CSE373: Data Structure & Algorithms

Lecture 21: More Comparison Sorting

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The main problem, stated carefully

For now, assume we have *n* comparable elements in an array and we want to rearrange them to be in increasing order

Input:

- An array **`A`** of data records
- A key value in each data record
- A comparison function (consistent and total)

Effect:

- Reorganize the elements of **`A`** such that for any **`i`** and **`j`**, if **`i < j`** then **`A[i] ≤ A[j]`**
- (Also, **`A`** must have exactly the same data it started with)
- Could also sort in reverse order, of course

An algorithm doing this is a comparison sort

Sorting: The Big Picture

Surprising amount of neat stuff to say about sorting:

Simple algorithms: $O(n^2)$

Insertion sort
Selection sort
Shell sort
…

Fancier algorithms: $O(n \log n)$

Heap sort
Merge sort
Quick sort (avg)
…

Comparison lower bound: $\Omega(n \log n)$

Specialized algorithms: $O(n)$

Bucket sort
Radix sort

Handling huge data sets

External sorting

Mergesort Analysis

Having defined an algorithm and argued it is correct, we should analyze its running time and space:

To sort n elements, we:

- Return immediately if n=1
- Else do 2 subproblems of size $n/2$ and then an $O(n)$ merge

Recurrence relation:

$T(1) = c_1$

$T(n) = 2T(n/2) + c_2 n$

One of the recurrence classics…

For simplicity let constants be 1 – no effect on asymptotic answer

$T(1) = 1$

$T(n) = 2T(n/2) + n$

$= 2(2T(n/4) + n/2) + n$

$= 4T(n/4) + 2n$

$= 4(2T(n/8) + n/4) + 2n$

$= 8T(n/8) + 3n$

….

$= 2^{\mathbf{k}}T(n/2^{\mathbf{k}}) + kn$

So total is $2^{\mathbf{k}}T(n/2^{\mathbf{k}}) + kn$ where

$n/2^{\mathbf{k}} = 1$, i.e., $\log n = k$

That is, $2^{\mathbf{log\ n}}\, T(1) + n \log n$

$= n + n \log n$

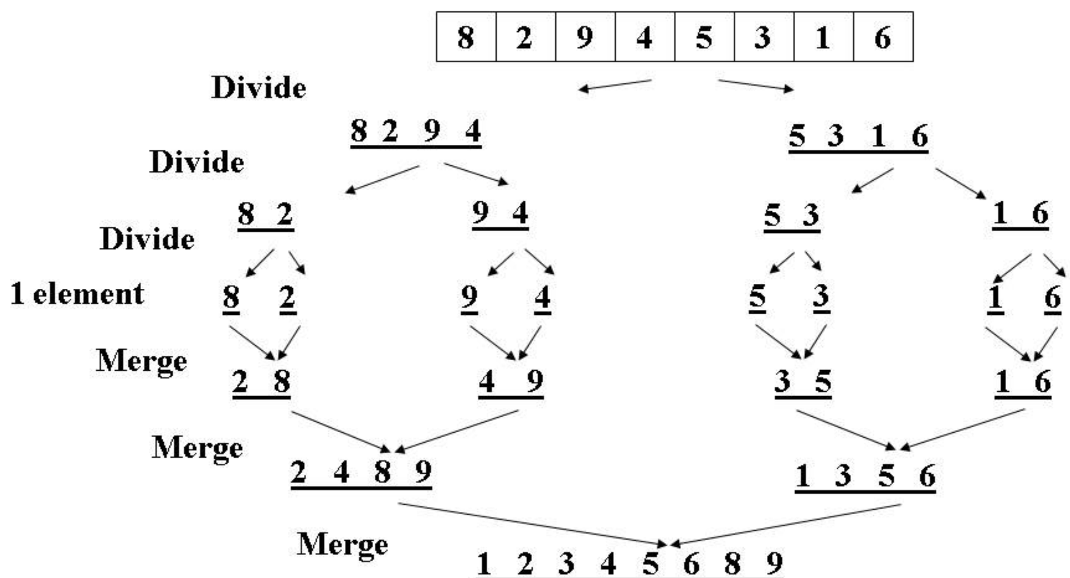
$= O(n \log n)$

Or more intuitively…

This recurrence is common you just "know" it's $O(n \log n)$

Merge sort is relatively easy to intuit (best, worst, and average):

- The recursion "tree" will have `log` *n* height
- At each level we do a *total* amount of merging equal to *n*



Quicksort

- Also uses divide-and-conquer
  - Recursively chop into two pieces
  - Instead of doing all the work as we merge together, we will do all the work as we recursively split into halves
  - Unlike merge sort, does not need auxiliary space
- $O(n \log n)$ on average ☺, but $O(n^2)$ worst-case ☹
- Faster than merge sort in practice?
  - Often believed so
  - Does fewer copies and more comparisons, so it depends on the relative cost of these two operations!

