CSE 373

AVL trees

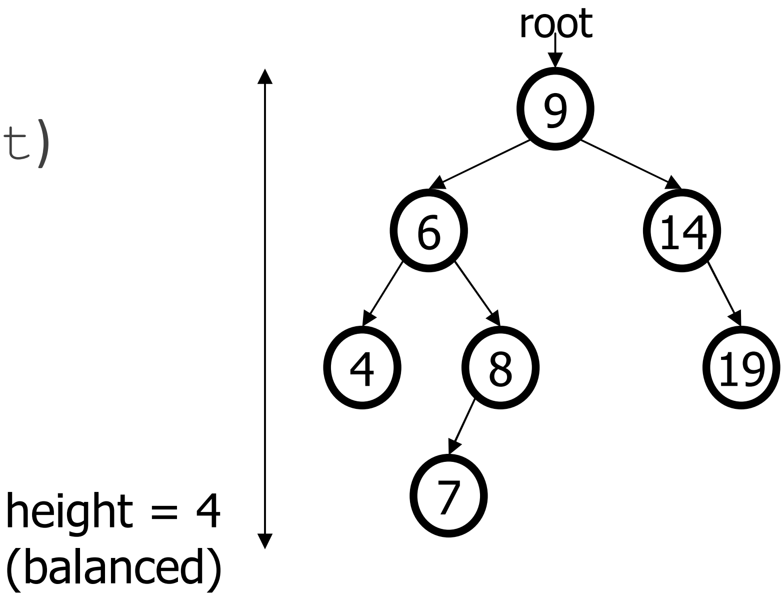
read: Weiss Ch. 4, section 4.1 - 4.4

slides created by Marty Stepp

http://www.cs.washington.edu/373/

Trees and balance

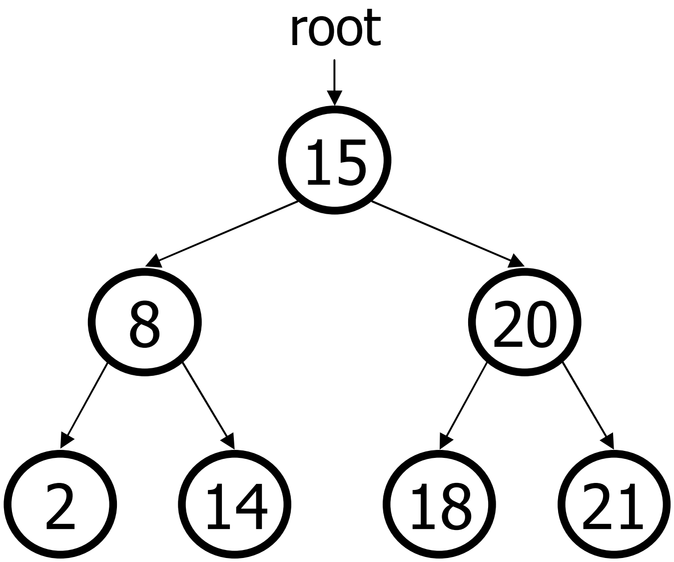
- **balanced tree**: One whose subtrees differ in height by at most 1 and are themselves balanced.
  - A balanced tree of *N* nodes has a height of ~ $\log_2 N$.
  - A very unbalanced tree can have a height close to *N*.
  - The runtime of adding to / searching a BST is closely related to height.
  - Some tree collections (e.g. `TreeSet`) contain code to balance themselves as new nodes are added.



Some height numbers

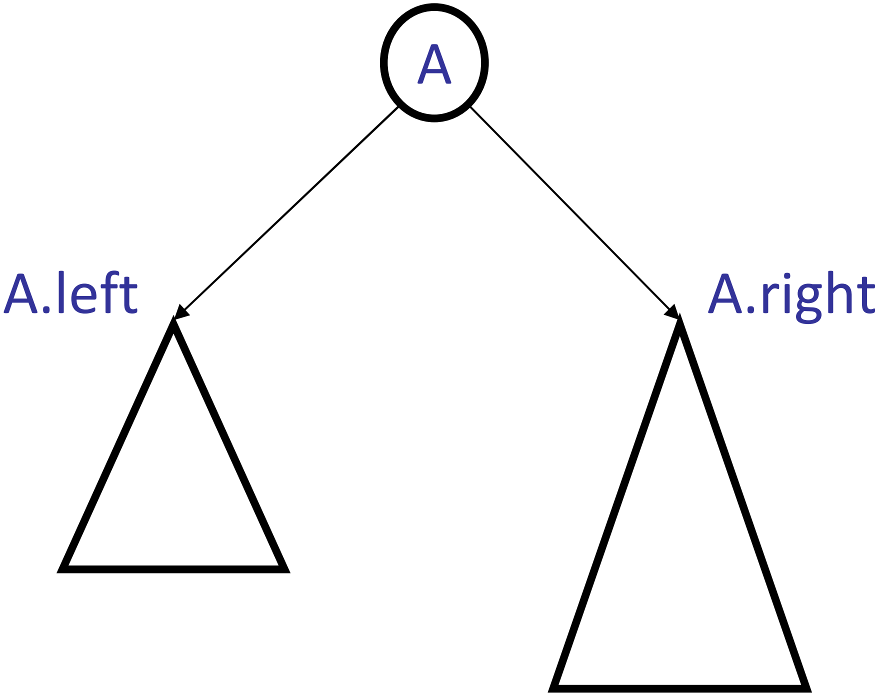
- *Observation:* The shallower the BST the better.
  - Average case height is O(log *N*)
  - Worst case height is O(*N*)
  - Simple cases such as adding (1, 2, 3, ..., *N*), or the opposite order, lead to the worst case scenario: height O(*N*).

- For binary tree of height *h*:
  - max # of leaves: 2^{h-1}
  - max # of nodes: $2^h - 1$
  - min # of leaves: 1
  - min # of nodes: h



Calculating tree height

- Height is max number of nodes in path from root to any leaf.
  - height(null) = 0
  - height(a leaf) = ?
  - height(A) = ?
  - *Hint:* it's recursive!
  - height(a leaf) = 1
  - height(A) = 1 + max(
    height(A.left), height(A.right))

