CSE 373

Binary search trees; tree height and balance

read: Weiss Ch. 4, section 4.1 - 4.3

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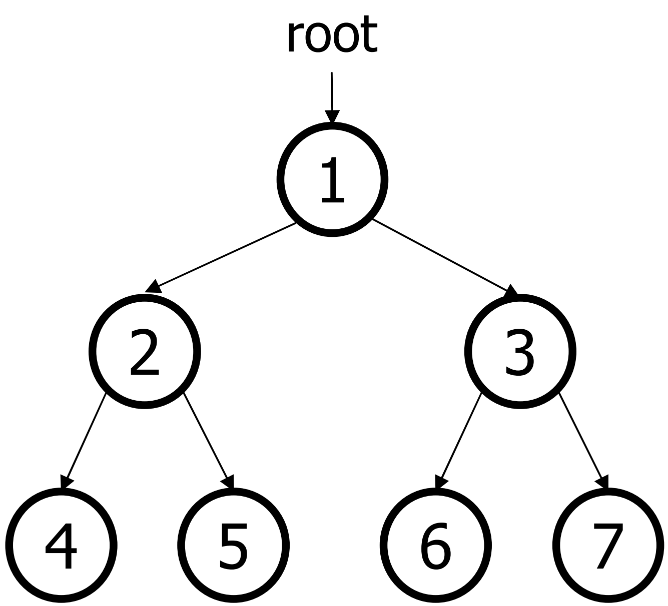
http://www.cs.washington.edu/373/

Trees

- **tree**: A directed, acyclic structure of linked nodes.
  - *directed* : Has one-way links between nodes.
  - *acyclic* : No path wraps back around to the same node twice.

– **binary tree**: One where each node has at most two children.

- *Recursive definition:* A tree is either:
  - empty (`null`), or
  - a **root** node that contains:
    - **data**,
    - a **left** subtree, and
    - a **right** subtree.
      - (The left and/or right subtree could be empty.)

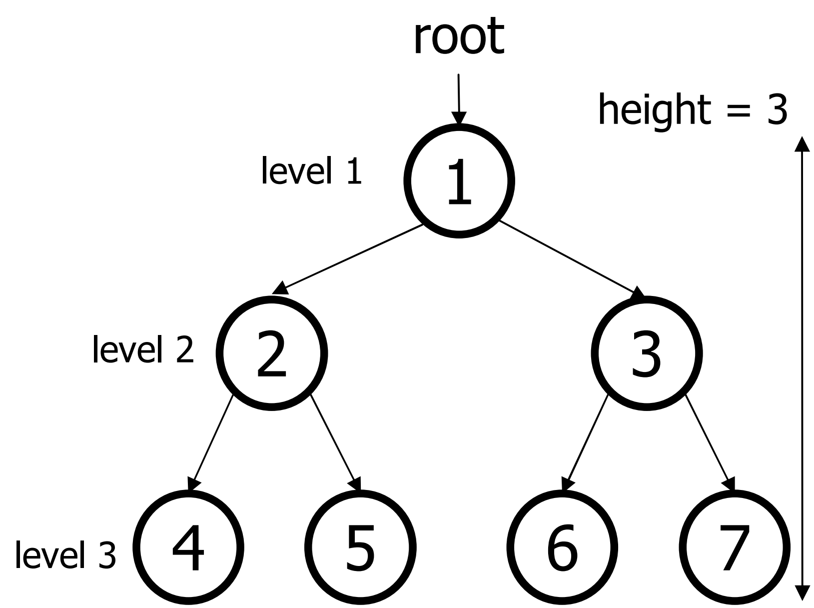


Tree terminology

- **node**: an object containing a data value and left/right children
  - **root**: topmost node of a tree
  - **leaf**: a node that has no children
  - **branch**: any internal node; neither the root nor a leaf

  - **parent**: a node that refers to this one
  - **child**: a node that this node refers to
  - **sibling**: a node with a common

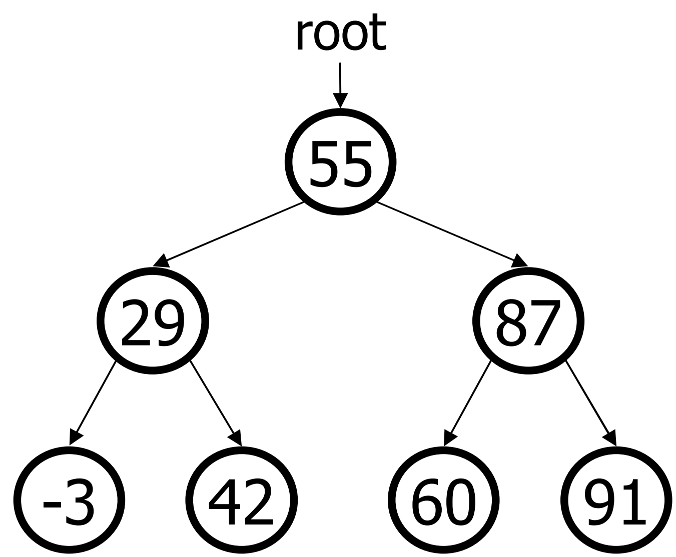
- **subtree**: the smaller tree of nodes on
  the left or right of the current node
- **height**: length of the longest path
  from the root to any node
- **level** or **depth**: length of the path
  from a root to a given node



