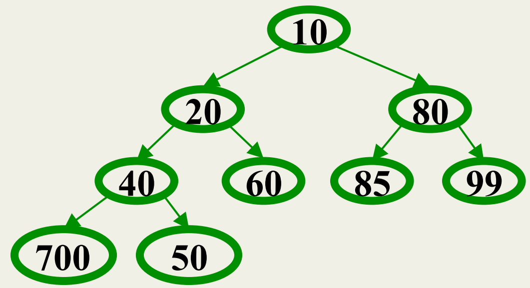
A priority queue is just an **abstraction** for an ordered queue.

A **binary heap** is a simple and concrete implementation of a priority queue

It's just one of many possible implementations!

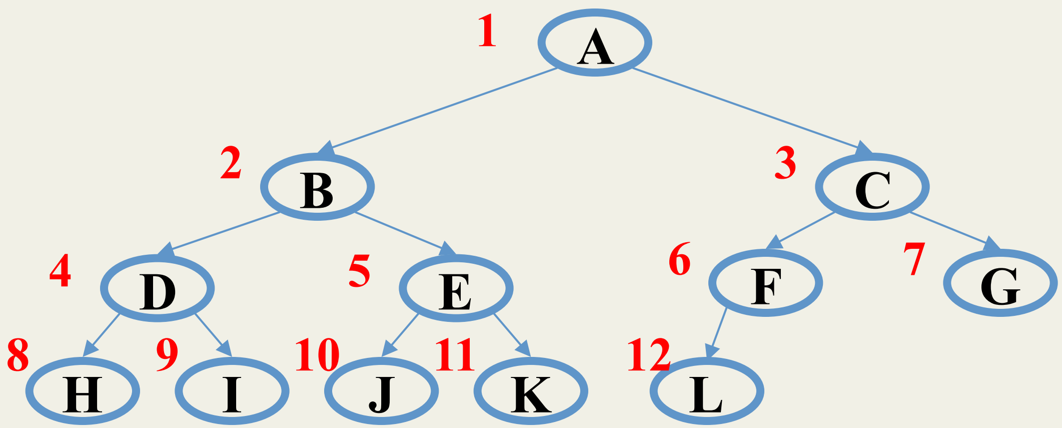
Review





- **Priority Queue ADT**: **`insert`** comparable object, **`deleteMin`**
- **Binary heap data structure**: Complete binary tree where each node has priority value greater than its parent
- *O*(height-of-tree) = *O*(**`log`** *n*) **`insert`** and **`deleteMin`** operations
  - **`insert`**: put at new last position in tree and percolate-up
  - **`deleteMin`**: remove root, put 'last' element at root and percolate-down
- But: tracking the "last position" is painful and we can do better