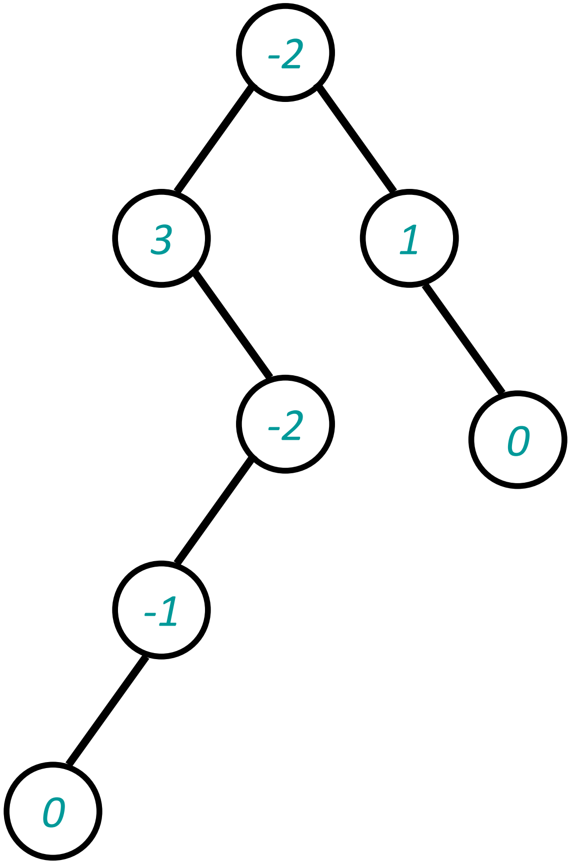


AVL trees

- **AVL tree**: a binary search tree that uses modified add and remove operations to stay balanced as its elements change
  - one of several kinds of auto-balancing trees (others in book)
  - invented in 1962 by two Russian mathematicians
    - (Adelson-Velskii and Landis)
    - A-V & L proved that an AVL tree's height is always O(log *N*).
  - *basic idea:* When nodes are added to / removed from the tree, if the tree becomes unbalanced, repair the tree until balance is restored.
    - rebalancing operations are relatively efficient (O(1))
    - overall tree maintains a balanced O(log *N*) height, fast to add/search

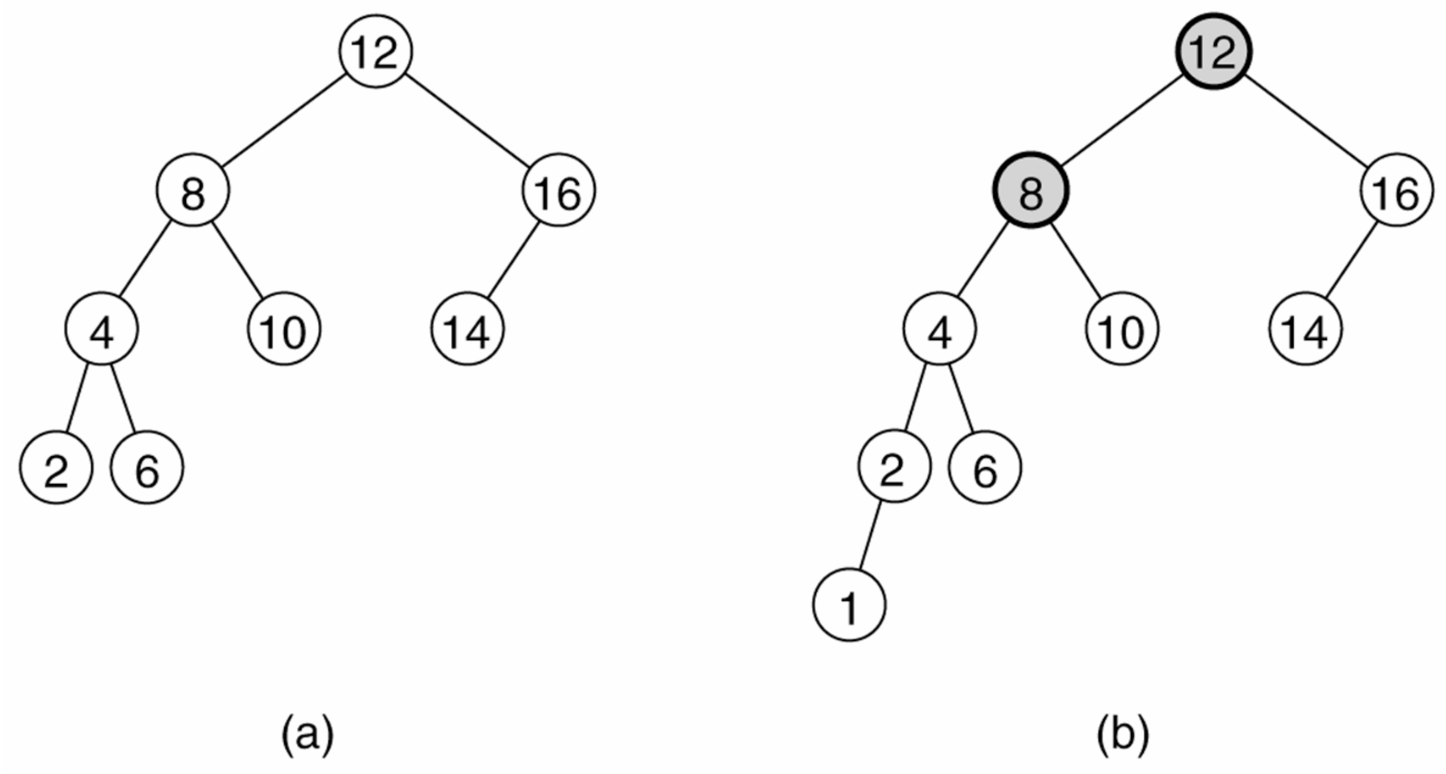
Balance factor

- **balance factor**, for a tree node *T* :
  - = height of *T*'s right subtree minus height of *T*'s left subtree.
  - BF(*T*) = Height(*T*.right) - Height(*T*.left)
    - (the tree at right shows BF of each node)
  - an AVL tree maintains a "balance factor" in each node of 0, 1, or -1
    - i.e. no node's two child subtrees differ in height by more than 1
  - it can be proven that the height of an AVL tree with *N* nodes is O(log *N*)