Quicksort Overview

1. Pick a pivot element

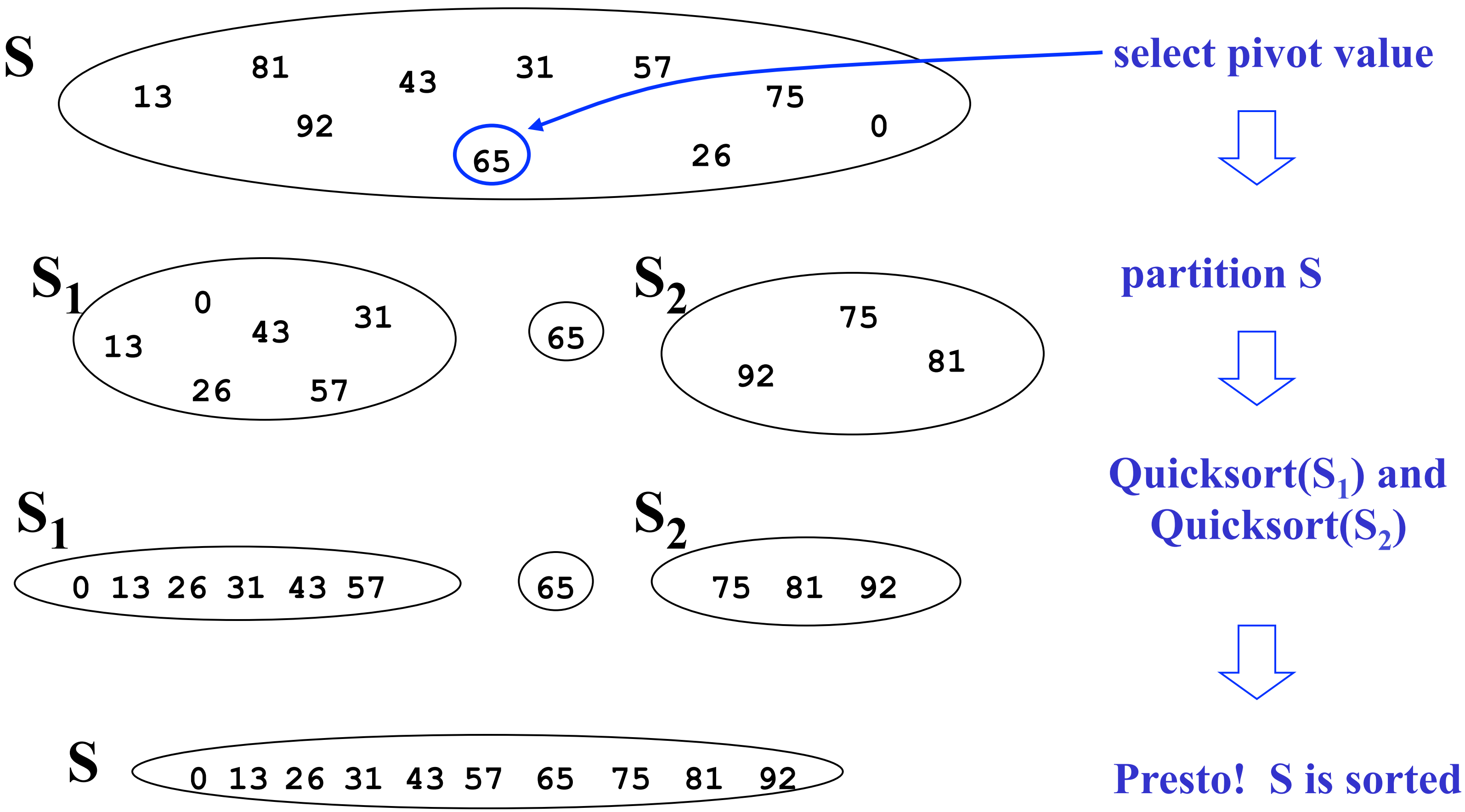
2. Partition all the data into:
   A. The elements less than the pivot
   B. The pivot
   C. The elements greater than the pivot