Array Representation of Binary Trees



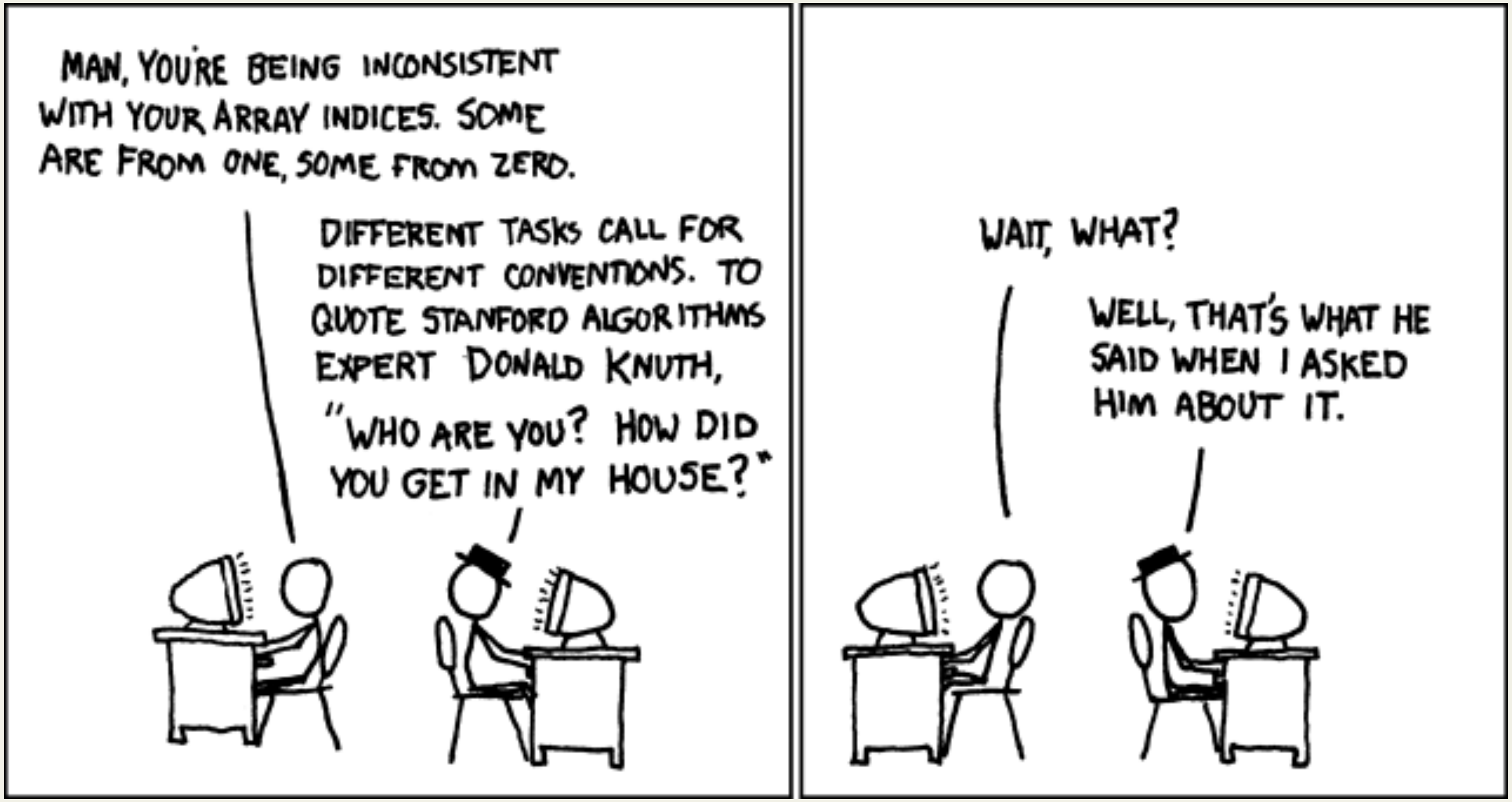
Starting at node `i`

left child: `i*2`
right child: `i*2+1`
parent: `i/2`

(wasting index 0 is convenient for the index arithmetic)

implicit (array) implementation:

	A	B	C	D	E	F	G	H	I	J	K	L	
0	1	2	3	4	5	6	7	8	9	10	11	12	13
		parent		i = 4				left	right				



http://xkcd.com/163

Judging the array implementation

Positives:

- Non-data space is minimized: just index 0 and unused space on right
  - In conventional tree representation, one edge per node (except for root), so *n*-1 wasted space (like linked lists)
  - Array would waste more space if tree were not complete

- Multiplying and dividing by 2 is very fast (shift operations in hardware)
- Last used position is just index `size`

Negatives:

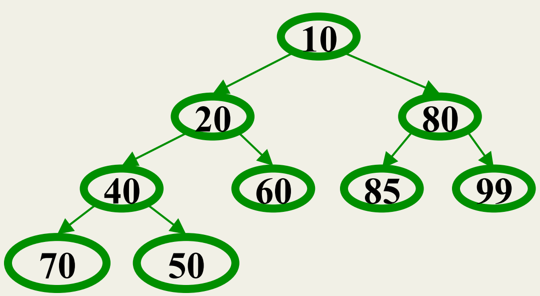
- Same might-by-empty or might-get-full problems we saw with stacks and queues (resize by doubling as necessary)

Plusses outweigh minuses: “this is how people do it”

Pseudocode: insert

```
void insert(int val) {
   if(size == arr.length-1)
     resize();
   size++;
   i=percolateUp(size,val);
   arr[i] = val;
}
```

```
int percolateUp(int hole,int val) {
  while(hole > 1 &&
         val < arr[hole/2])
    arr[hole] = arr[hole/2];
    hole = hole / 2;
  }
  return hole;
}
```



This pseudocode uses ints. In real use, you will have data nodes with priorities.

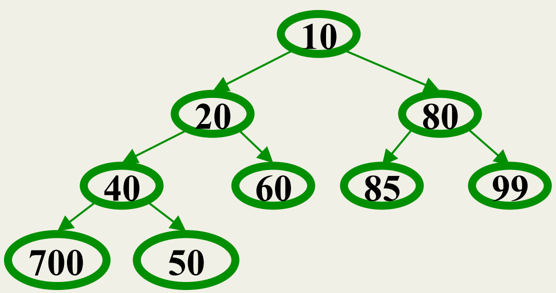
0	1	2	3	4	5	6	7	8	9	10	11	12	13
	10	20	80	40	60	85	99	700	50				

Pseudocode: deleteMin

This pseudocode uses ints. In real use, you will have data nodes with priorities.

```
int deleteMin() {
  if(isEmpty()) throw…
  ans = arr[1];
  hole = percolateDown
           (1,arr[size]);
  arr[hole] = arr[size];
  size--;
  return ans;
}
```

```
int percolateDown(int hole,int val){
 while(2*hole <= size) {
   left  = 2*hole;
   right = left + 1;
   if(arr[left] < arr[right]
      || right > size)
     target = left;
   else
     target = right;
   if(arr[target] < val) {
     arr[hole] = arr[target];
     hole = target;
   } else
     break;
 }
 return hole;
}
```

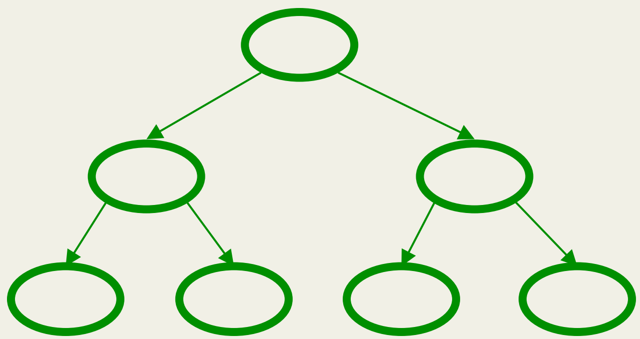


	10	20	80	40	60	85	99	700	50				
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Example