Binary search trees

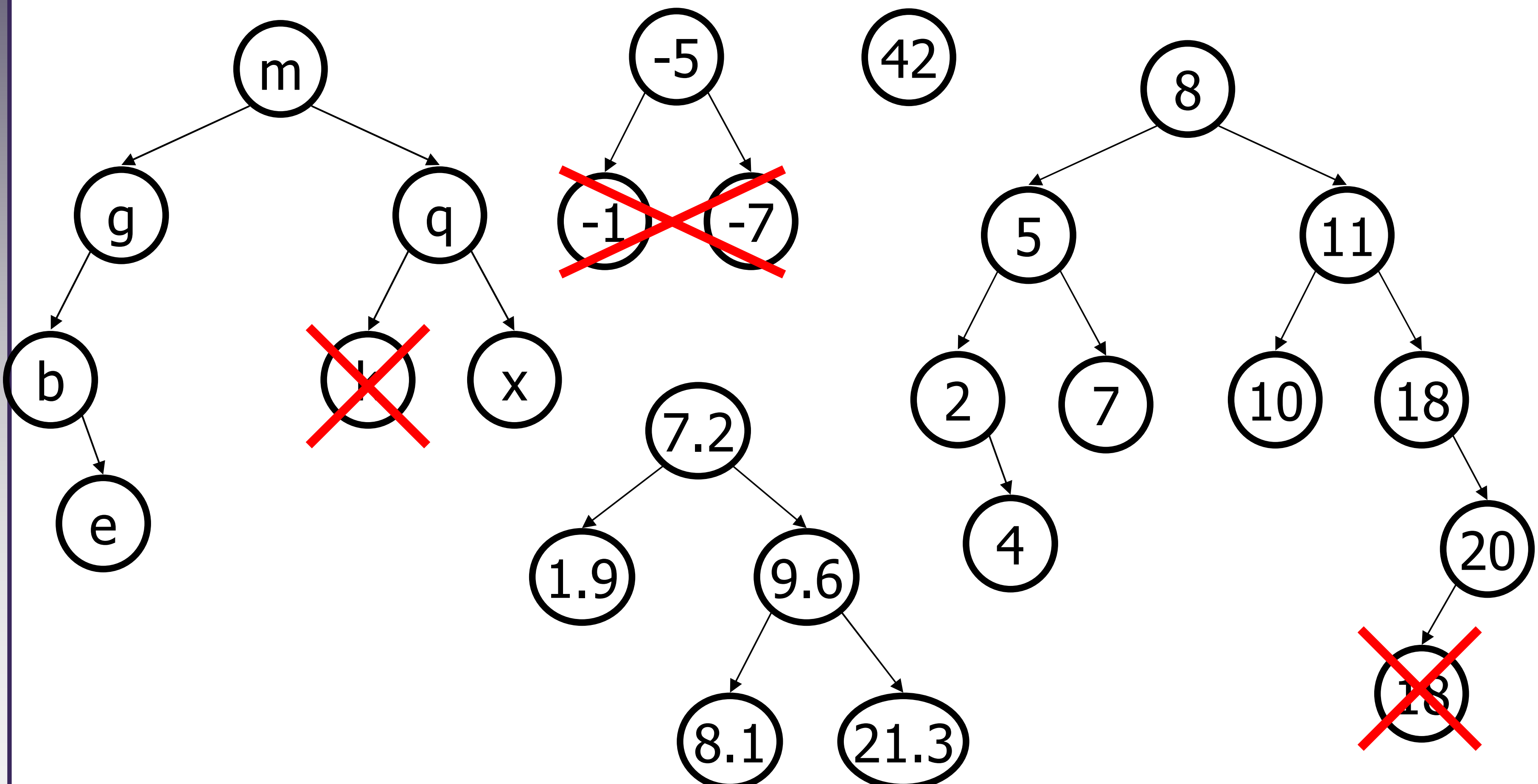
- **binary search tree** ("BST"): a binary tree where each non-empty node *R* has the following properties:
  - every element of *R*'s left subtree contains data "less than" *R*'s data,
  - every element of *R*'s right subtree contains data "greater than" *R*'s,
  - *R*'s left and right subtrees are also binary search trees.

- BSTs store their elements in sorted order, which is helpful for searching/sorting tasks.



BST examples

- Which of the trees shown are legal binary search trees?



