Graphs: Definitions and Representations
(Chapter 9)

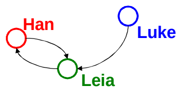
CSE 373
Data Structures and Algorithms

Today's Outline

- **Admin**:
  - HW #4 due Tuesday, Feb 21 at 11pm
  - Midterm 2, Fri Feb 24
- **Memory hierarchy**
- **Graphs**
  - **Representations**

Graphs

- A graph is a formalism for representing relationships among items
  - Very general definition because very general concept
- A graph is a pair
  G = (V,E)
  - A set of vertices, also known as nodes
    V = {v_1,v_2,…,v_n}
  - A set of edges
    E = {e_1,e_2,…,e_m}
    - Each edge **e_i** is a pair of vertices
      (v_j,v_k)
    - An edge "connects" the vertices
- Graphs can be directed or undirected



V = {Han,Leia,Luke}
E = {(Luke,Leia),
(Han,Leia),
(Leia,Han)}

An ADT?

- Can think of graphs as an ADT with operations like
  isEdge((v_j,v_k))
- But what the "standard operations" are is unclear
- Instead we tend to develop algorithms over graphs and then use data structures that are efficient for those algorithms
- Many important problems can be solved by:
  1. Formulating them in terms of graphs
  2. Applying a standard graph algorithm
- To make the formulation easy and standard, we have a lot of *standard terminology* about graphs

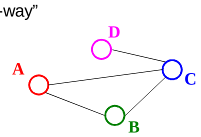
Some graphs

For each, what are the vertices and what are the edges?

- Web pages with links
- Facebook friends
- "Input data" for the Kevin Bacon game
- Methods in a program that call each other
- Road maps (e.g., Google maps)
- Airline routes
- Family trees
- Course pre-requisites
- …

Undirected Graphs

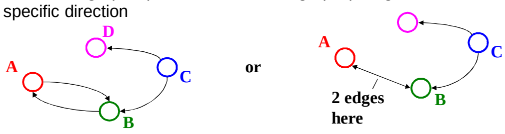
- In undirected graphs, edges have no specific direction
  - Edges are always "two-way"



- Thus, **(u,v) ∈ E** implies **(v,u) ∈ E**.
  - Only one of these edges needs to be in the set; the other is implicit
- Degree of a vertex: number of edges containing that vertex
  - Put another way: the number of adjacent vertices

Directed graphs

- In directed graphs (sometimes called digraphs), edges have a specific direction

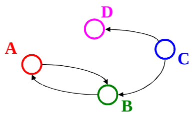


- Thus, **(u,v) ∈ E** does *not* imply **(v,u) ∈ E**.
  - Let **(u,v) ∈ E** mean u → v and call **u** the source and **v** the destination
- In-Degree of a vertex: number of in-bound edges, i.e., edges where the vertex is the destination
- Out-Degree of a vertex: number of out-bound edges, i.e., edges where the vertex is the source

Self-edges, connectedness, etc.

- A self-edge a.k.a. a loop is an edge of the form **(u,u)**
  - Depending on the use/algorithm, a graph may have:
    - No self edges
    - Some self edges
    - All self edges (in which case often implicit, but we will be explicit)
- A node can have a degree / in-degree / out-degree of zero
- A graph does not have to be connected (In an undirected graph, this means we can follow edges from any node to every other node), even if every node has non-zero degree

More notation



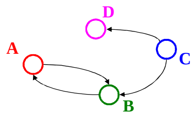
For a graph **G = (V,E)**:

- **|V|** is the number of vertices
- **|E|** is the number of edges
  - Minimum?
  - Maximum for undirected?
  - Maximum for directed?

```
V = {A, B, C, D}
E = {(C, B),
     (A, B),
     (B, A)
     (C, D)}
```

- If **(u,v) ∈ E**
  - Then **v** is a neighbor of **u**, i.e., **v** is adjacent to **u**
  - Order matters for directed edges

More notation



For a graph **G = (V,E)**:

- **|V|** is the number of vertices
- **|E|** is the number of edges
  - Minimum? **0**
  - Maximum for undirected? **|V||V+1|/2 ∈ *O*(|V|²)**
  - Maximum for directed? **|V|² ∈ *O*(|V|²)**
    (assuming self-edges allowed, else subtract **|V|**)
- If **(u,v) ∈ E**
  - Then **v** is a neighbor of **u**, i.e., **v** is adjacent to **u**
  - Order matters for directed edges: In this example **v** is adjacent to **u**, but **u** is not adjacent to **v** (unless **(v,u) ∈ E**)

Examples again

Which would use directed edges? Which would have self-edges? Which could have 0-degree nodes?

- Web pages with links
- Facebook friends
- “Input data” for the Kevin Bacon game
- Methods in a program that call each other
- Road maps (e.g., Google maps)
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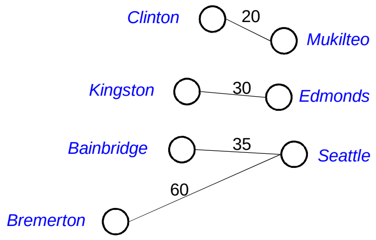
Weighted graphs

- In a weighed graph, each edge has a weight a.k.a. cost
  - Typically numeric (most examples will use ints)
  - *Orthogonal* to whether graph is directed
  - Some graphs allow *negative weights*; many don’t



Examples

What, if anything, might weights represent for each of these? Do negative weights make sense?

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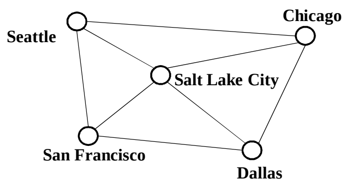
Paths and Cycles

- A path is a list of vertices $[v_0, v_1, \ldots, v_n]$ such that $(v_i, v_{i+1}) \in E$ for all $0 \leq i < n$. Say *"a path from* v_0 *to* v_n*"*
- A cycle is a path that begins and ends at the same node ($v_0 == v_n$)

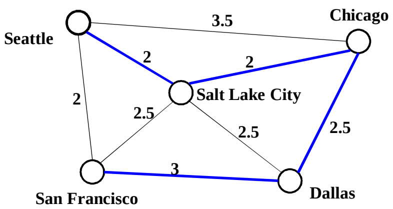


Example path (that also happens to be a cycle):
[Seattle, Salt Lake City, Chicago, Dallas, San Francisco, Seattle]

Path Length and Cost

- Path length: Number of *edges* in a path (also called "unweighted cost")
- Path cost: sum of the weights of each edge

Example where:
P= [Seattle, Salt Lake City, Chicago, Dallas, San Francisco]



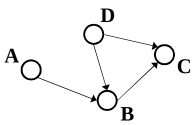
length(P) = 4
cost(P) = 9.5

Simple paths and cycles

- A simple path repeats no vertices, (except the first might be the last):
  [Seattle, Salt Lake City, San Francisco, Dallas]
  [Seattle, Salt Lake City, San Francisco, Dallas, Seattle]
- Recall, a cycle is a path that ends where it begins:
  [Seattle, Salt Lake City, San Francisco, Dallas, Seattle]
  [Seattle, Salt Lake City, Seattle, Dallas, Seattle]
- A simple cycle is a cycle and a simple path:
  [Seattle, Salt Lake City, San Francisco, Dallas, Seattle]

Paths/cycles in directed graphs

Example:

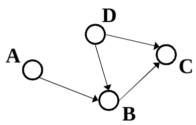


Is there a path from A to D?

Does the graph contain any cycles?