AVL tree examples

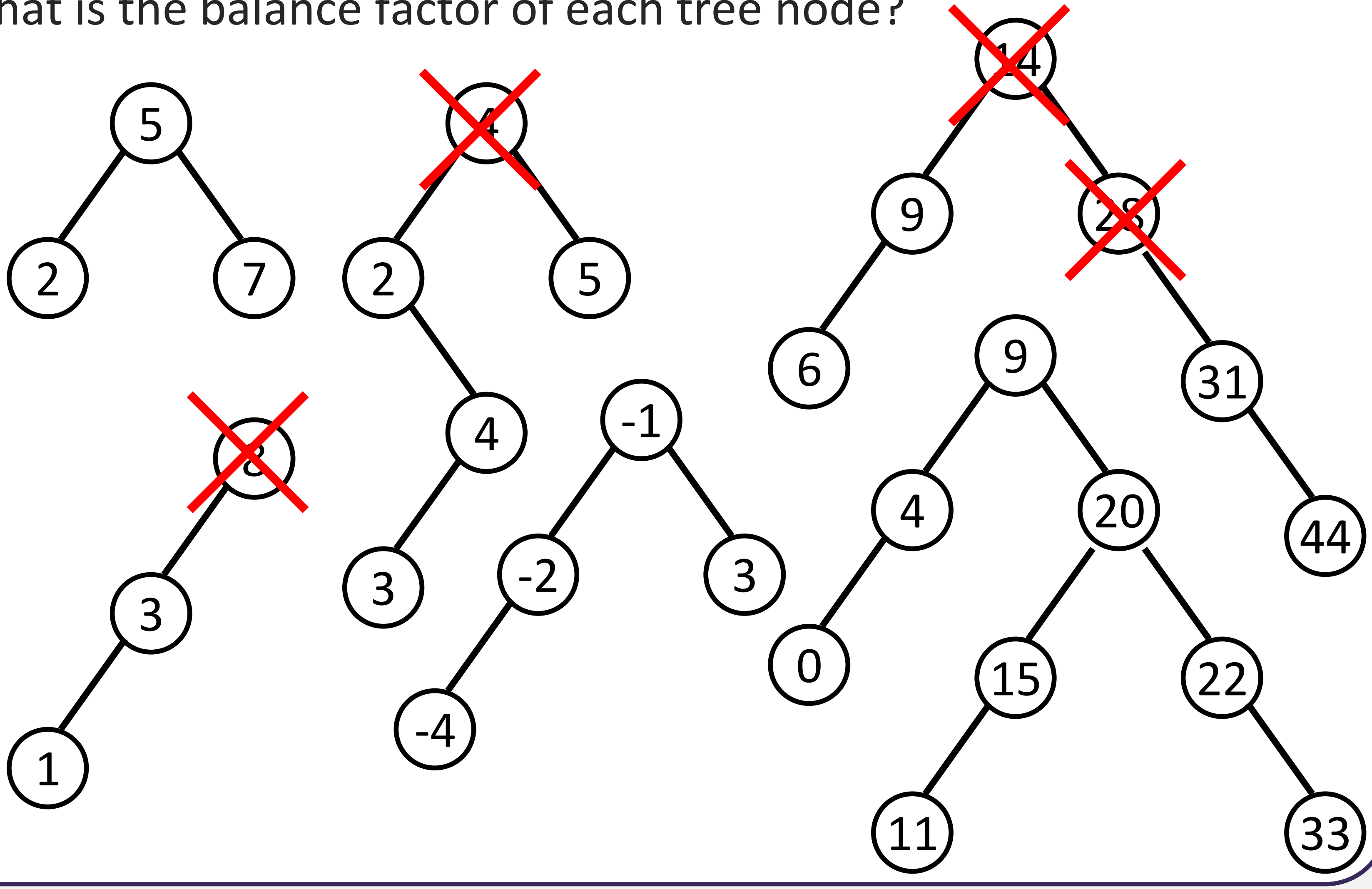
- Two binary search trees:
  - (a) an AVL tree
  - (b) not an AVL tree (unbalanced nodes are darkened)



(a) (b)

Which are valid AVL trees?

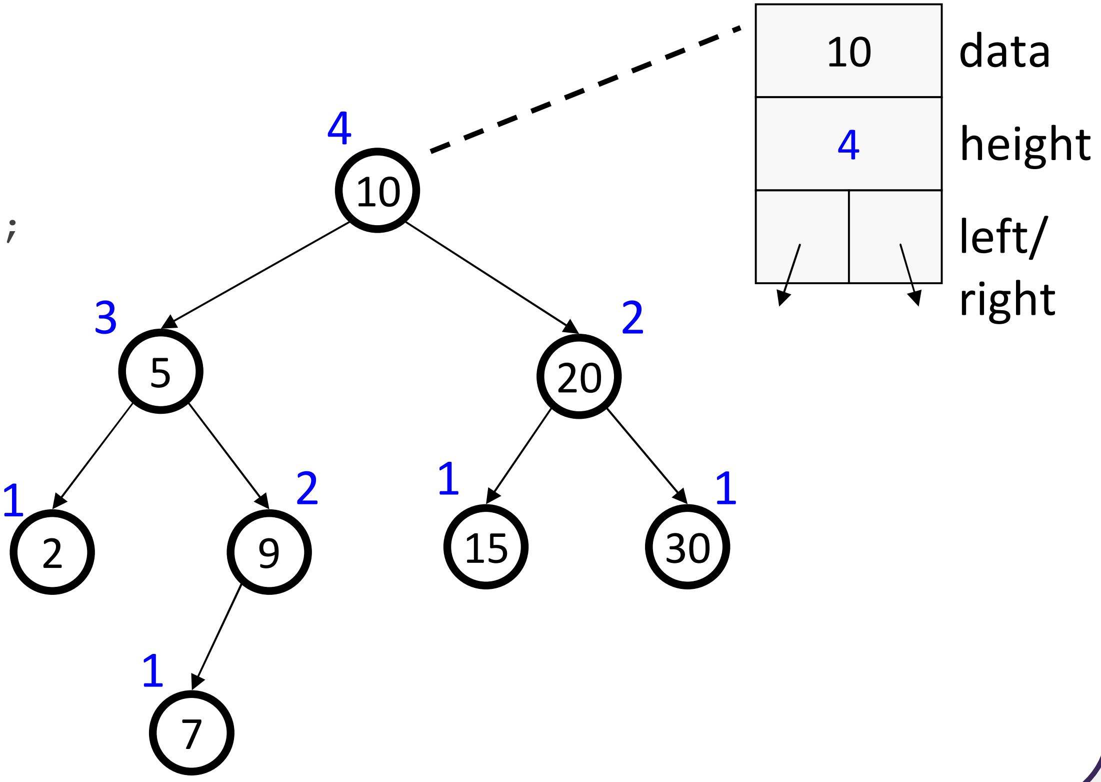
- What is the balance factor of each tree node?



Tracking subtree height

- Many of the AVL tree operations depend on height.
  - Height can be computed recursively by walking the tree; *too slow*.
  - Instead, each node can keep track of its subtree height as a field:

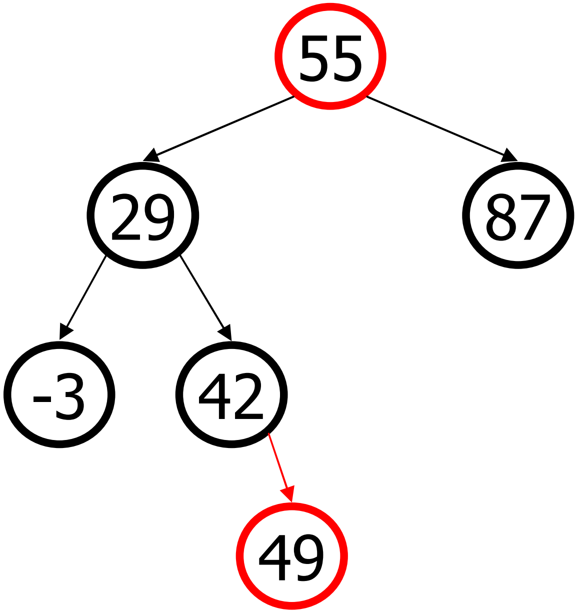
```
private class TreeNode {
  private E data;
  private int height;
  private TreeNode left;
  private TreeNode right;
}
```



AVL add operation

- For all AVL operations, we assume the tree *was* balanced before the operation began.
  - Adding a new node begins the same as with a typical BST, traversing left and right to find the proper location and attaching the new node.
  - But adding this new node may unbalance the tree by 1:

