CSE373: Data Structures and Algorithms

Lecture 2: Math Review; Algorithm Analysis

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Today

- Finish discussing stacks and queues
- Review math essential to algorithm analysis
  - Proof by induction
  - Powers of 2
  - Binary numbers
  - Exponents and logarithms
- Begin analyzing algorithms
  - Using asymptotic analysis (continue next time)

Mathematical induction

Suppose *P(n)* is some predicate (mentioning integer *n*)

- Example: P(*n*) ≥ *n*/2 + 1

To prove *P(n)* for all integers $n \geq n_0$, it suffices to prove:

1. $P(n_0)$, called the basis or **base case**
2. If *P(k)* then *P(k+1)* , called the "induction step" or **inductive case**

Why we will care:

To show an algorithm is correct or has a certain running time, *no matter how big a data structure or input value is*

(Our ***"n"*** will be the data structure or input size.)

Example

P(n) = “the sum of the first *n* powers of 2 (starting at 0) is $2^n - 1$”

Theorem: *P(n)* holds for all $n \geq 1$

Proof: By induction on *n*

- Base case: $n = 1$:
  - Sum of first power of 2 is 2^0 , which equals 1.

    For $n = 1$: $2^n - 1 = 1$.

- Inductive case:
  - **Assumption**: the sum of the first *k* powers of 2 is $2^k - 1$
  - Show the sum of the first $(k + 1)$ powers of 2 is $2^{k+1} - 1$ using our assumption:
    Therefore, the sum of of the first $(k + 1)$ powers of 2 is:

    $=(2^k - 1) + 2^{(k+1)-1}$

    $=(2^k - 1) + 2^k$

    $= 2^{k+1} - 1$

Assumption

k+1’th term

Powers of 2

- A bit is 0 or 1 (just two different "letters" or "symbols")
- A sequence of *n* bits can represent 2^n distinct things
  - For example, the numbers 0 through 2^n-1
- 2^{10} is 1024 ("about a thousand", kilo in CSE speak)
- 2^{20} is "about a million", mega in CSE speak
- 2^{30} is "about a billion", giga in CSE speak

Java: an **`int`** is 32 bits and signed, so "max int" is "about 2 billion"

a **`long`** is 64 bits and signed, so "max long" is 2^{63}-1

Therefore…

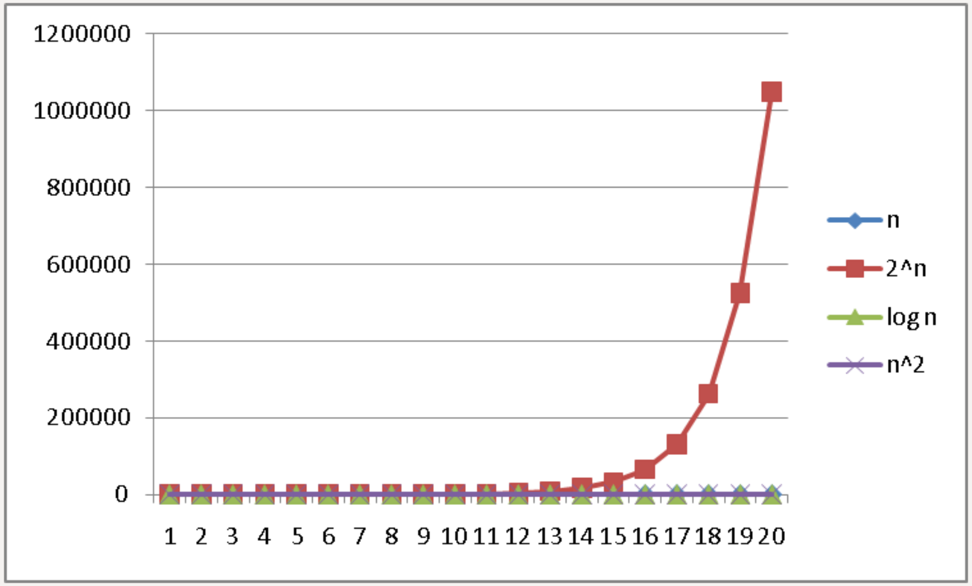
Could give a unique id to…

- Every person in the U.S. with 29 bits
- Every person in the world with 33 bits
- Every person to have ever lived with 38 bits (estimate)
- Every atom in the universe with 250-300 bits

So if a password is 128 bits long and randomly generated…
how long would it take to crack?