Paths/cycles in directed graphs

Example:



Is there a path from A to D? No

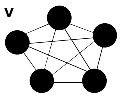
Does the graph contain any cycles? No

Undirected graph connectivity

- An undirected graph is connected if for all pairs of vertices **u,v**, there exists a *path* from **u** to **v**

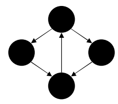
Connected graph **Disconnected graph**

- An undirected graph is complete, a.k.a. fully connected if for all pairs of vertices **u,v**, there exists an *edge* from **u** to **v**

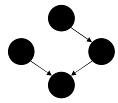


Directed graph connectivity

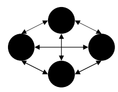
- A directed graph is strongly connected if there is a path from every vertex to every other vertex



- A directed graph is weakly connected if there is a path from every vertex to every other vertex *ignoring direction of edges*



- A complete a.k.a. fully connected directed graph has an edge from every vertex to every other vertex



Examples

For undirected graphs: connected?
For directed graphs: strongly connected? weakly connected?

- Web pages with links
- Facebook friends
- "Input data" for the Kevin Bacon game
- Methods in a program that call each other
- Road maps (e.g., Google maps)
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- …

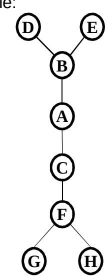
Trees as graphs

When talking about graphs, we say a tree is a graph that is:
- undirected
- acyclic
- connected

So all trees are graphs, but not all graphs are trees

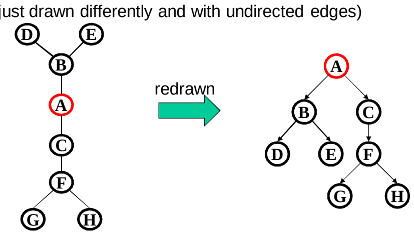
How does this relate to the trees we know and love?...

Example:



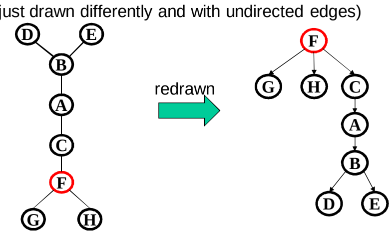
Rooted Trees

- We are more accustomed to rooted trees where:
  - We identify a unique ("special") root
  - We think of edges as directed: parent to children
- Given a tree, once you pick a root, you have a unique rooted tree (just drawn differently and with undirected edges)



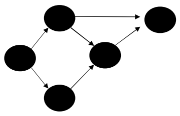
Rooted Trees (Another example)

- We are more accustomed to rooted trees where:
  - We identify a unique ("special") root
  - We think of edges as directed: parent to children
- Given a tree, once you pick a root, you have a unique rooted tree (just drawn differently and with undirected edges)

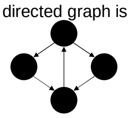


Directed acyclic graphs (DAGs)

- A DAG is a directed graph with no (directed) cycles
  - Every rooted directed tree is a DAG
  - But not every DAG is a rooted directed tree:



  - Every DAG is a directed graph
  - But not every directed graph is a DAG:



Examples

Which of our directed-graph examples do you expect to be a DAG?

- Web pages with links
- “Input data” for the Kevin Bacon game
- Methods in a program that call each other
- Airline routes
- Family trees
- Course pre-requisites
- …

Density / sparsity

- Recall: In an undirected graph, $0 \le |E| < |V|^2$
- Recall: In a directed graph: $0 \le |E| \le |V|^2$
- So for any graph, $|E|$ is $O(|V|^2)$
- One more fact: If an undirected graph is *connected*, then $|E| \ge |V|-1$
- Because $|E|$ is often much smaller than its maximum size, we do not always approximate as $|E|$ as $O(|V|^2)$
  - This is a correct bound, it just is often not tight
  - If it is tight, i.e., $|E|$ is $\Theta(|V|^2)$ we say the graph is dense
    - More sloppily, dense means “lots of edges”
  - If $|E|$ is $O(|V|)$ we say the graph is sparse
    - More sloppily, sparse means “most (possible) edges missing”

What’s the data structure?

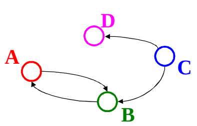
Things we might want to do:
- iterate over vertices
- iterate over edges
- iterate over vertices adj. to a vertex
- check whether an edge exists
- find the lowest-cost path from x to y

Which data structure is “best” can depend on:
- properties of the graph (e.g., dense versus sparse)
- the common queries (e.g., “is **(u,v)** an edge?” versus “what are the neighbors of node **u**?”)