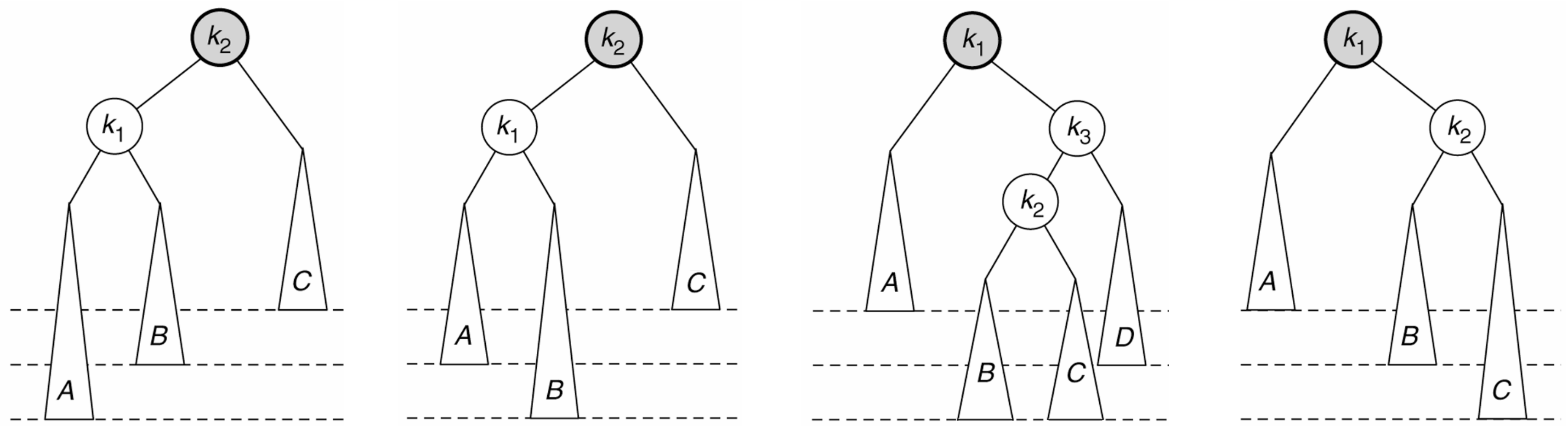
```
set.add(49);
```



AVL add cases

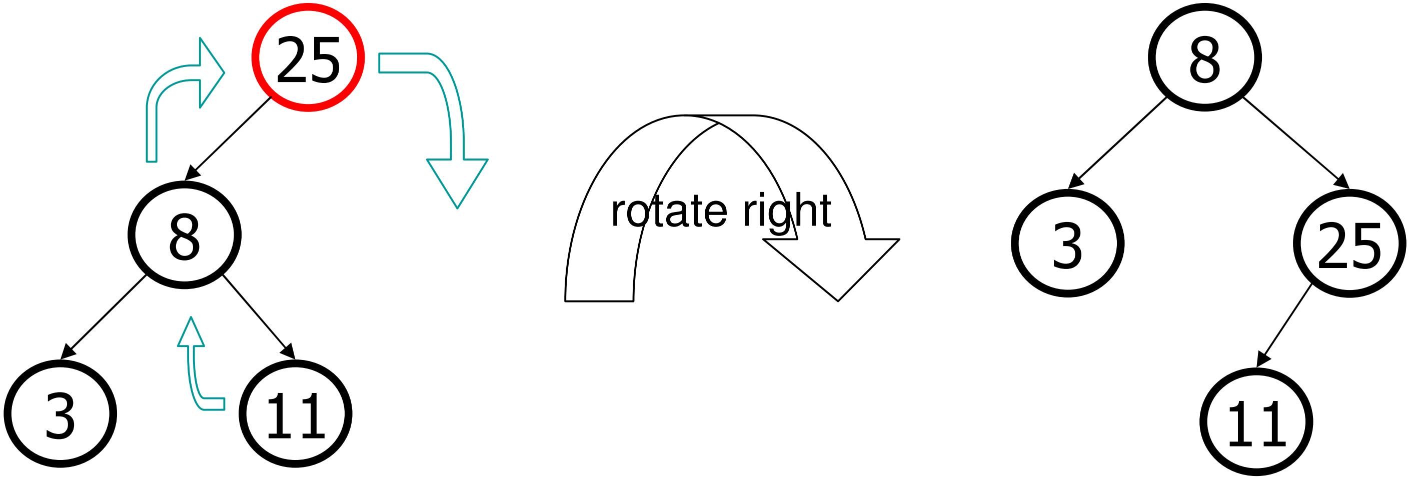
- Consider the lowest node k_2 that has now become unbalanced.
  - The new offending node could be in one of the four following grandchild subtrees, relative to k_2:

1) Left-Left, 2) Left-Right, 3) Right-Left, 4) Right-Right



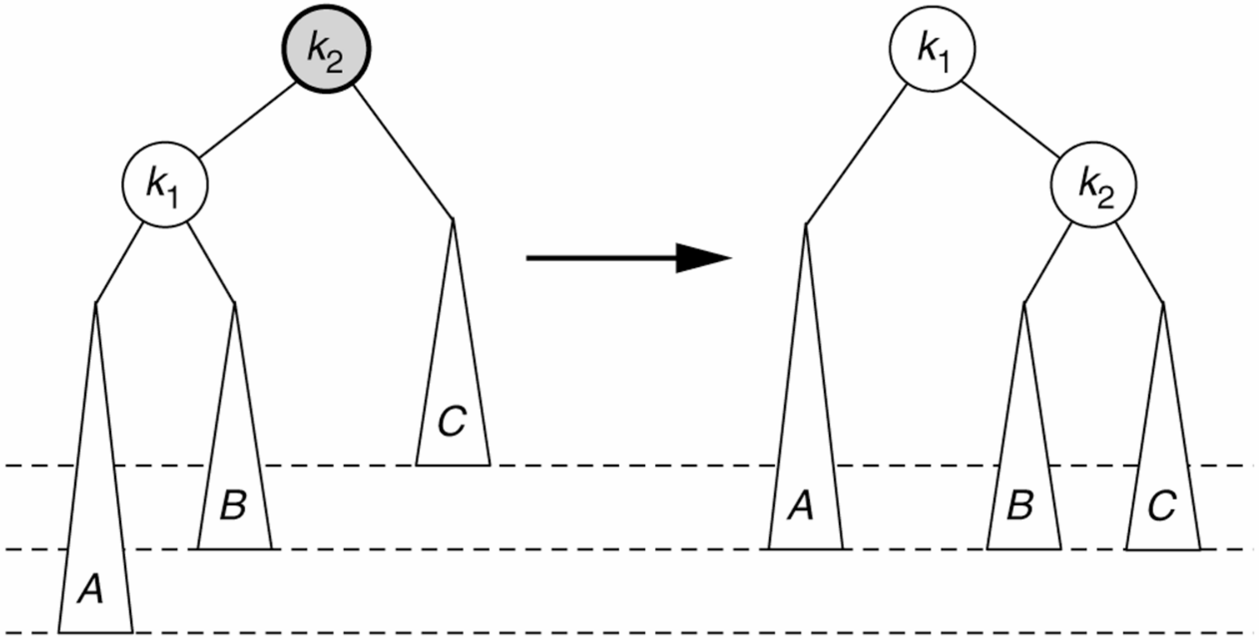
Key idea: rotations

- If a node has become out of balanced in a given direction, *rotate* it in the opposite direction.
  - *rotation:* A swap between parent and left or right child, maintaining proper BST ordering.