3. Recursively sort A and C

4. The answer is, “as simple as A, B, C”

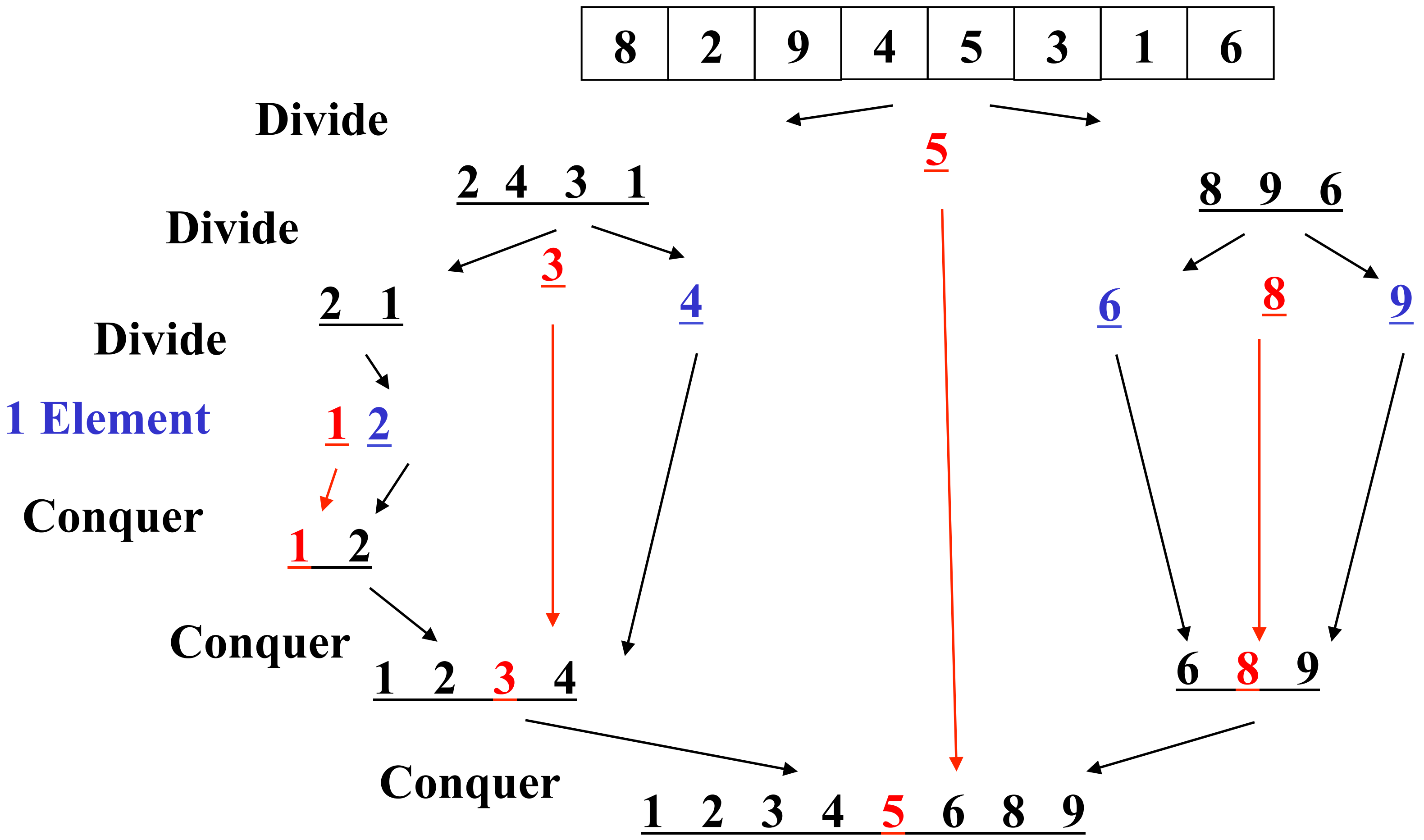
(Alas, there are some details lurking in this algorithm)

Think in Terms of Sets



[Weiss]