1. **insert:** 16, 32, 4, 69, 105, 43, 2
2. **deleteMin**

0	1	2	3	4	5	6	7



Example

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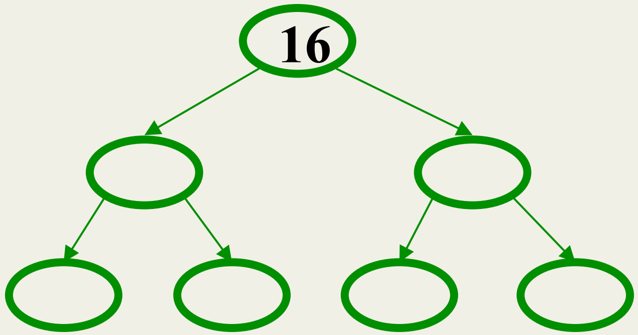
2. **deleteMin**

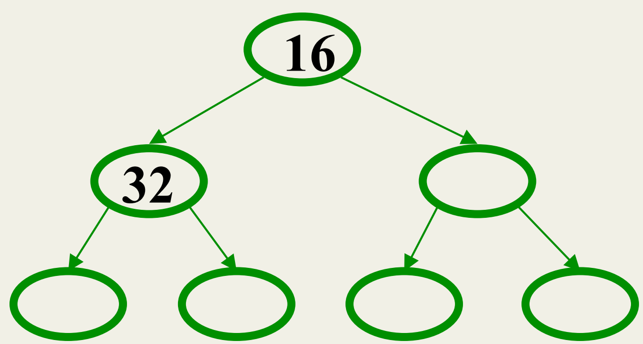
	16						
0	1	2	3	4	5	6	7



Example

1. **insert:** 16, 32, 4, 69, 105, 43, 2
2. **deleteMin**

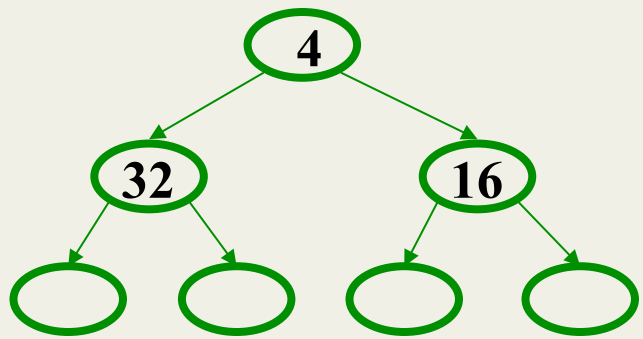
	16	**32**					
0	1	2	3	4	5	6	7



Example

1. **insert:** 16, 32, 4, 69, 105, 43, 2
2. **deleteMin**

	4	**32**	**16**				
0	1	2	3	4	5	6	7



Example

1. **insert:** 16, 32, 4, 69, 105, 43, 2
2. **deleteMin**

	4	**32**	**16**	**69**			
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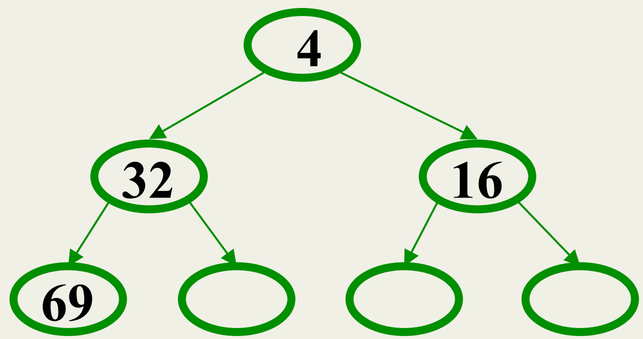
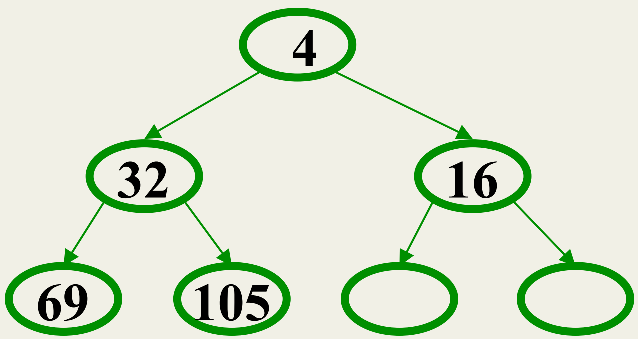


Example

1. **insert:** 16, 32, 4, 69, 105, 43, 2
2. **deleteMin**

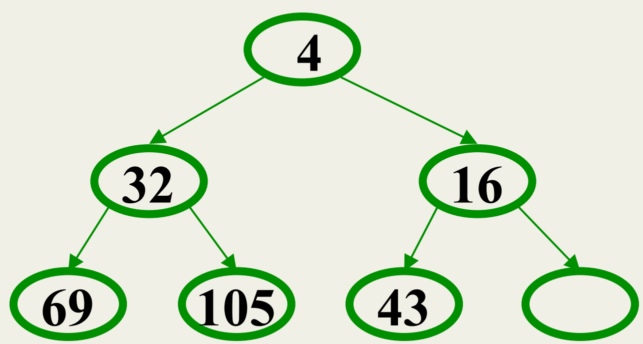
	4	**32**	**16**	**69**	**105**		
0	1	2	3	4	5	6	7