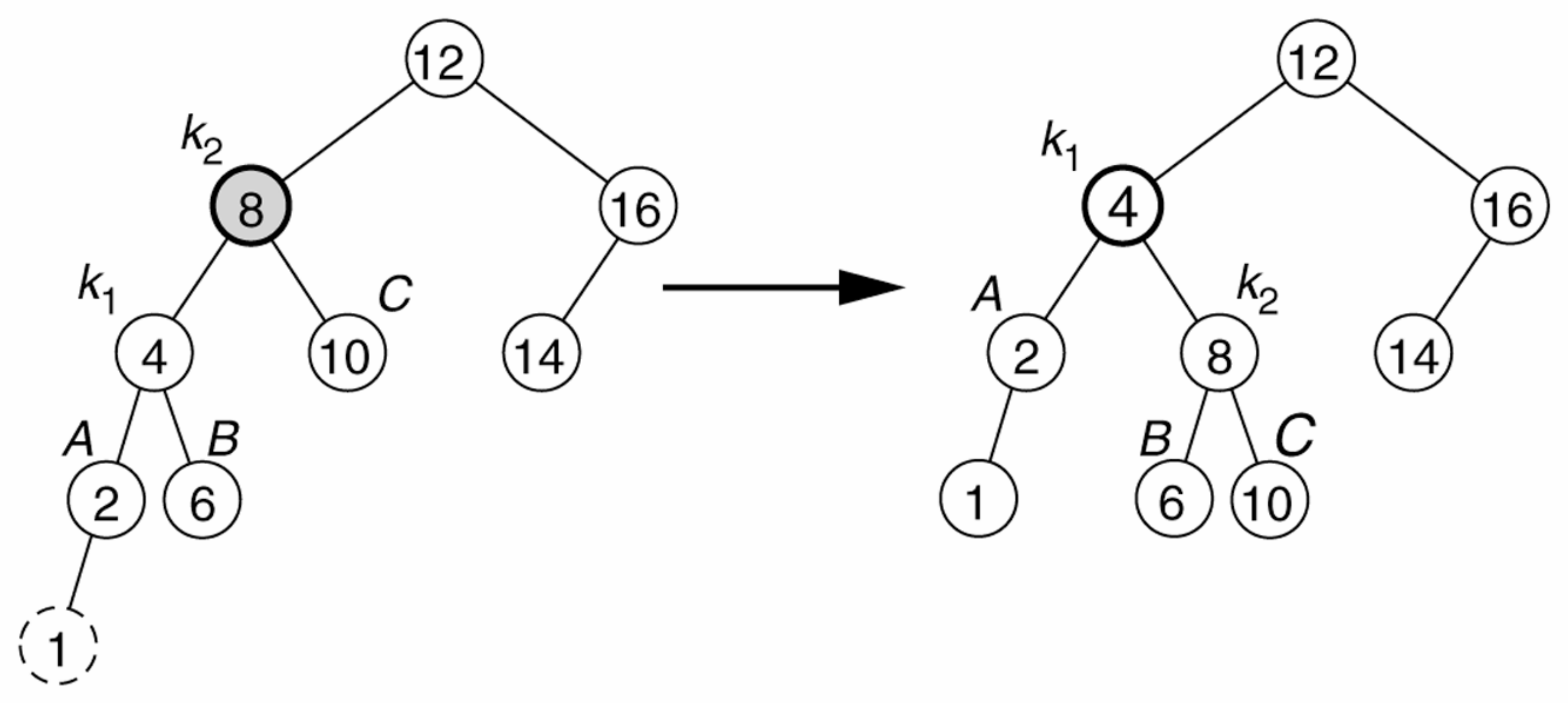
Right rotation

- **right rotation** (clockwise): *(fixes Case 1 (LL))*
    - left child k_1 becomes parent
    - original parent k_2 demoted to right
    - k_1's original right subtree *B* (if any) is attached to k_2 as left subtree



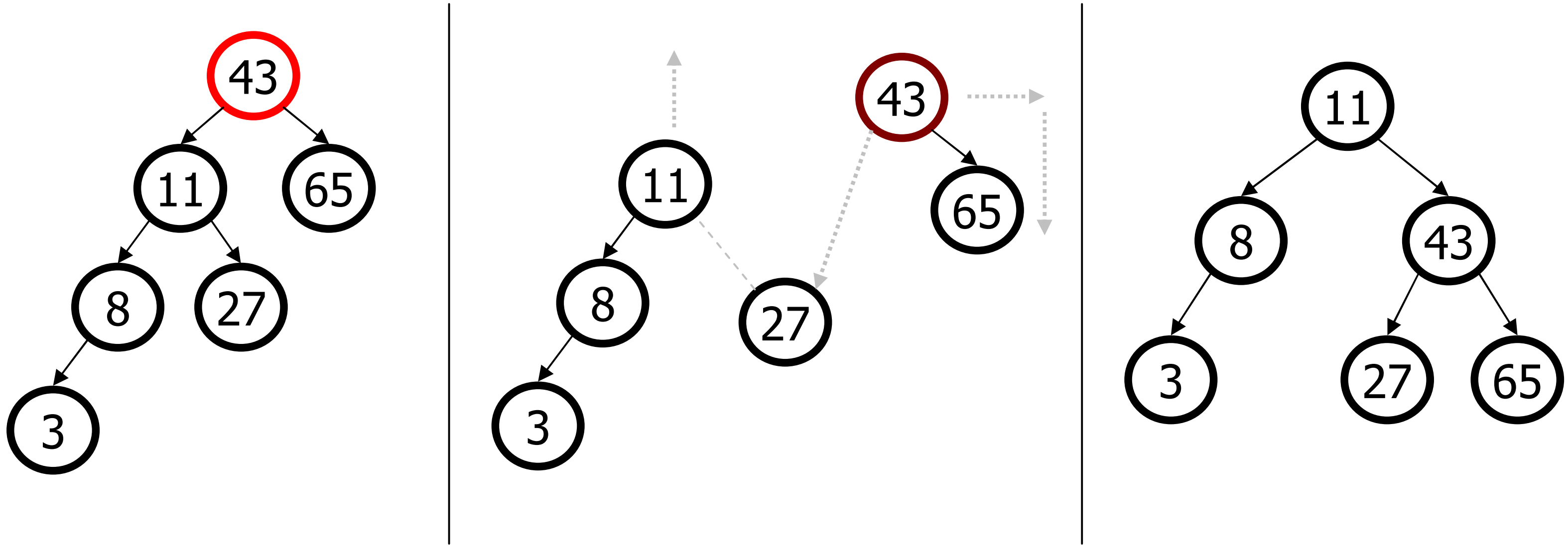
Right rotation example

- What is the balance factor of k_2 before and after rotating?