Example, Showing Recursion



Details

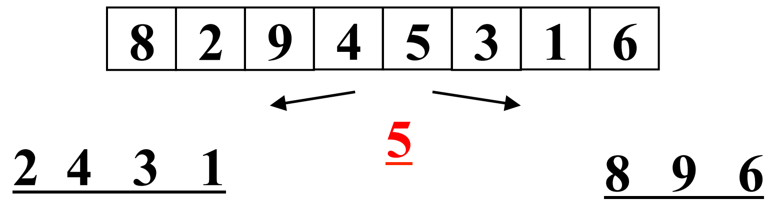
Have not yet explained:

- How to pick the pivot element
  - Any choice is correct: data will end up sorted
  - But as analysis will show, want the two partitions to be about equal in size

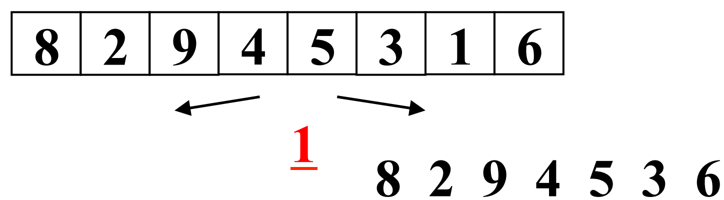
- How to implement partitioning
  - In linear time
  - In place

Pivots

- Best pivot?
  - Median
  - Halve each time



- Worst pivot?
  - Greatest/least element
  - Problem of size n - 1
  - $O(n^2)$



Potential pivot rules

While sorting **`arr`** from **`lo`** (inclusive) to **`hi`** (exclusive)…

- Pick **`arr[lo]`** or **`arr[hi-1]`**
  - Fast, but worst-case occurs with mostly sorted input

- Pick random element in the range
  - Does as well as any technique, but (pseudo)random number generation can be slow
  - Still probably the most elegant approach

- Median of 3, e.g., **`arr[lo], arr[hi-1], arr[(hi+lo)/2]`**
  - Common heuristic that tends to work well

Partitioning

- Conceptually simple, but hardest part to code up correctly
  – After picking pivot, need to partition in linear time in place

- One approach (there are slightly fancier ones):
  1. Swap pivot with **`arr[lo]`**
  2. Use two fingers **`i`** and **`j`**, starting at **`lo+1`** and **`hi-1`**

```
3. while (i < j)
       if (arr[j] > pivot) j--
       else if (arr[i] < pivot) i++
       else swap arr[i] with arr[j]
```

  4. Swap pivot with **`arr[i]`** *

*skip step 4 if pivot ends up being least element

Example

- Step one: pick pivot as median of 3
  - **`lo`** = 0, **`hi`** = 10

0	1	2	3	4	5	6	7	8	9
8	1	4	9	0	3	5	2	7	6

- Step two: move pivot to the **`lo`** position

0	1	2	3	4	5	6	7	8	9
6	1	4	9	0	3	5	2	7	8

Example

Often have more than one swap during partition – this is a short example

Now partition in place

6	1	4	9	0	3	5	2	7	8

Move fingers

6	1	4	9	0	3	5	2	7	8

Swap

6	1	4	2	0	3	5	9	7	8

Move fingers

6	1	4	2	0	3	5	9	7	8

Move pivot

5	1	4	2	0	3	6	9	7	8

Analysis

- Best-case: Pivot is always the median

  T(0)=T(1)=1

  T(n)=2T(n/2) + n -- linear-time partition

  Same recurrence as mergesort: $O(n \log n)$

- Worst-case: Pivot is always smallest or largest element

  T(0)=T(1)=1

  T(n) = 1T(n-1) + n

  Basically same recurrence as selection sort: $O(n^2)$

- Average-case (e.g., with random pivot)
  - O(n `log` n), not responsible for proof (in text)

Cutoffs

- For small n, all that recursion tends to cost more than doing a quadratic sort
  - Remember asymptotic complexity is for large n

- Common engineering technique: switch algorithm below a cutoff
  - Reasonable rule of thumb: use insertion sort for $n < 10$

- Notes:
  - Could also use a cutoff for merge sort
  - Cutoffs are also the norm with parallel algorithms
    - Switch to sequential algorithm
  - None of this affects asymptotic complexity

Cutoff skeleton

```
void quicksort(int[] arr, int lo, int hi) {
  if(hi - lo < CUTOFF)
     insertionSort(arr,lo,hi);
  else
     …
}
```

Notice how this cuts out the vast majority of the recursive calls

- Think of the recursive calls to quicksort as a tree
- Trims out the bottom layers of the tree

Visualizations

- http://www.cs.usfca.edu/~galles/visualization/Algorithms.html

How Fast Can We Sort?