Example

1. **insert:** 16, 32, 4, 69, 105, 43, 2
2. **deleteMin**

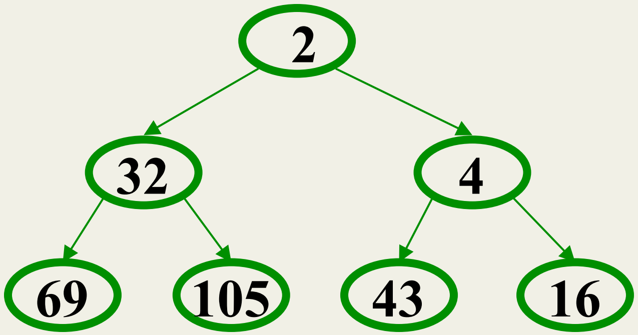
	4	**32**	**16**	**69**	**105**	**43**	
0	1	2	3	4	5	6	7



Example

1. **insert:** 16, 32, 4, 69, 105, 43, 2
2. **deleteMin**

	2	**32**	**4**	**69**	**105**	**43**	**16**
0	1	2	3	4	5	6	7



Other operations

- `decreaseKey`: given pointer to object in priority queue (e.g., its array index), lower its priority value*. Remember **lower priority value** is *better* (higher in tree).*
  - Change priority and percolate up

- `increaseKey`: given pointer to object in priority queue (e.g., its array index), raise its priority value.
  - Change priority and percolate down

- `remove`: given pointer to object in priority queue (e.g., its array index), remove it from the queue.
  - Percolate up to top and removeMin

Running time for all these operations?

Build Heap

- Suppose you have *n* items to put in a new (empty) priority queue
  - Call this operation **`buildHeap`**

- *n* distinct **`insert`**s works (slowly)
  - Only choice if ADT doesn't provide **`buildHeap`** explicitly
  - $O(n \log n)$

- Why would an ADT provide this unnecessary operation?
  - Convenience
  - Efficiency: an $O(n)$ algorithm called Floyd's Method
  - Common issue in ADT design: how many specialized operations

Floyd's Method

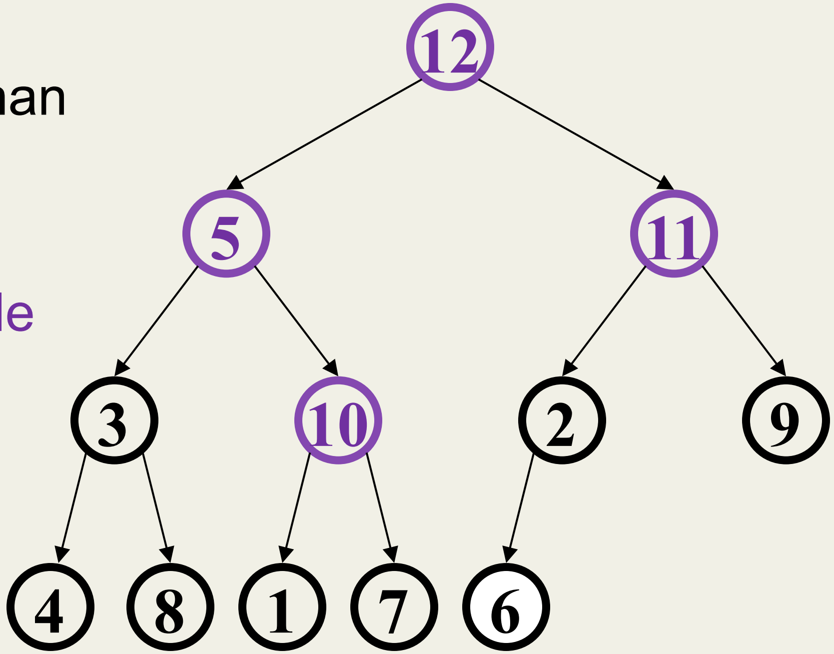
1. Use *n* items to make any complete tree you want
   – That is, put them in array indices 1,…,*n*

2. Treat it as a heap and fix the heap-order property
   – Bottom-up: leaves are already in heap order, work up toward the root one level at a time