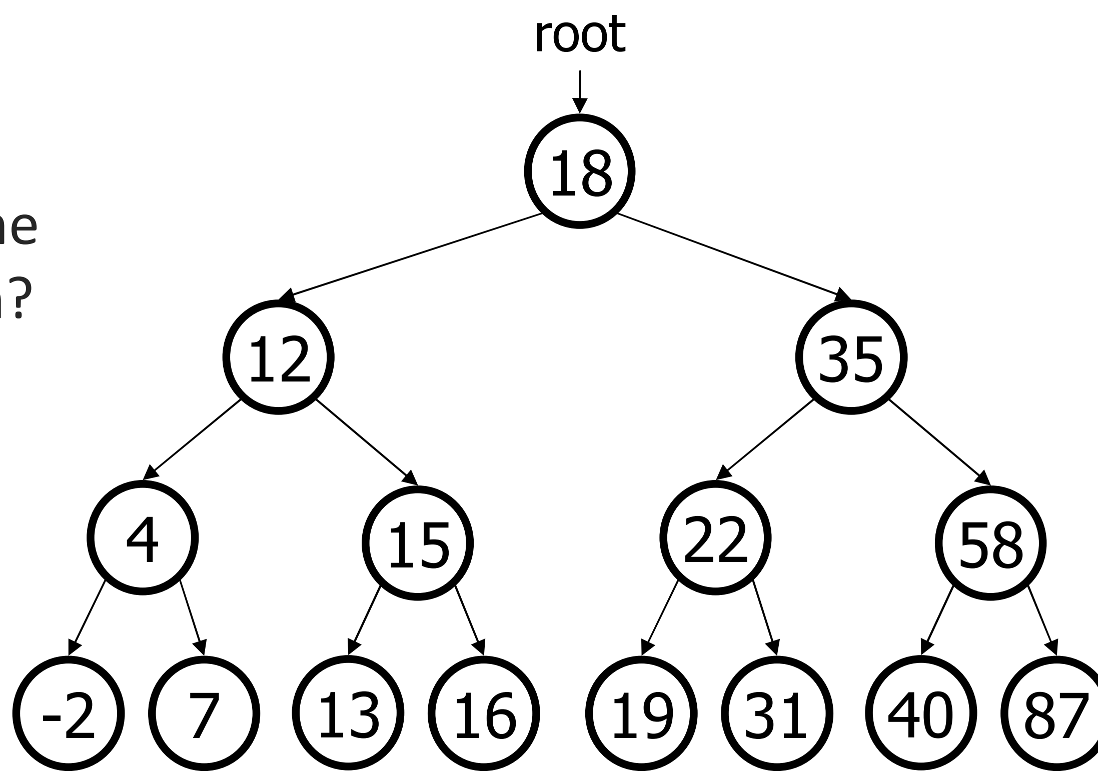
A TreeSet class

```
public class TreeSet<E extends Comparable<E>>
        implements Set<E> {
    private TreeNode root;    // null for an empty tree

    public TreeSet() {
        root = null;
    }

    ...

    private class TreeNode {
        private E data;
        private TreeNode left;
        private TreeNode right;
        ...
    }
}
```

Searching a BST

- Describe an algorithm for searching a binary search tree.
  - Try searching for the value 31, then 6.

- What is the maximum number of nodes you would need to examine to perform any search?



Template for tree methods

```
public type name(parameters) {
    name(root, parameters);
}

private type name(TreeNode node, parameters) {
    ...
}
```

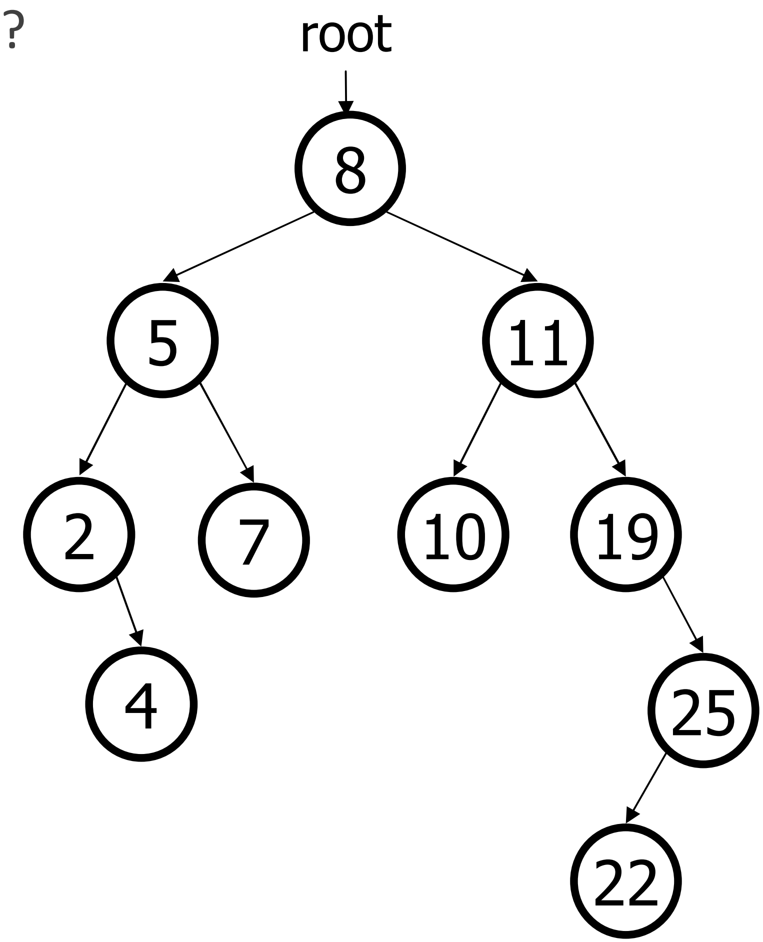
- Tree methods are often implemented recursively
  - with a public/private pair
  - the private version accepts the root node to process