Right rotation steps

1. Detach left child (11)'s right subtree (27) *(don't lose it!)*
2. Consider left child (11) be the new parent.
3. Attach old parent (43) onto right of new parent (11).
4. Attach new parent (11)'s old right subtree (27) as left subtree of old parent (43).



Right rotation code

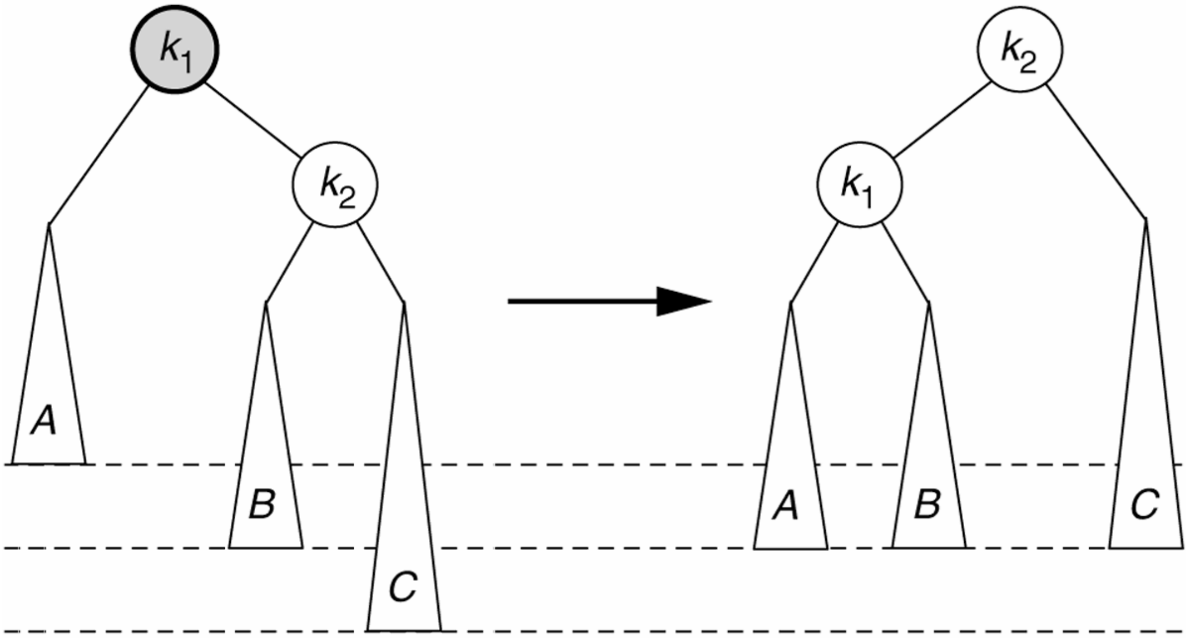
```
private TreeNode rightRotate(TreeNode oldParent) {
    // 1. detach left child's right subtree
    TreeNode orphan = oldParent.left.right;

    // 2. consider left child to be the new parent
    TreeNode newParent = oldParent.left;

    // 3. attach old parent onto right of new parent
    newParent.right = oldParent;

    // 4. attach new parent's old right subtree as
    //    left subtree of old parent
    oldParent.left = orphan;

    oldParent.height = height(oldParent);  // update nodes'
    newParent.height = height(newParent);  // height values

    return newParent;
}
```

Right rotation code

```
private int height(TreeNode node) {
    if (node == null) {
        return 0;
    }
    int left  = (node.left  == null) ? 0 : node.left.height;
    int right = (node.right == null) ? 0 : node.right.height;
    return Math.max(left, right) + 1;
}
```

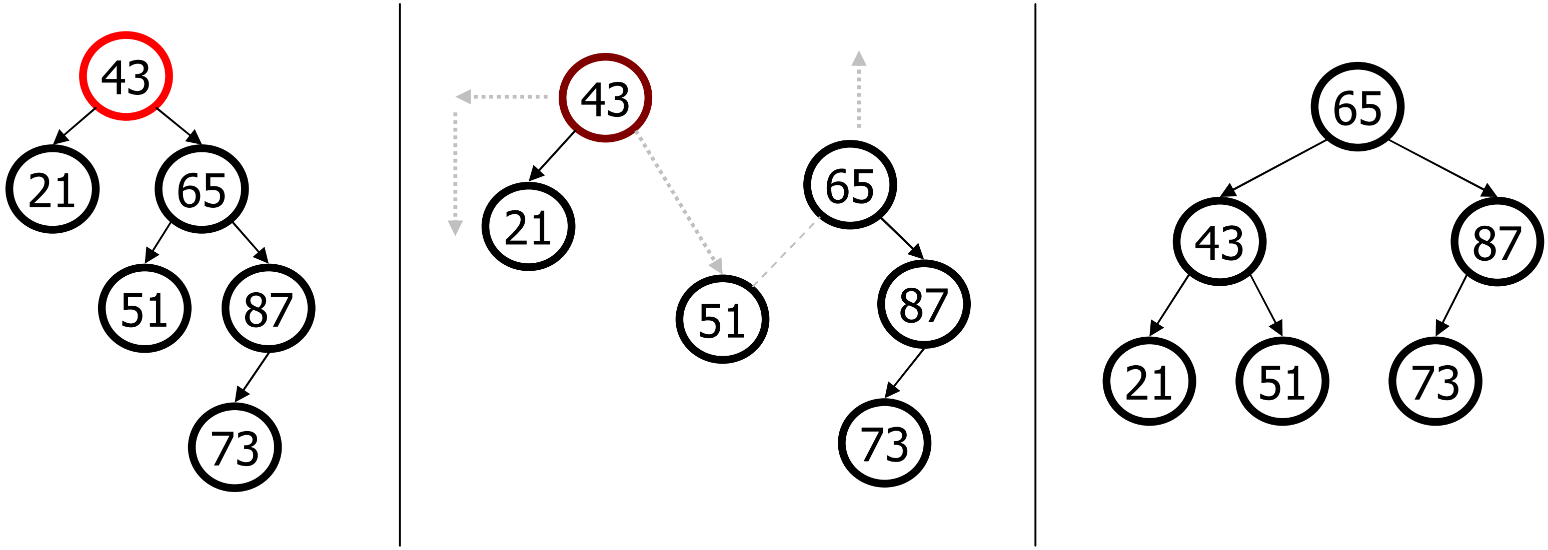
Left rotation

- **left rotation** (counter-clockwise): *(fixes Case 4 (RR))*
  - right child k_2 becomes parent
  - original parent k_1 demoted to left
  - k_2's original left subtree B (if any) is attached to k_1 as left subtree



Left rotation steps

1. Detach right child (65)'s left subtree (51) *(don't lose it!)*
2. Consider right child (65) be the new parent.
3. Attach old parent (43) onto left of new parent (65).
4. Attach new parent (65)'s old left subtree (51)
   as right subtree of old parent (43).