- Heapsort & mergesort have $O(n \log n)$ worst-case running time
- Quicksort has $O(n \log n)$ average-case running time
- These bounds are all tight, actually $\Theta(n \log n)$
- So maybe we need to dream up another algorithm with a lower asymptotic complexity, such as $O(n)$ or $O(n \log \log n)$
  – Instead: we *know* that this is *impossible*
    - *Assuming* our comparison *model*: The only operation an algorithm can perform on data items is a 2-element comparison

A General View of Sorting

- Assume we have *n* elements to sort
  - For simplicity, assume none are equal (no duplicates)

- How many *permutations* of the elements (possible orderings)?

- Example, *n*=3

  a[0]<a[1]<a[2]   a[0]<a[2]<a[1]   a[1]<a[0]<a[2]

  a[1]<a[2]<a[0]   a[2]<a[0]<a[1]   a[2]<a[1]<a[0]

- In general, *n* choices for least element, *n*-1 for next, *n*-2 for next, …
  - *n*(*n*-1)(*n*-2)…(2)(1) = *n*! possible orderings

Counting Comparisons

- So *every* sorting algorithm has to "find" the right answer among the n! possible answers
  - Starts "knowing nothing", "anything is possible"
  - Gains information with each comparison
  - Intuition: Each comparison can *at best* eliminate *half* the remaining possibilities
  - Must narrow answer down to a single possibility

- What we can show:

Any sorting algorithm must do *at least* $(1/2)n \log n - (1/2)n$

(which is $\Omega(n \log n)$) comparisons

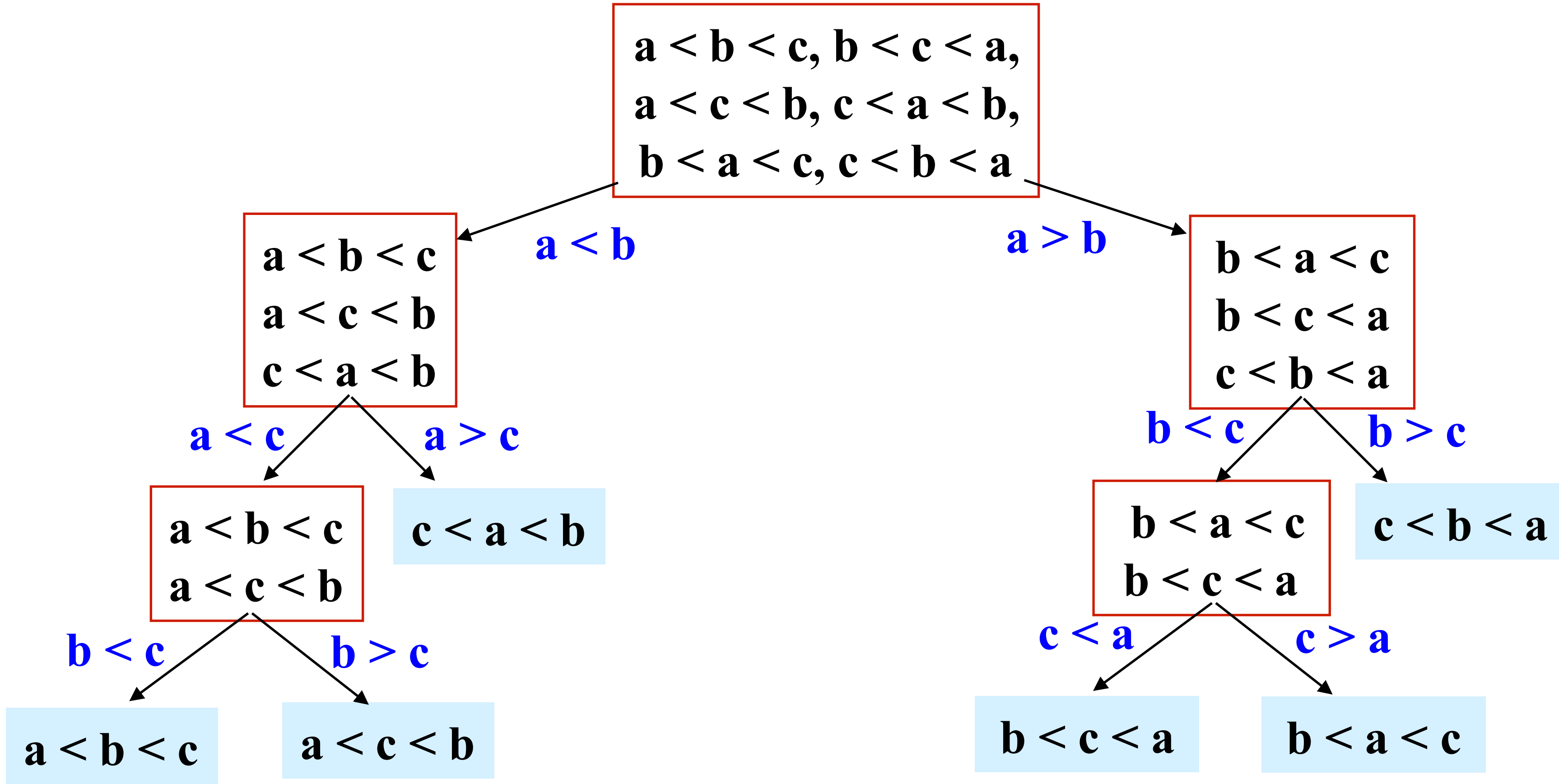
  - Otherwise there are at least two permutations among the n! possible that cannot yet be distinguished, so the algorithm would have to guess and could be wrong [incorrect algorithm]