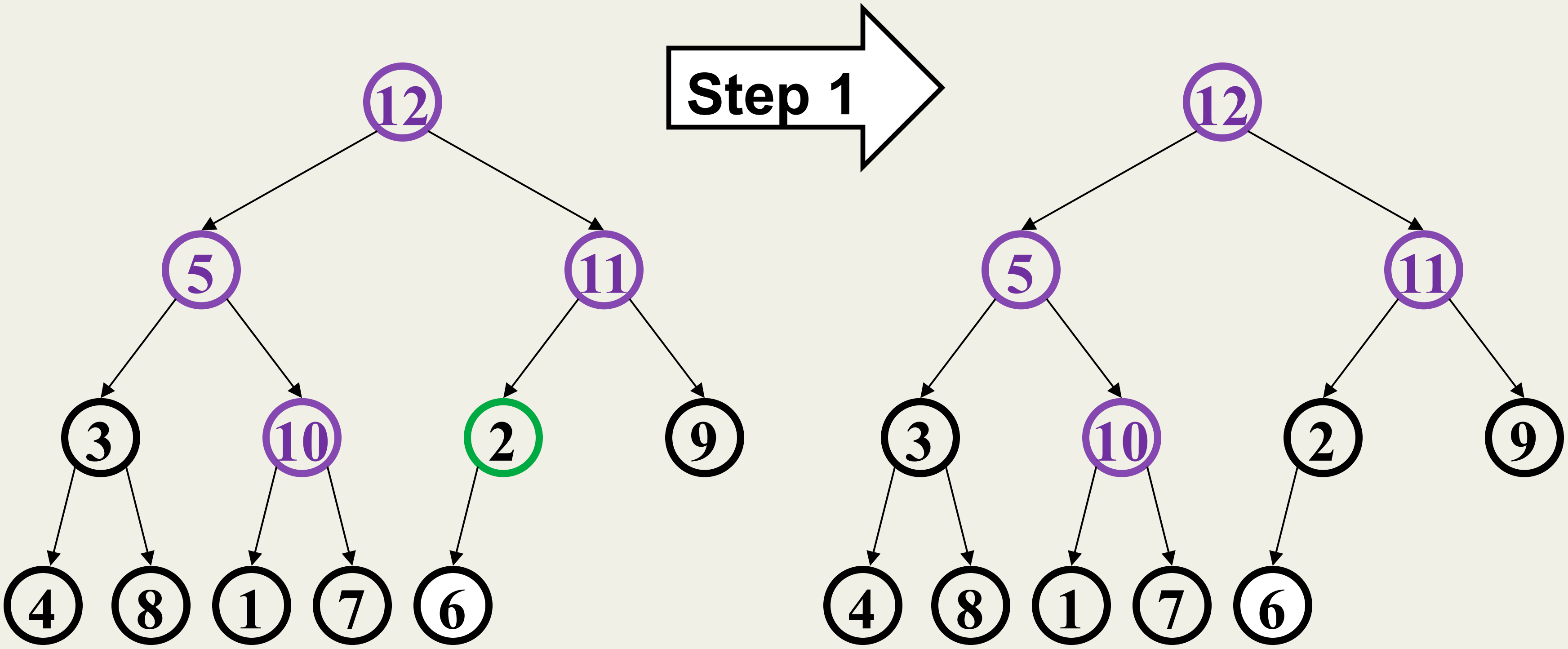
```
void buildHeap() {
   for(i = size/2; i>0; i--) {
     val  = arr[i];
     hole = percolateDown(i,val);
     arr[hole] = val;
   }
}
```

Example

- In tree form for readability
  - Purple for node not less than descendants
    - heap-order problem
  - Notice no leaves are purple
  - Check/fix each non-leaf bottom-up (6 steps here)