The contains method

```
// Returns whether this BST contains the given integer.
public boolean contains(E value) {
    return contains(root, value);
}

private boolean contains(TreeNode node, E value) {
    if (node == null) {
        return false;   // base case: not found here
    } else {
        int comp = node.data.compareTo(value);
        if (comp == 0) {
            return true;    // base case: found here
        } else if (comp > 0) {
            return contains(node.left, value);
        } else {   // comp < 0
            return contains(node.right, value);
        }
    }
}
```

Adding to a BST

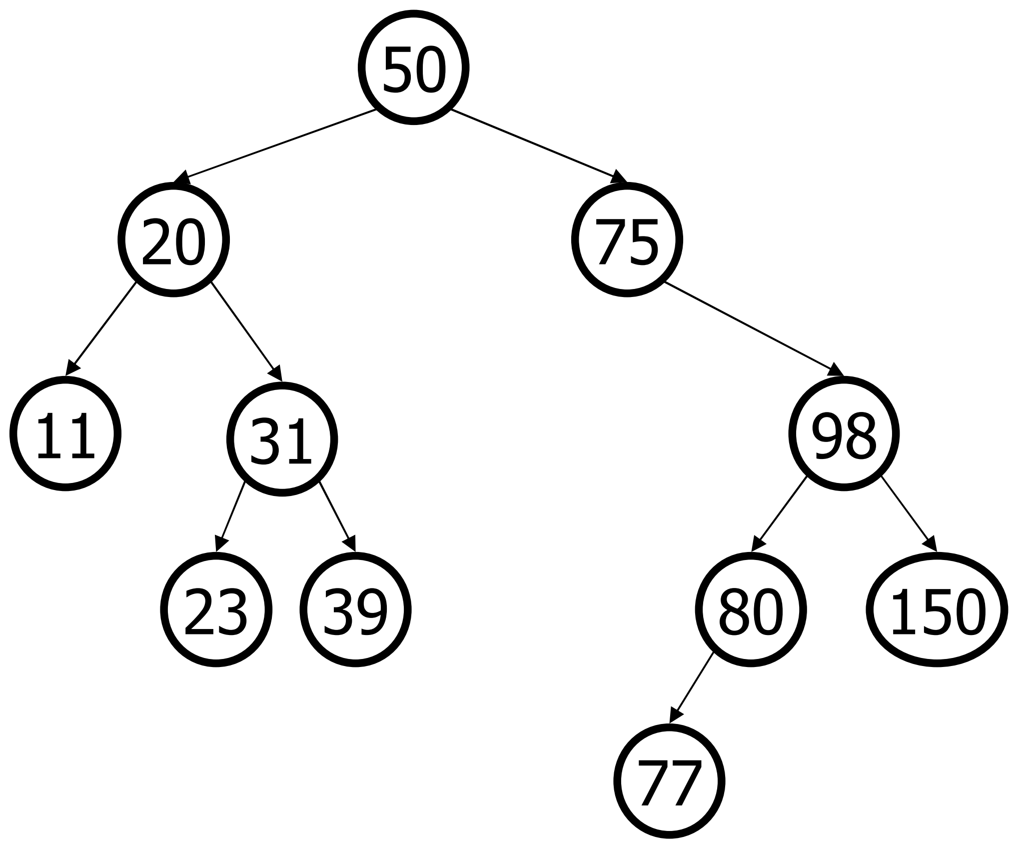
- Suppose we want to add new values to the BST below.
  - Where should the value 14 be added?
  - Where should 3 be added? 7?
  - If the tree is empty, where should a new value be added?
- What is the general algorithm?



Adding exercise

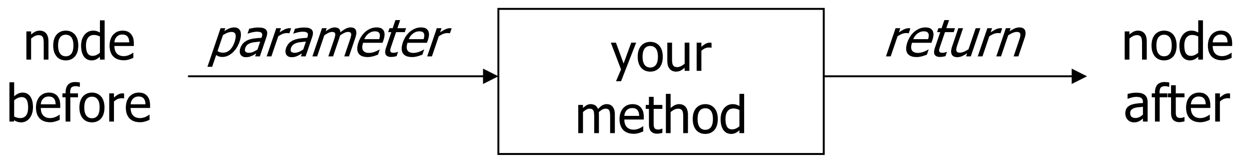
- Draw what a binary search tree would look like if the following values were added to an initially empty tree in this order:

50
20
75
98
80
31
150
39
23
11
77



The x = change(x) pattern

- Methods that modify a tree should have the following pattern:
  - input (parameter): old state of the node
  - output (return): new state of the node



- In order to actually change the tree, you must reassign:

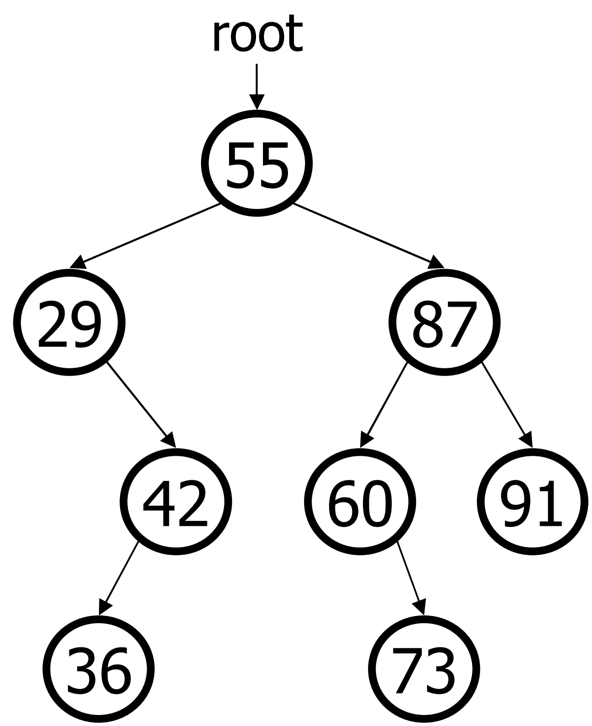
```
node        = change(node, parameters);
node.left   = change(node.left, parameters);
node.right  = change(node.right, parameters);
overallRoot = change(overallRoot, parameters);
```