Optional: *Counting Comparisons*

- Don't know what the algorithm is, but it cannot make progress without doing comparisons
  - Eventually does a first comparison "is $a < b$?"
  - Can use the result to decide what second comparison to do
  - Etc.: comparison k can be chosen based on first k-1 results

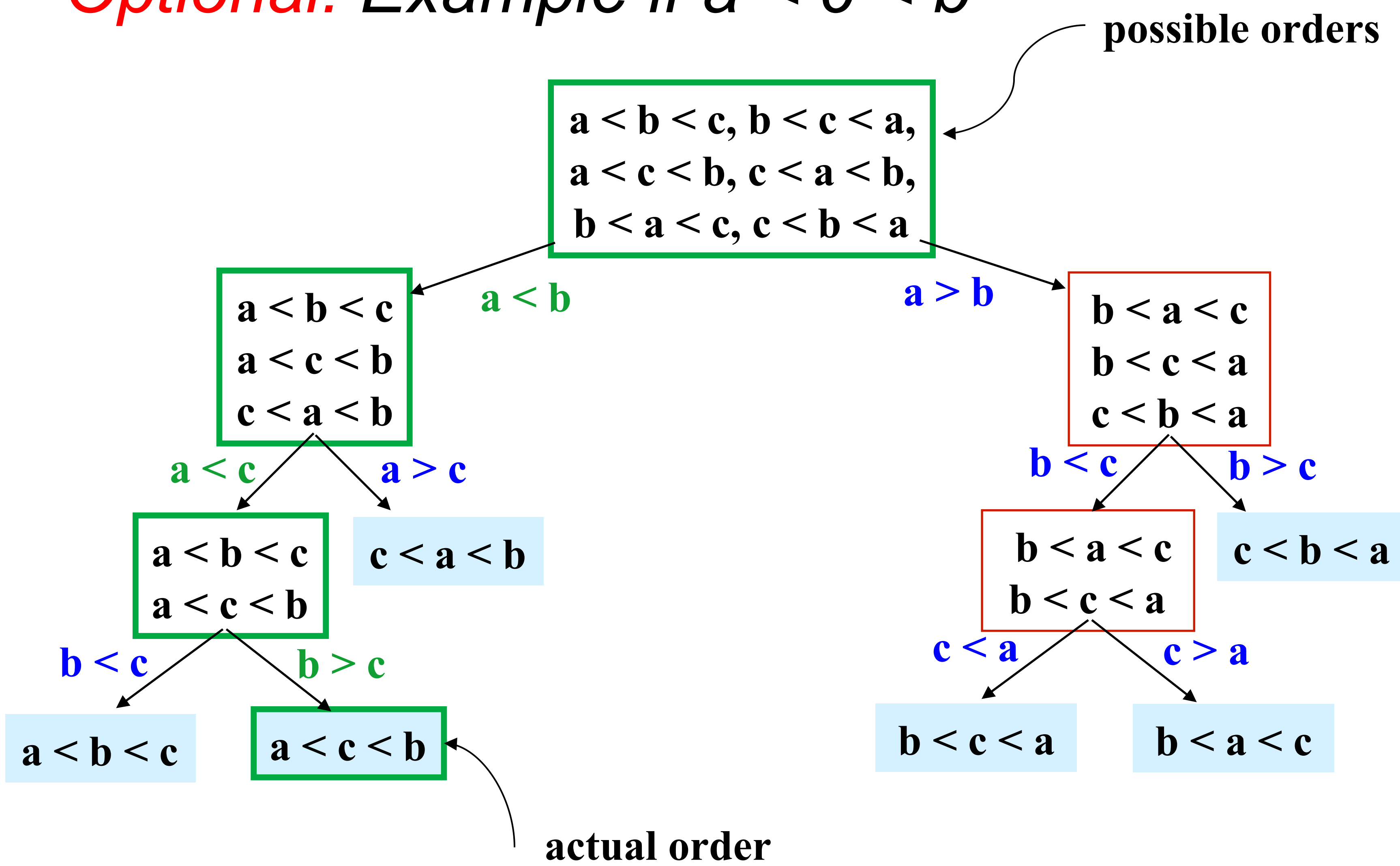
- Can represent this process as a *decision tree*
  - Nodes contain "set of remaining possibilities"
    - Root: None of the $n!$ options yet eliminated
  - Edges are "answers from a comparison"
  - The algorithm does not actually build the tree; it's what our *proof* uses to represent "the most the algorithm could know so far" as the algorithm progresses