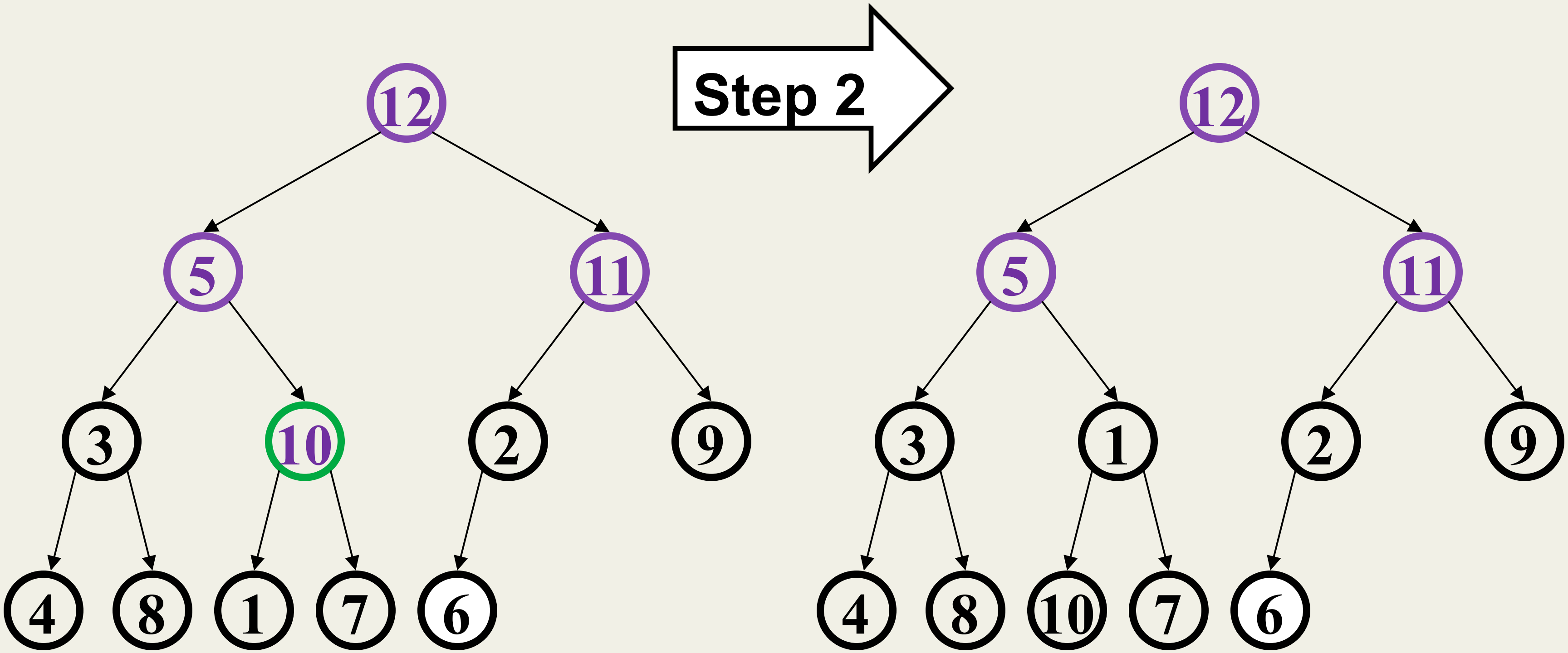


Example



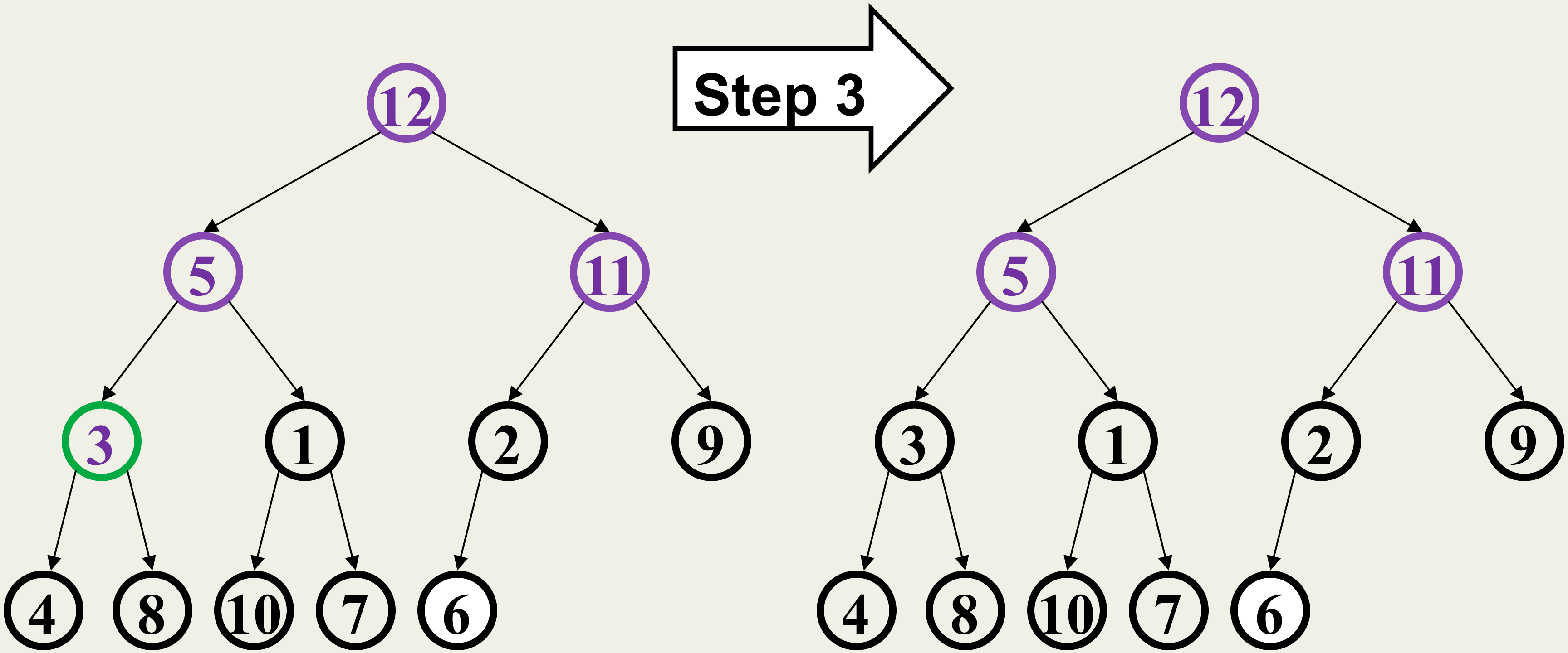
- Happens to already be less