The add method

```
// Adds the given value to this BST in sorted order.
public void add(E value) {
    root = add(root, value);
}

private TreeNode add(TreeNode node, E value) {
    if (node == null) {
        node = new TreeNode(value);
    } else {
        int comp = node.data.compareTo(value);
        if (comp > 0) {
            node.left = add(node.left, value);
        } else if (comp < 0) {
            node.right = add(node.right, value);
        } // else a duplicate; do nothing
    }

    return node;
}
```

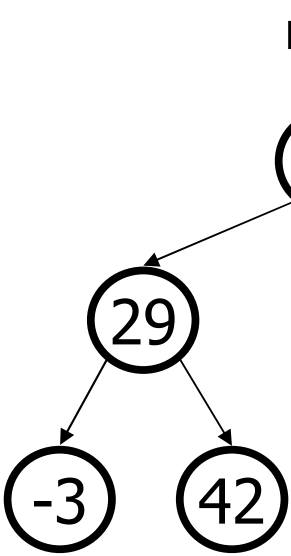
Removing from a BST

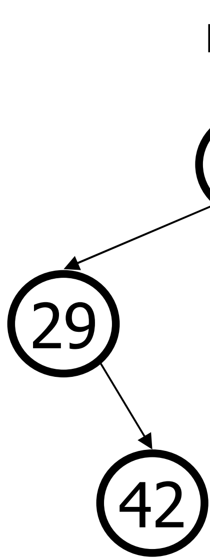
- How can we remove a value from a BST in such a way as to maintain proper BST ordering?
  - `tree.remove(73);`
  - `tree.remove(29);`
  - `tree.remove(87);`
  - `tree.remove(55);`

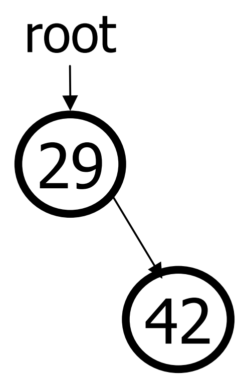


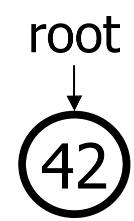
Cases for removal 1

1. a **leaf**: replace with `null`
2. a node with a **left child only**: replace with left child
3. a node with a **right child only**: replace with right child









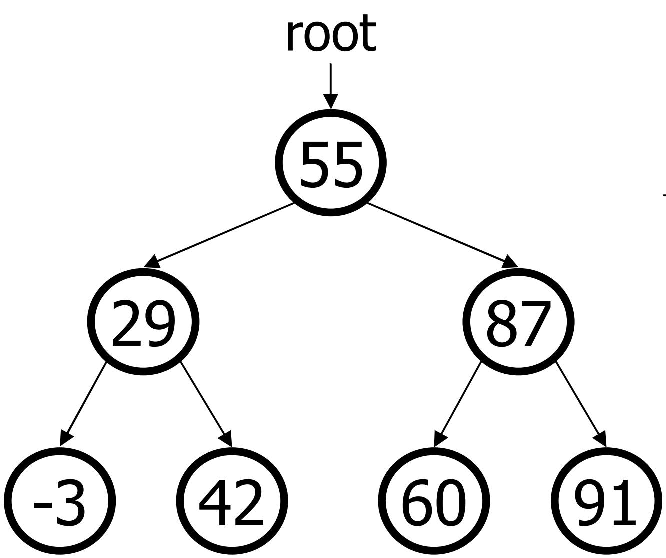

```
tree.remove(29);
```

```
tree.remove(-3);
```

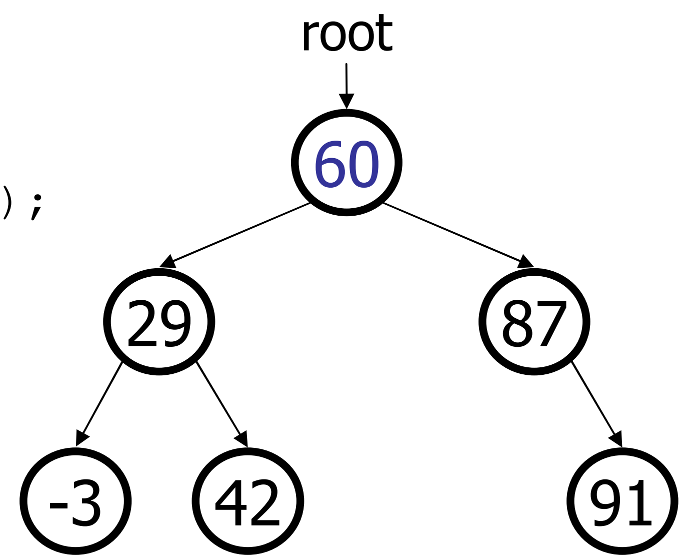
```
tree.remove(55);
```

Cases for removal 2

4. a node with **both** children: replace with **min from right**
   - (replacing with max from left would also work)