Logarithms and Exponents

- Since so much is binary `log` in CS almost always means $\log_2$
- Definition: $\log_2 x = y$ if $x = 2^y$
- So, $\log_2$ 1,000,000 = "a little under 20"
- Just as exponents grow *very* quickly, logarithms grow *very* slowly



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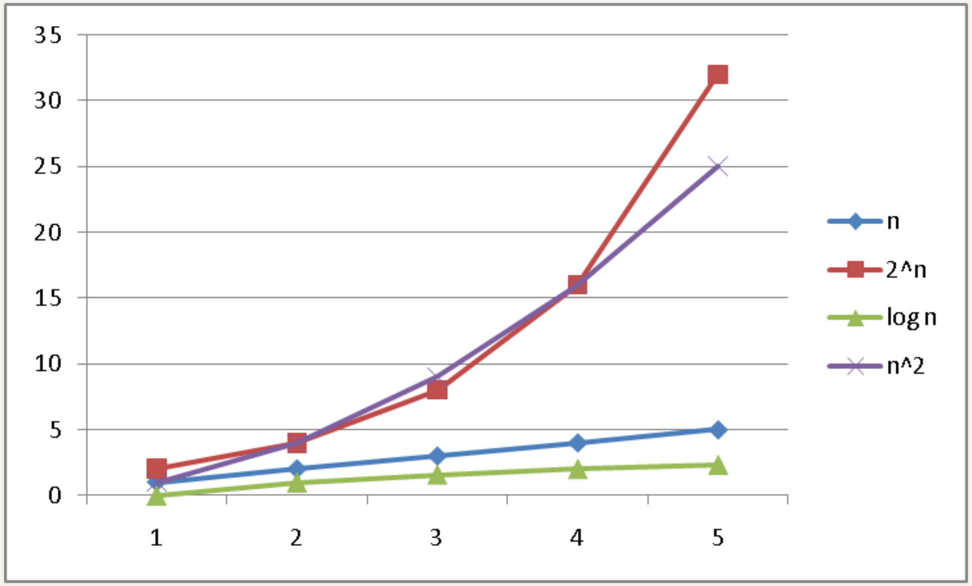

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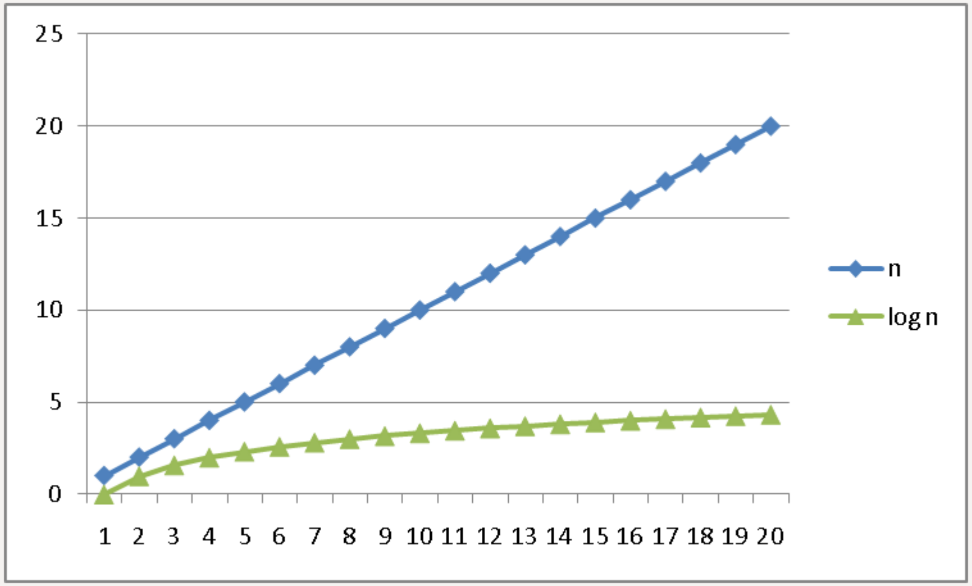