than children (er, child)

Example



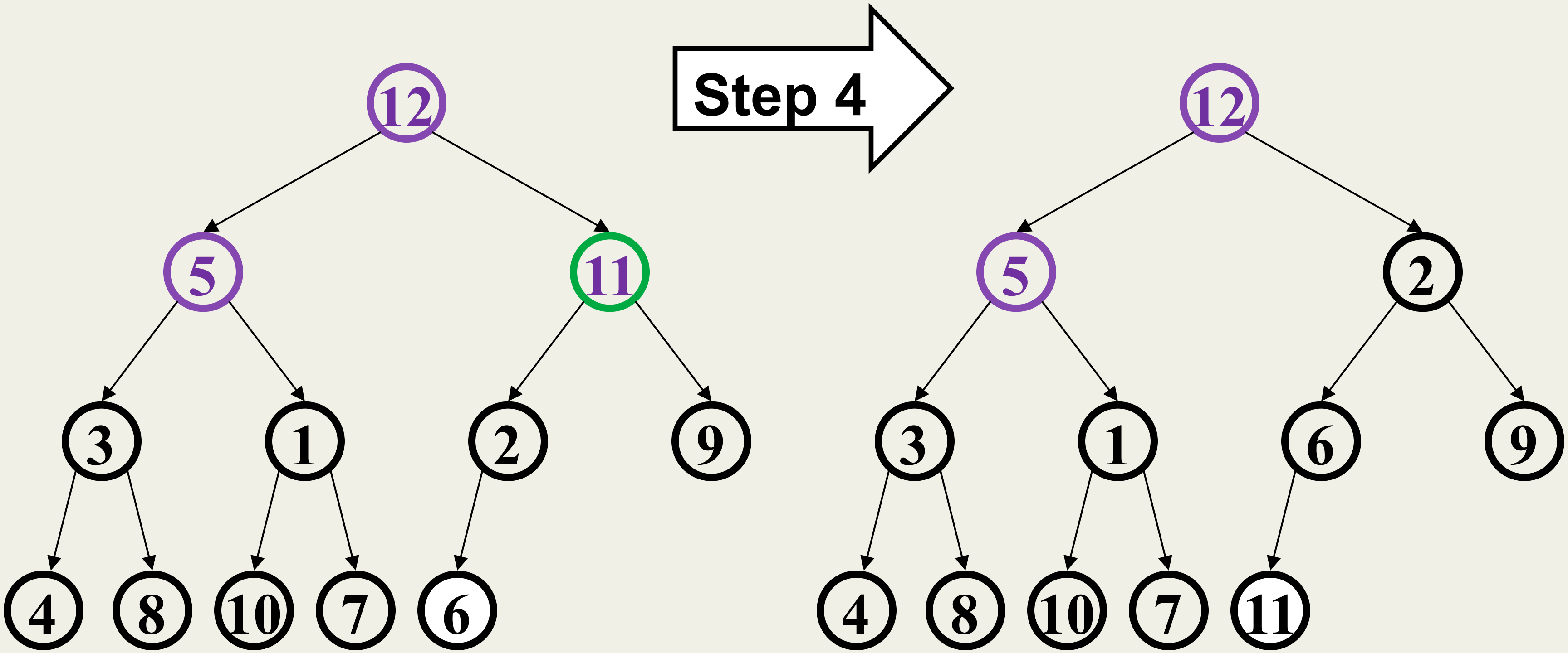
- Percolate down (notice that moves 1 up)