`tree.remove(55);`

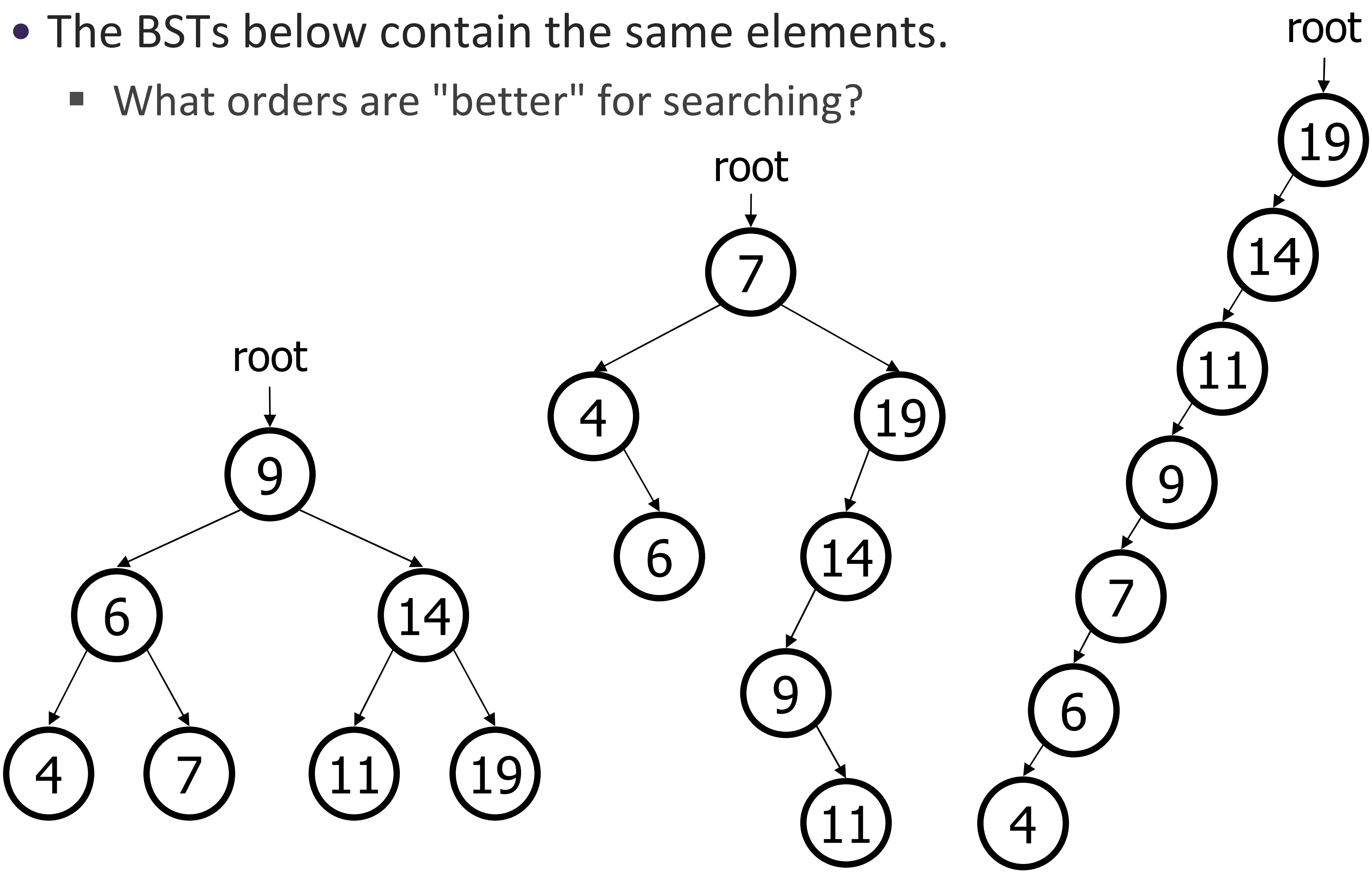


The remove method

```
// Removes the given value from this BST, if it exists.
public void remove(E value) {
    root = remove(root, value);
}

private TreeNode remove(TreeNode node, E value) {
    if (node == null) {
        return null;
    } else {
        int comp = root.data.compareTo(value);
        if (comp > 0) {
            root.left = remove(root.left, value);
        } else if (comp < 0) {
            root.right = remove(root.right, value);
        } else {  // comp == 0; remove this node
            if (root.right == null) {
                return root.left;     // replace w/ L
            } else if (root.left == null) {
                return root.right;    // replace w/ R
            } else {
                // both children; replace w/ min from R
                root.data = getMin(root.right);
                root.right = remove(root.right, root.data);
            }
        }
    }
    return root;
}
```