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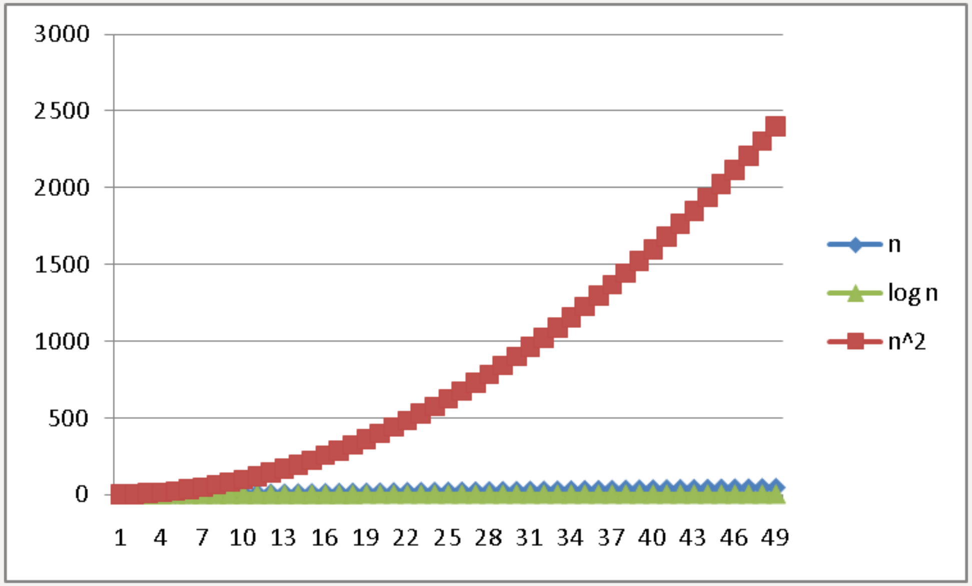


Properties of logarithms

- **`log(A*B) = log(A) + log(B)`**
  - So **`log(N`**^{k}**`)= k log(N)`**
- **`log(A/B) = log(A) – log(B)`**
- **`log log x`** is written **`log(log(x))`**
- **`log(x)log(x)`** is written **`log`**^{2}**`x`**
  - It is greater than **`log(x)`** for all **`x > 2`**
  - It is not the same as **`log(log(x))`**

Log base doesn't matter much!

"Any base *B* log is equivalent to base 2 log within a constant factor"

- And we are about to stop worrying about constant factors!
- In particular, $\log_2(x) \approx 3.22\log_{10}(x)$
- In general,

$$\log_B(x) = \log_A(x) \;/\; \log_A(B)$$

Floor and ceiling

$\lfloor X \rfloor$ Floor function: the largest integer $\leq$ X

$$\lfloor 2.7 \rfloor = 2 \qquad \lfloor -2.7 \rfloor = -3 \qquad \lfloor 2 \rfloor = 2$$

$\lceil X \rceil$ Ceiling function: the smallest integer $\geq$ X

$$\lceil 2.3 \rceil = 3 \qquad \lceil -2.3 \rceil = -2 \qquad \lceil 2 \rceil = 2$$

Floor and ceiling properties

1. $X - 1 < \lfloor X \rfloor \leq X$
2. $X \leq \lceil X \rceil < X + 1$
3. $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n \quad$ if n is an integer

Algorithm Analysis

As the "size" of an algorithm's input grows (integer, length of array, size of queue, etc.), we analyze:

- How much longer does the algorithm take? **(time)**
- How much more memory does the algorithm need? **(space)**

Because the curves we saw are so different, often care about only **which curve we resemble**

Separate issue: **Algorithm *correctness*** – does it produce the right answer for all input?