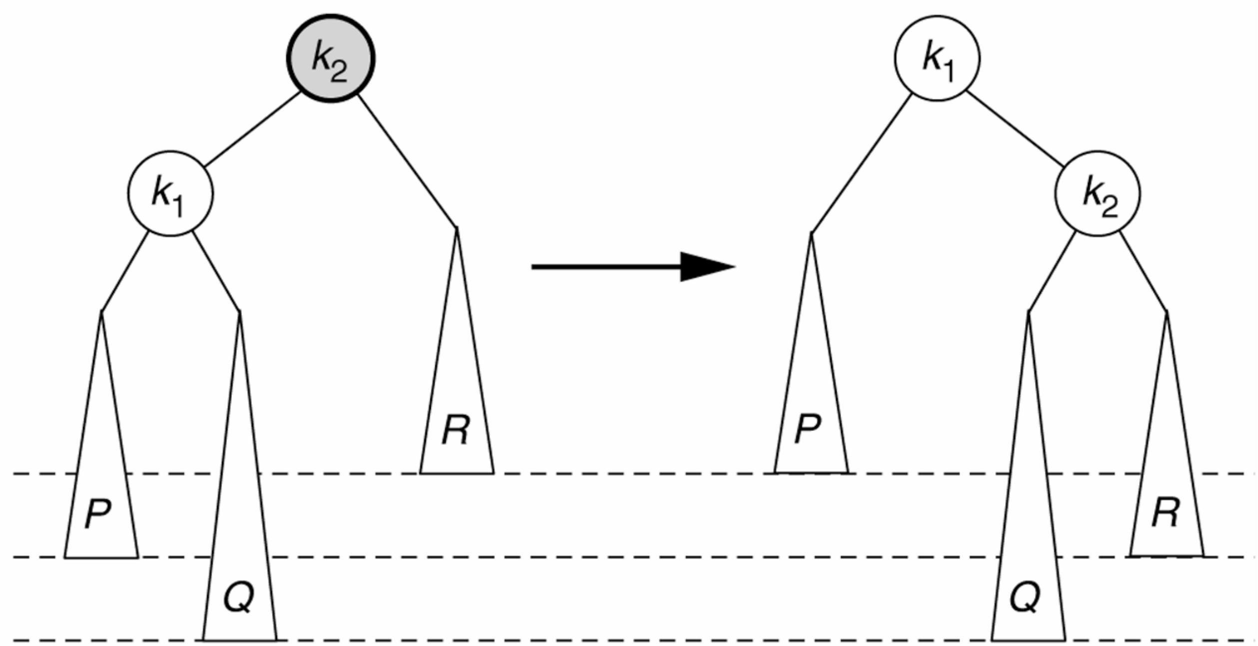


Left rotation code

```
private TreeNode leftRotate(TreeNode oldParent) {
    // 1. detach right child's left subtree
    TreeNode orphan = oldParent.right.left;

    // 2. consider right child to be the new parent
    TreeNode newParent = oldParent.right;

    // 3. attach old parent onto left of new parent
    newParent.left = oldParent;

    // 4. attach new parent's old left subtree as
    //    right subtree of old parent
    oldParent.right = orphan;

    oldParent.height = height(oldParent);  // update nodes'
    newParent.height = height(newParent);  // height values

    return newParent;
}
```

Problem cases

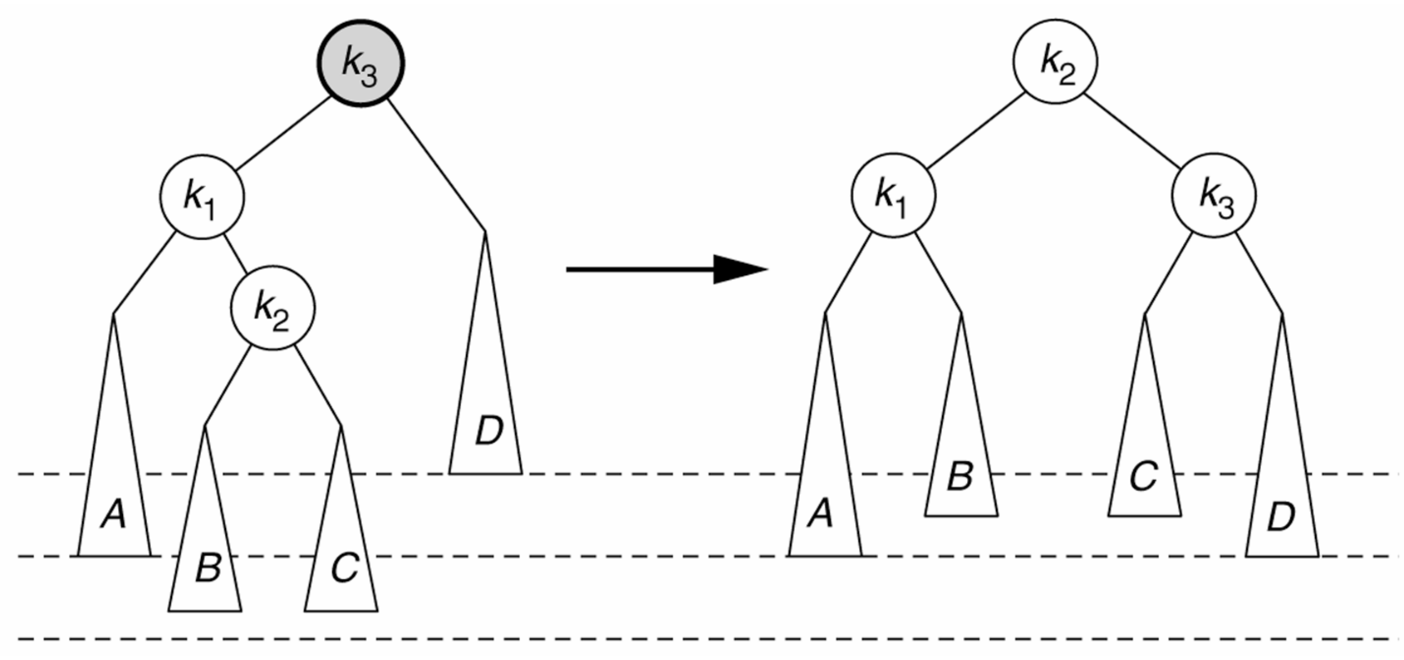
- A single right rotation does not fix Case 2 (LR).



  - (Similarly, a single left rotation does not fix Case 3 (RL).)