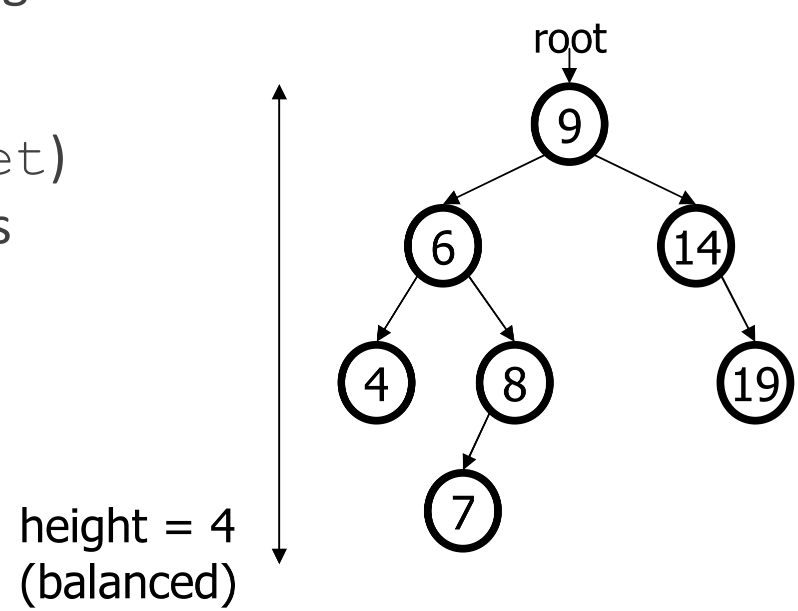
Searching BSTs

- The BSTs below contain the same elements.
  - What orders are "better" for searching?



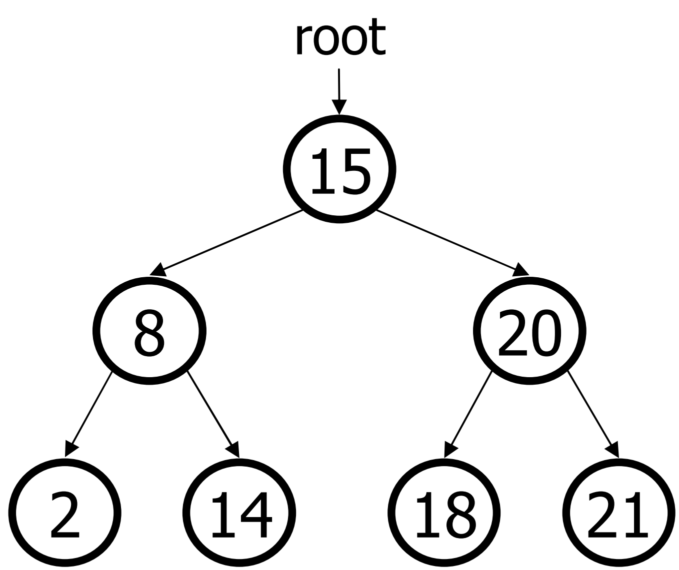
Trees and balance

- **balanced tree**: One whose subtrees differ in height by at most 1 and are themselves balanced.
  - A balanced tree of *N* nodes has a height of ~ $\log_2 N$.
  - A very unbalanced tree can have a height close to *N*.
  - The runtime of adding to / searching a BST is closely related to height.
  - Some tree collections (e.g. `TreeSet`) contain code to balance themselves as new nodes are added.



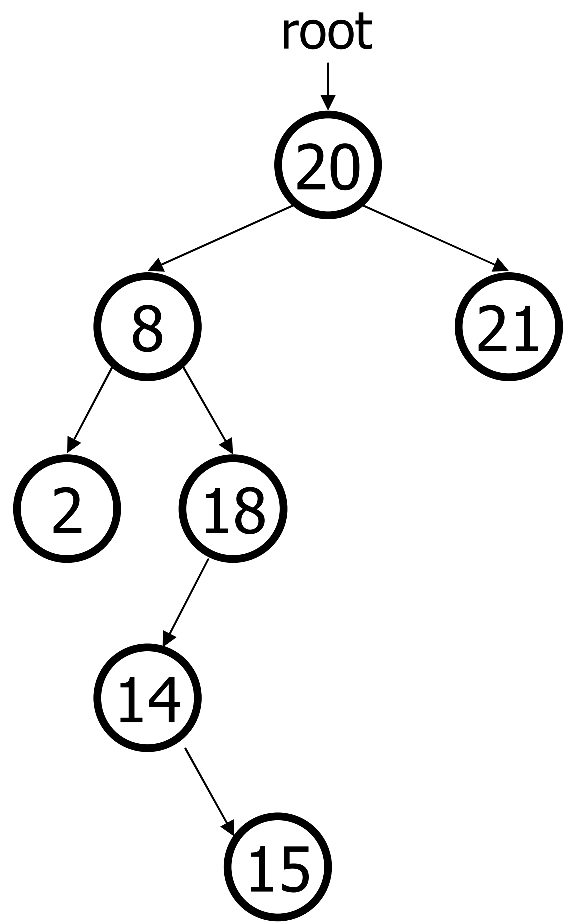
A balanced tree

- Values: 2, 8, 14, 15, 18, 20, 21
  - Order added: 15, 8, 2, 20, 21, 14, 18
- Different tree structures possible; depends on order inserted
- 7 nodes, expected height log 7 ≈ 3
- Perfectly balanced