- Usually more important

```
//Sorts the given input array of 'ints'
public int[] miracleSort(int[] input){
        /*for (int i=0; i<10000; i++) {
            pray
        }*/
        return input;
}
```

Example

- What does this pseudocode return?

```
x := 0;
for i=1 to N do
  for j=1 to i do
     x := x + 3;
return x;
```

- Correctness: For any N ≥ 0, it returns…

Example

- What does this pseudocode return?

```
x := 0;
for i=1 to N do
  for j=1 to i do
     x := x + 3;
return x;
```

- Correctness: For any N ≥ 0, it returns 3N(N+1)/2

- Proof: By induction on *n*
  - *P(n)* = after outer for-loop executes *n* times, **`P(n)`** holds: 3n(n+1)/2

  - **Base case**: n=0, returns 0

  - **Inductive case**: Assume *P(k)* holds for 3k(k+1)/2 after *k* iterations. Next iteration adds 3(k+1). Show that it hold for (k + 1):

    = 3k(k+1)/2 + 3(k+1)

    = (3k(k+1) + 6(k+1))/2

    = (k+1)(3k+6)/2

    = 3(k+1)(k+2)/2

Example

- How long does this pseudocode run?

```
x := 0;
for i=1 to N do
  for j=1 to i do
     x := x + 3;
return x;
```

- Running time: For any N ≥ 0,
  - Assignments, additions, returns take "1 unit time"
  - Loops take the sum of the time for their iterations