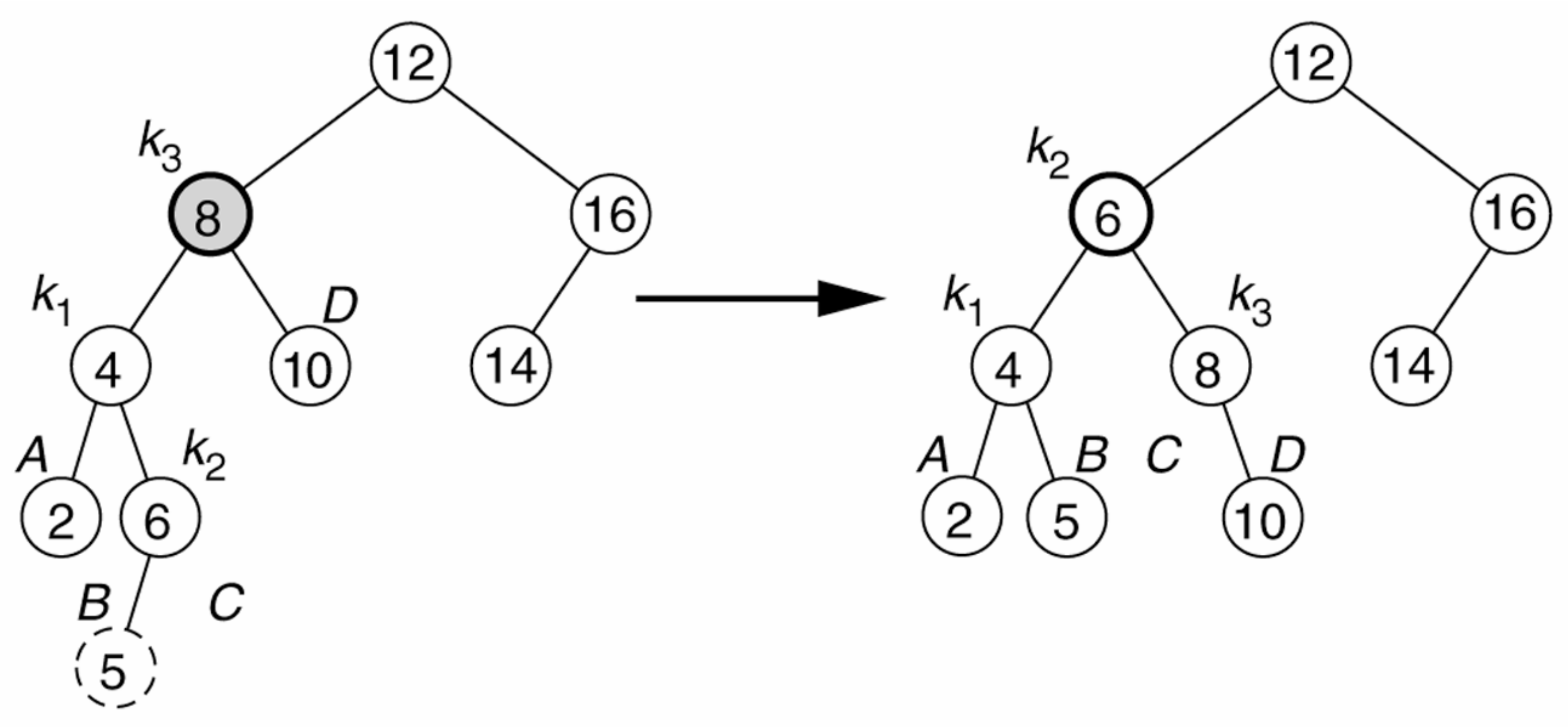
Left-right double rotation

- **left-right double rotation**: *(fixes Case 2 (LR))*
  - 1) left-rotate k_3's left child ... reduces Case 2 into Case 1
  - 2) right-rotate k_3 to fix Case 1



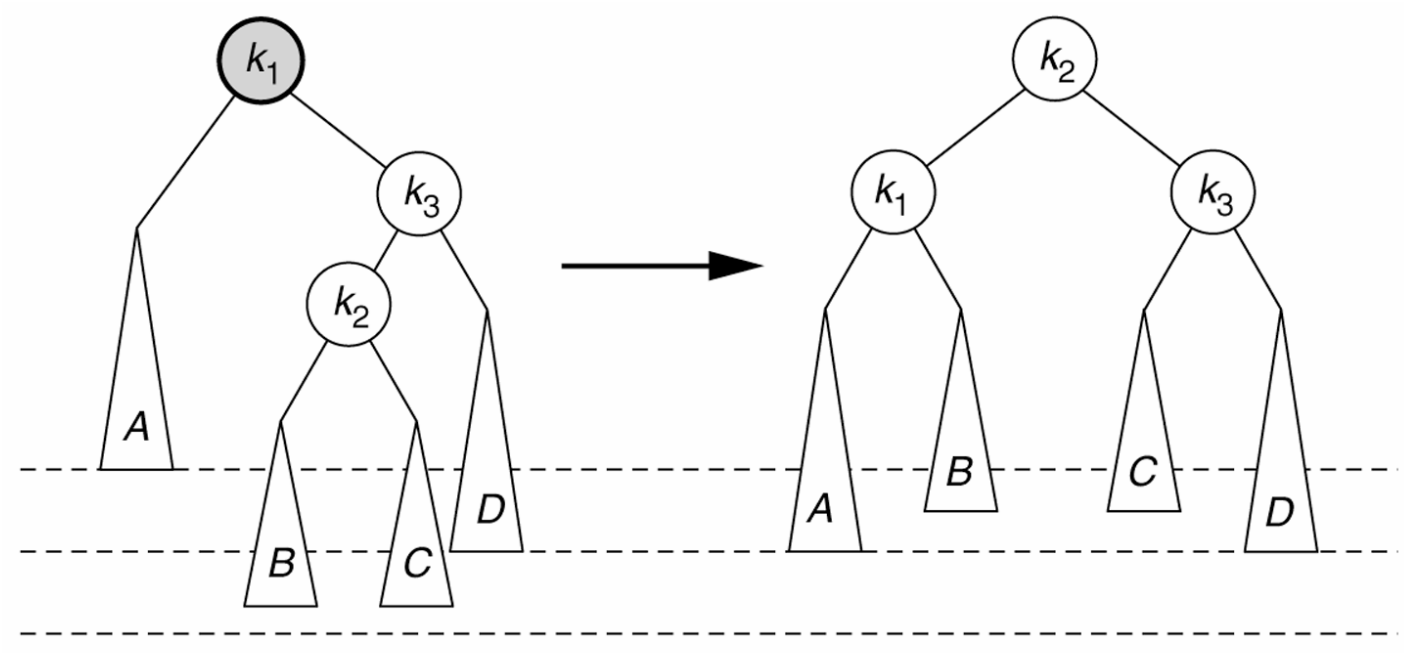
Left-right rotation example

- What is the balance factor of k_1, k_2, k_3 before and after rotating?



Right-left double rotation

- **right-left double rotation**: *(fixes Case 3 (RL))*
  - 1) right-rotate k_1's right child ... reduces Case 3 into Case 4
  - 2) left-rotate k_1 to fix Case 4



AVL add example

- Draw the AVL tree that would result if the following numbers were added in this order to an initially empty tree:
  - 20, 45, 90, 70, 10, 40, 35, 30, 99, 60, 50, 80