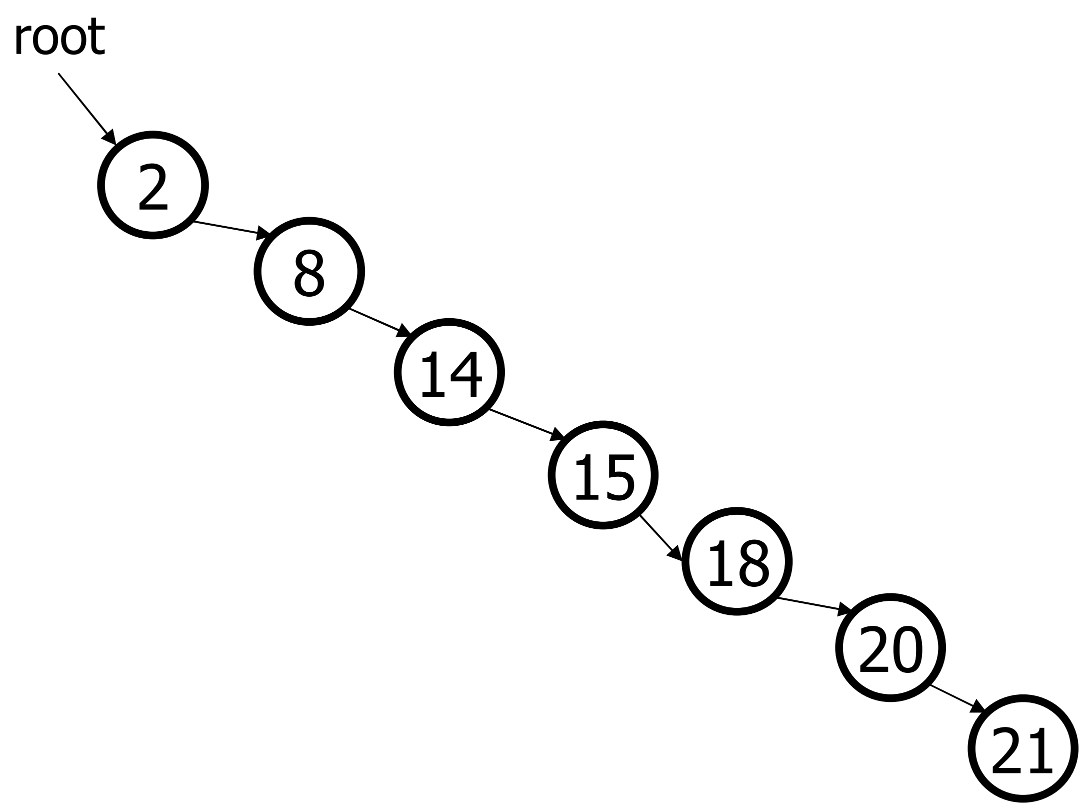
Mostly balanced tree

- Same Values: 2, 8, 14, 15, 18, 20, 21
  - Order added: 20, 8, 21, 18, 14, 15, 2
- Somewhat balanced; height 5



Degenerate tree

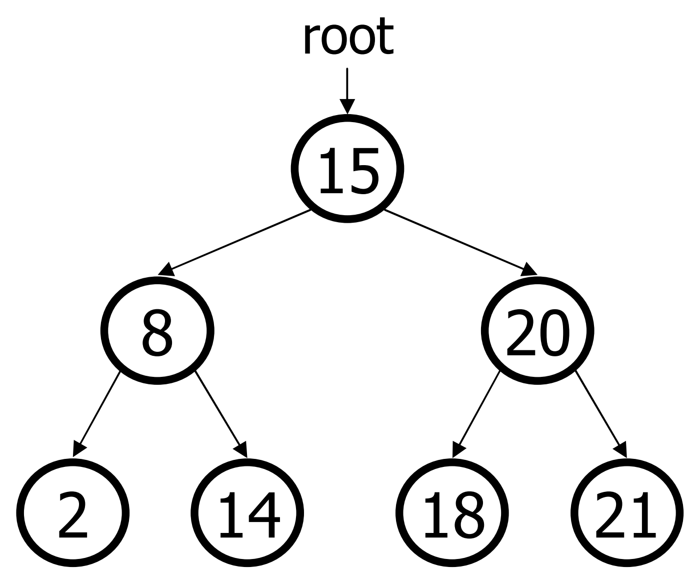
- Same Values: 2, 8, 14, 15, 18, 20, 21
  - Order added: 2, 8, 14, 15, 18, 20, 21
- Totally unbalanced; height 7



Some height numbers

- *Observation:* The shallower the BST the better.
  - Average case height is O(log *N*)
  - Worst case height is O(*N*)
  - Simple cases such as adding (1, 2, 3, ..., *N*), or the opposite order, lead to the worst case scenario: height O(*N*).

- For binary tree of height *h*:
  - max # of leaves: 2^{h-1}
  - max # of nodes: $2^h - 1$
  - min # of leaves: 1
  - min # of nodes: h



Calculating tree height

- Height is max number of nodes in path from root to any leaf.
  - height(null) = 0
  - height(a leaf) = ?
  - height(A) = ?
  - *Hint:* it's recursive!
  - height(a leaf) = 1
  - height(A) = 1 + max(
    height(A.left), height(A.right))