Optional: *One Decision Tree for n=3*



- **The leaves contain all the possible orderings of a, b, c**
- **A different algorithm would lead to a different tree**

Optional: Example if a < c < b



Optional: What the Decision Tree Tells Us

- A binary tree because each comparison has 2 outcomes
  - (We assume no duplicate elements)
  - (Would have 1 outcome if algorithm asks redundant questions)

- Because any data is possible, any algorithm needs to ask enough questions to produce all *n*! answers
  - Each answer is a different leaf
  - So the tree must be big enough to have *n*! leaves
  - Running *any* algorithm on *any* input will *at best* correspond to a root-to-leaf path in *some* decision tree with *n*! leaves
  - So no algorithm can have worst-case running time better than the height of a tree with *n*! leaves
    - Worst-case number-of-comparisons for an algorithm is an input leading to a longest path in algorithm's decision tree

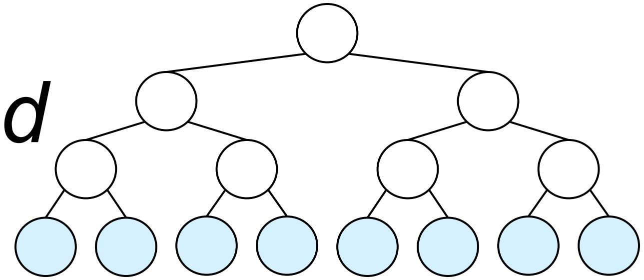
Optional: *Where are we*

- Proven: No comparison sort can have worst-case running time better than the height of a binary tree with n! leaves
  - A comparison sort could be worse than this height, but it cannot be better

- Now: a binary tree with n! leaves has height $\Omega(n \texttt{ log } n)$
  - Height could be more, but cannot be less
  - Factorial function grows very quickly

- Conclusion: Comparison sorting is $\Omega(n \texttt{ log } n)$
  - An amazing computer-science result: proves all the clever programming in the world cannot comparison-sort in linear time

Optional: *Height lower bound*



- The height of a binary tree with L leaves is at least $\log_2 L$
- So the height of our decision tree, h:

$h \geq \log_2 (n!)$	property of binary trees
$= \log_2 (n*(n-1)*(n-2)\ldots(2)(1))$	definition of factorial
$= \log_2 n + \log_2 (n-1) + \ldots + \log_2 1$	property of logarithms
$\geq \log_2 n + \log_2 (n-1) + \ldots + \log_2 (n/2)$	drop smaller terms (≥0)
$\geq \log_2 (n/2) + \log_2 (n/2) + \ldots + \log_2 (n/2)$	shrink terms to $\log_2 (n/2)$
$= (n/2)\log_2 (n/2)$	arithmetic
$= (n/2)(\log_2 n - \log_2 2)$	property of logarithms
$= (1/2)n\log_2 n - (1/2)n$	arithmetic
"=" $\Omega(n \log n)$	