Example



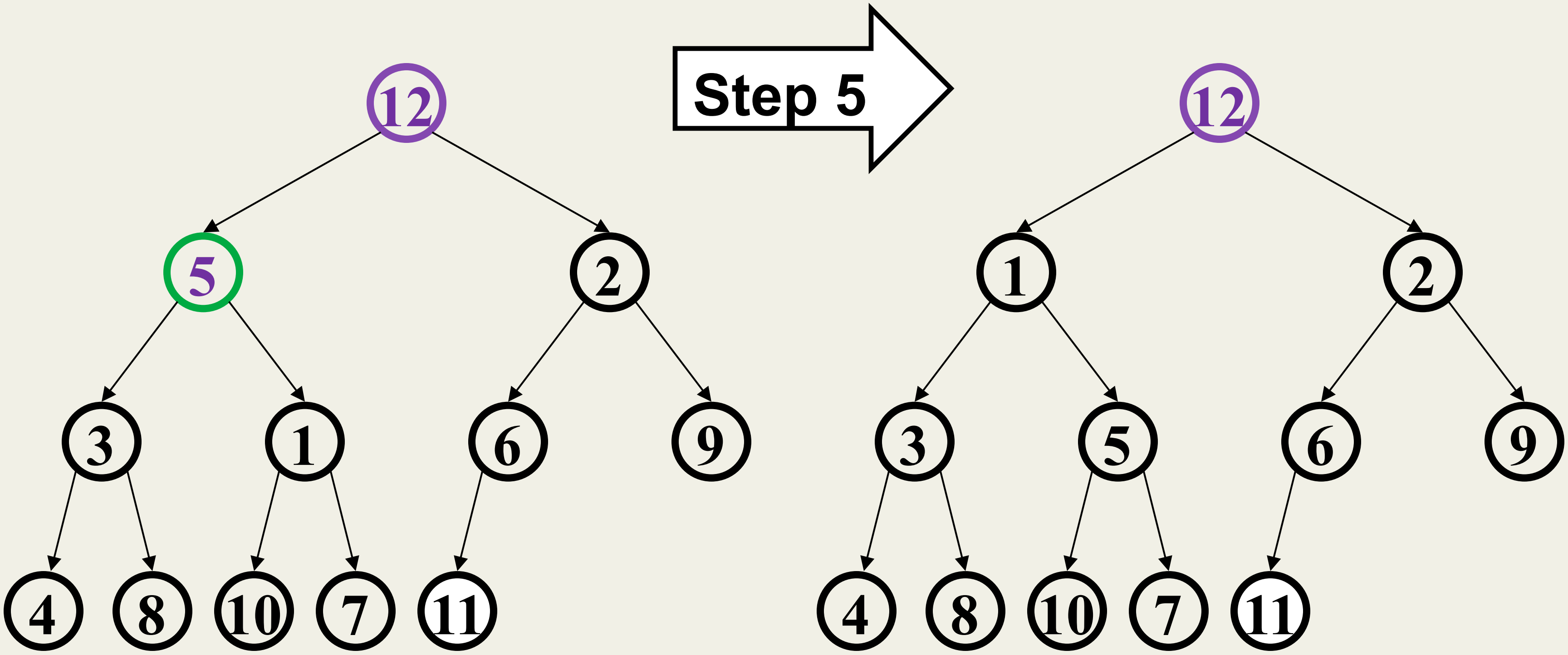
- Another nothing-to-do step

Example

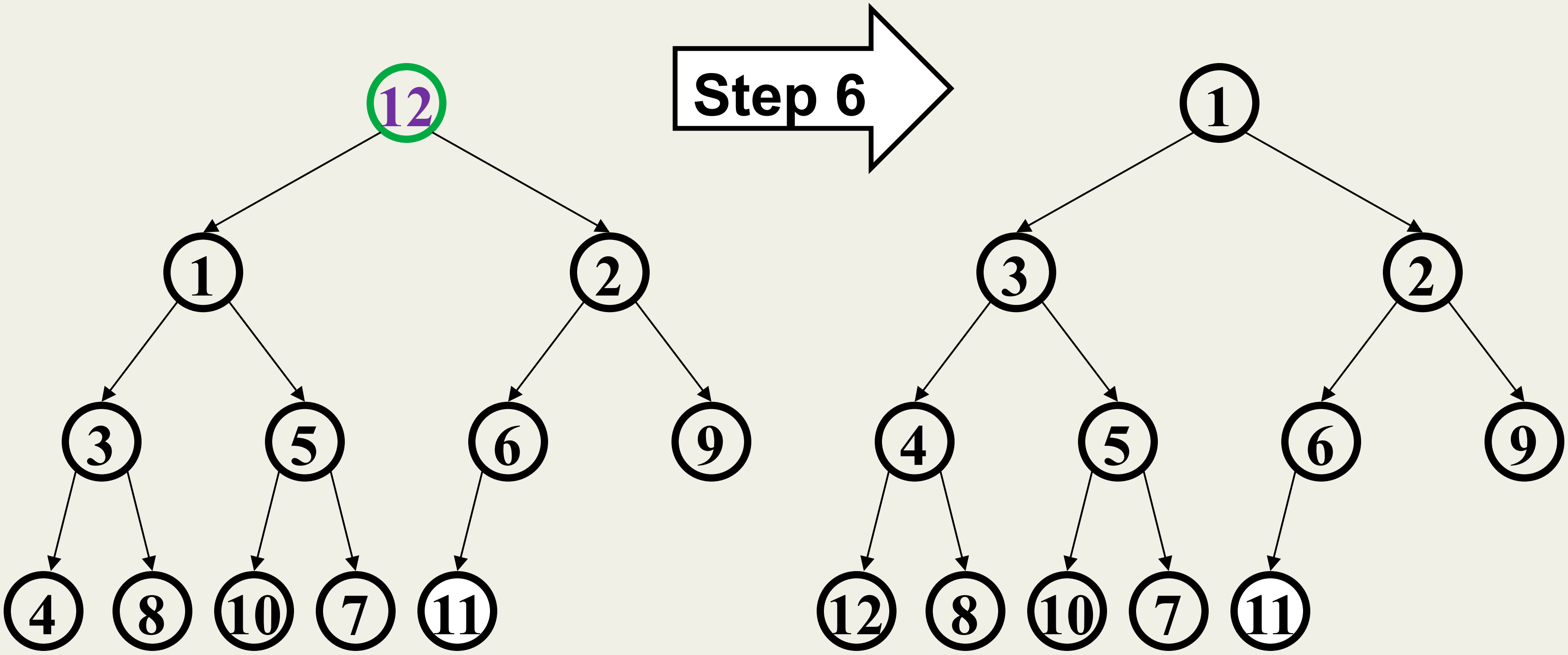


- Percolate down as necessary (steps 4a and 4b)

Example



Example



But is it right?

- "Seems to work"
  - Let's *prove* it restores the heap property (correctness)
  - Then let's *prove* its running time (efficiency)

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Correctness

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Loop Invariant: For all `j>i`, `arr[j]` is less than its children

- True initially: If `j > size/2`, then `j` is a leaf
  - Otherwise its left child would be at position > `size`
- True after one more iteration: loop body and `percolateDown` make `arr[i]` less than children without breaking the property for any descendants

So after the loop finishes, all nodes are less than their children

Efficiency

```
void buildHeap() {
   for(i = size/2; i>0; i--) {
     val  = arr[i];
     hole = percolateDown(i,val);
     arr[hole] = val;
   }
}
```

Easy argument: **`buildHeap`** is $O(n$ **`log`** $n)$ where n is **`size`**

- **`size/2`** loop iterations
- Each iteration does one **`percolateDown`**, each is $O($**`log`** $n)$

This is correct, but there is a more precise (“tighter”) analysis of the algorithm…

Efficiency

```
void buildHeap() {
   for(i = size/2; i>0; i--) {
     val  = arr[i];
     hole = percolateDown(i,val);
     arr[hole] = val;
   }
}
```

Better argument: **`buildHeap`** is $O(n)$ where n is **`size`**

- **`size/2`** total loop iterations: $O(n)$
- 1/2 the loop iterations percolate at most 1 step
- 1/4 the loop iterations percolate at most 2 steps
- 1/8 the loop iterations percolate at most 3 steps
- …
- ((1/2) + (2/4) + (3/8) + (4/16) + (5/32) + …) < 2 (page 4 of Weiss)
  - So at most **`2(size/2)`** *total* percolate steps: $O(n)$

Lessons from `buildHeap`

- Without **`buildHeap`**, our ADT already let clients implement their own in $O(n \texttt{ log } n)$ worst case
  - Worst case is inserting better priority values later
- By providing a specialized operation internal to the data structure (with access to the internal data), we can do $O(n)$ worst case
  - Intuition: Most data is near a leaf, so better to percolate down
- Can analyze this algorithm for:
  - Correctness:
    - Non-trivial inductive proof using loop invariant
  - Efficiency:
    - First analysis easily proved it was $O(n \texttt{ log } n)$
    - Tighter analysis shows same algorithm is $O(n)$

What we are skipping

- **`merge:`** given two priority queues, make one priority queue
  - How might you merge binary heaps:
    - If one heap is much smaller than the other?
    - If both are about the same size?
  - Different pointer-based data structures for priority queues support logarithmic time **`merge`** operation (impossible with binary heaps)
    - Leftist heaps, skew heaps, binomial queues
    - Worse constant factors
    - Trade-offs!