We need a data structure that represents graphs:
- List of vertices + list of edges (rarely good enough)
- Adjacency Matrix
- Adjacency List

Adjacency matrix

- Assign each node a number from `0` to `|V|-1`
- A |V| x |V| matrix (i.e., 2-D array) of booleans (or 1 vs. 0)
  - If `M` is the matrix, then `M[u][v] == true` means there is an edge from **u** to **v**



	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	F	T	F	T
D	F	F	F	F

Adjacency matrix properties

	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	F	T	F	T
D	F	F	F	F

- Running time to:
  - Get a vertex’s out-edges:
  - Get a vertex’s in-edges:
  - Decide if some edge exists:
  - Insert an edge:
  - Delete an edge:
- Space requirements:
- Best for sparse or dense graphs?

Adjacency matrix properties

- Running time to:
  - Get a vertex's out-edges: $O(|V|)$
  - Get a vertex's in-edges: $O(|V|)$
  - Decide if some edge exists: $O(1)$
  - Insert an edge: $O(1)$
  - Delete an edge: $O(1)$
- Space requirements:
  - $|V|^2$ bits
- Best for dense graphs

	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	F	T	F	T
D	F	F	F	F

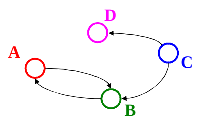
Adjacency matrix properties (cont.)

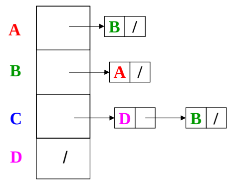
- How will the adjacency matrix vary for an **undirected graph**?
  - Undirected: Will be symmetric about diagonal axis
- How can we adapt the representation for **weighted graphs**?
  - Instead of a boolean, store an int/double in each cell
  - Need some value to represent 'not an edge'
    - Say -1 or 0

	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	F	T	F	T
D	F	F	F	F

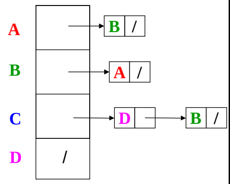
Adjacency List

- Assign each node a number from `0` to `|V|-1`
- An array of length `|V|` in which each entry stores a list (e.g., linked list) of all adjacent vertices



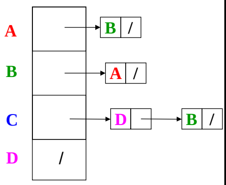


Adjacency List Properties



- Running time to:
  - Get all of a vertex's out-edges:
  - Get all of a vertex's in-edges:
  - Decide if some edge exists:
  - Insert an edge:
  - Delete an edge:
- Space requirements:
  -
- Best for dense or sparse graphs?

Adjacency List Properties

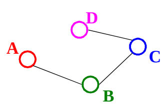


- Running time to:
  - Get all of a vertex's out-edges: $O(d)$ where d is out-degree of vertex
  - Get all of a vertex's in-edges: $O(|E|)$ (but could keep a second adjacency list for this!)
  - Decide if some edge exists: $O(d)$ where d is out-degree of source
  - Insert an edge: $O(1)$
  - Delete an edge: $O(d)$ where d is out-degree of source
- Space requirements:
  - $O(|V|+|E|)$
- Best for sparse graphs: so usually just stick with linked lists

Undirected graphs

Adjacency matrices & adjacency lists both do fine for undirected graphs
- Matrix: Could save space; only ~1/2 the array is used
- Lists: Each edge in two lists to support efficient "get all neighbors"

Example:



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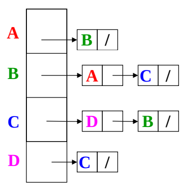

Next…

Okay, we can represent graphs

Now let's implement some useful and non-trivial algorithms

- Topological sort: Given a DAG, order all the vertices so that every vertex comes before all of its neighbors
- Shortest paths: Find the shortest or lowest-cost path from x to y
  - Related: Determine if there even is such a path