Cost of assigning x and returning x

- So: 2 + 2*(number of times inner loop runs)
  - And how many times is that…

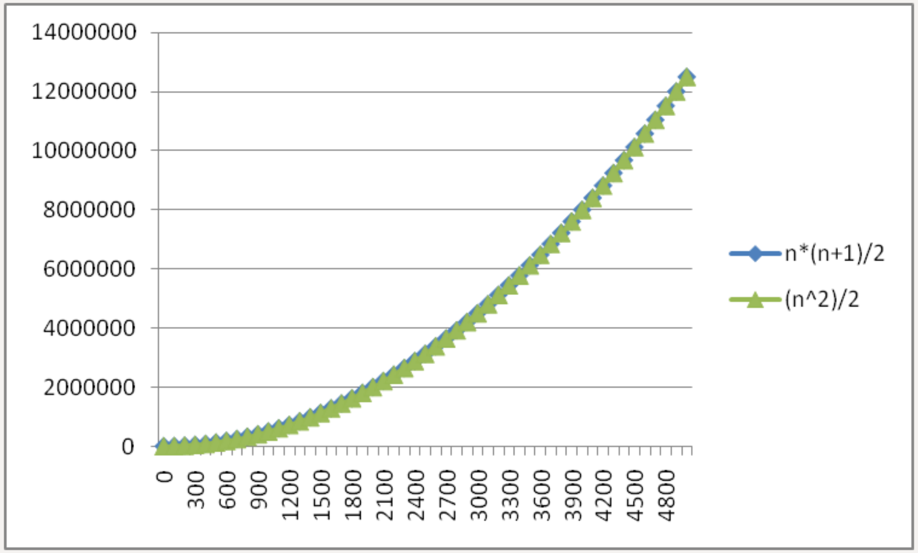
Example

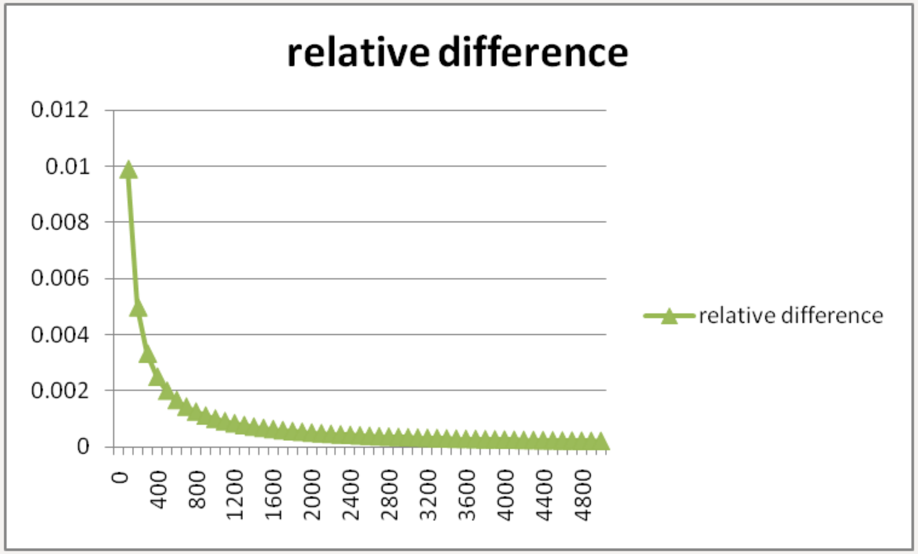
- How long does this pseudocode run?

```
x := 0;
for i=1 to N do
  for j=1 to i do
     x := x + 3;
return x;
```

- The total number of loop iterations is N*(N+1)/2
  - This is a very common loop structure, worth memorizing
  - Proof is by induction on N, known for centuries
  - This is *proportional to* N^2 , and we say $O(N^2)$, “big-Oh of”
    - For large enough N, the N and constant terms are irrelevant, as are the first assignment and return
    - See plot… N*(N+1)/2 vs. just $N^2/2$

Lower-order terms don’t matter

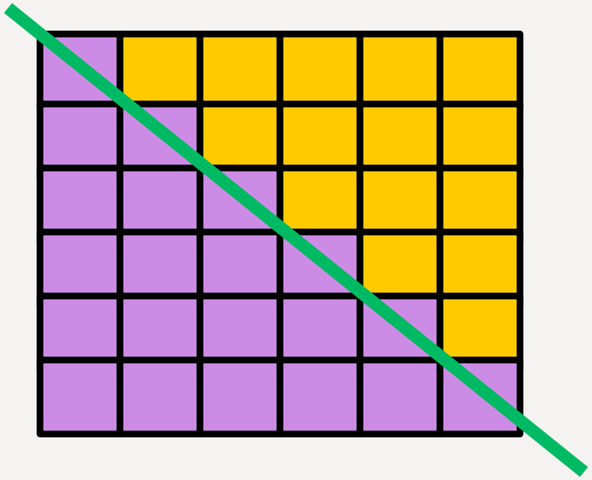




Geometric interpretation

$$\sum_{i=1}^{N} i = N^2/2+N/2$$

```
for i=1 to N do
  for j=1 to i do
     // small work
```



- Area of square: $\mathbf{N^2}$
- Area of lower triangle of square: $\mathbf{N^2/2}$
- Extra area from squares crossing the diagonal: **N/2**
- As N grows, fraction of “extra area” compared to lower triangle goes to zero (becomes insignificant)

Big-O: Common Names

$O(1)$	constant (same as $O(k)$ for constant k)
$O(\log n)$	logarithmic
$O(n)$	linear
$O(n \log n)$	"n $\log n$"
$O(n^2)$	quadratic
$O(n^3)$	cubic
$O(n^k)$	polynomial (where is k is any constant > 1)
$O(k^n)$	exponential (where k is any constant > 1)

exponential does not mean "grows really fast", it means "grows at rate proportional to k^n for some $k > 1$"!

- A savings account accrues interest exponentially(k=1.01?)
- If you don't know k, you probably don't know it's exponential