Asymptotic Analysis II

CSE 373
Data Structures & Algorithms
Ruth Anderson
Winter 2012

Today's Outline

- **Announcements**
  – Assignment #1, due Thurs, Jan 12 at 11pm
  – Assignment #2, posted today, due Fri Jan 20 at BEGINNING of lecture
- **Algorithm Analysis**
  – Big-Oh
  – Analyzing code

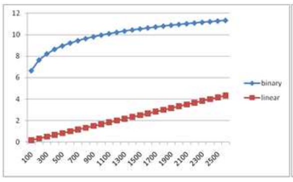
Ignoring constant factors

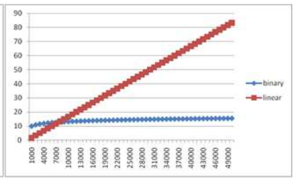
- So binary search is $O(\log n)$ and linear search is $O(n)$
  – But which is faster?
- Could depend on constant factors:
  – How *many* assignments, additions, etc. for each n
    - *E.g.* $T(n) = 5{,}000{,}000n$ vs. $T(n) = 5n^2$
  – And could depend on size of n *(if n is small then constant additive factors could be more important)*
    - *E.g.* $T(n) = 5{,}000{,}000 + \log n$ vs. $T(n) = 10 + n$
- **But** there exists ***some*** n_0 such that for all $n > n_0$ binary search wins
- Let's play with a couple plots to get some intuition…

Linear Search vs. Binary Search

Let's try to "help" linear search:
- Run it on a computer 100x as fast (say 2010 model vs. 1990)
- Use a new compiler/language that is 3x as fast
- Be a clever programmer to eliminate half the work
- So doing each iteration is 600x as fast as in binary search

For small n, linear search is faster! But eventually binary search wins.





Asymptotic notation

About to show formal definition of Big-O, which amounts to saying:
1. Eliminate low-order terms
2. Eliminate coefficients

Examples:
– $4n + 5$
– $0.5n \log n + 2n + 7$
– $n^3 + 2^n + 3n$
– $n \log (10n^2)$

Examples

True or false?

1. 4+3n is O(n)
2. n+2logn is O(logn)
3. logn+2 is O(1)
4. n^{50} is $O(1.1^n)$

Examples

True or false?

1. 4+3n is O(n) — True
2. n+2logn is O(logn) — False
3. logn+2 is O(1) — False
4. n^{50} is O(1.1^n) — True

Big-Oh relates functions

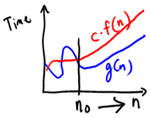
We use *O* on a function f(*n*) (for example n^2) to mean *the set of functions with asymptotic behavior* ***less than or equal to*** f(*n*)

So ($3n^2$+17) **is in** $O(n^2)$
- $3n^2$+17 and n^2 have the same asymptotic behavior

Confusingly, we also say/write:
- ($3n^2$+17) **is** $O(n^2)$
- ($3n^2$+17) ∈ $O(n^2)$
- ($3n^2$+17) = $O(n^2)$

But we would never say $O(n^2)$ = ($3n^2$+17)

Formally Big-Oh



Definition: **g(*n*)** is in O(**f(*n*)**) iff there exist positive constants ***c*** and n_0 such that

g(*n*) ≤ ***c*** **f(*n*)** for all $n \geq n_0$

To show **g(*n*)** is in O(**f(*n*)**), pick a ***c*** large enough to "cover the constant factors" and n_0 large enough to "cover the lower-order terms"

- Example: Let **g(*n*)** = $3n^2$+17 and **f(*n*)** = n^2

  c = 5 and n_0 = 10 is more than good enough

This is "less than or equal to"
- So $3n^2$+17 is also $O(n^5)$ and $O(2^n)$ etc.

Using the definition of Big-Oh (Example 1)

Def'n: **g(*n*)** is in **O(f(*n*))** iff there exist positive constants ***c*** and n_0 s.t. **g(*n*)** ≤ ***c*** **f(*n*)** for all $n \geq n_0$

Given: **g(n)** = 1000n & **f(n)** = n^2

Prove: **g(n)** is in O(**f(n)**)

- A valid proof is to find valid **c** & n_0
- Try: n_0 =1000, **c** =1
- Also: n_0 =1, **c** =1000

Using the definition of Big-Oh (Example 2)

Def'n: **g(*n*)** is in **O(f(*n*))** iff there exist positive constants ***c*** and n_0 s.t. **g(*n*)** ≤ ***c*** **f(*n*)** for all $n \geq n_0$

Given: **g(n)** = 4n & **f(n)** = n^2

Prove: **g(n)** is in O(**f(n)**)

- A valid proof is to find valid **c** & n_0
- When n=4, **g(n)** =16 & **f(n)** =16; this is the crossing over point
- So we can choose n_0 = 4, and **c** = 1
- Note: There are many possible choices: ex: n_0 = 78, and **c** = 42 works fine

Using the definition of Big-Oh (Example 3)

Def'n: **g(*n*)** is in **O(f(*n*))** iff there exist positive constants ***c*** and n_0 s.t. **g(*n*)** ≤ ***c*** **f(*n*)** for all $n \geq n_0$

Given: **g(n)** = n^4 & **f(n)** = 2^n,

Prove: **g(n)** is in O(**f(n)**)

- A valid proof is to find valid **c** & n_0
- One possible answer: n_0 = 20, and **c** = 1

What's with the c?

- To capture this notion of similar asymptotic behavior, we allow a constant multiplier (called **c**)
- Consider:
  - **g(n)** = 7n+5
  - **f(n)** = n
- These have the same asymptotic behavior (linear), so **g(n)** is in O(**f(n)**) even though **g(n)** is always larger
- There is no positive n_0 such that **g(n)** ≤ **f(n)** for all n ≥ n_0
- The 'c' in the definition allows for that: $g(n) \le c\, f(n)$ for all $n \ge n_0$
- To prove **g(n)** is in O(**f(n)**), have **c** = 12, $n_0 = 1$

Big Oh: Common Categories

From fastest to slowest:

$O(1)$	constant (same as $O(k)$ for constant k)
$O(\log n)$	logarithmic ($\log_k n$, $\log n^2$ is $O(\log n)$)
$O(n)$	linear
$O(n \log n)$	"n log n"
$O(n^2)$	quadratic
$O(n^3)$	cubic
$O(n^k)$	polynomial (where k is an constant)
$O(k^n)$	exponential (where k is any constant > 1)

Usage note: "exponential" does not mean "grows really fast", it means "grows at rate proportional to k^n for some k>1"

More Definitions

- **Upper bound**: $O($ **f(n)** $)$ is the set of all functions asymptotically less than or equal to **f(n)**
  - g(n) is in $O($ **f(n)** $)$ if there exist positive constants c and n_0 such that
    $g(n) \le c\, f(n)$ for all $n \ge n_0$
- **Lower bound**: $\Omega($ **f(n)** $)$ is the set of all functions asymptotically greater than or equal to **f(n)**
  - g(n) is in $\Omega($ **f(n)** $)$ if there exist positive constants c and n_0 such that
    $g(n) \ge c\, f(n)$ for all $n \ge n_0$
- **Tight bound**: $\theta($ **f(n)** $)$ is the set of all functions asymptotically equal to **f(n)**
  - g(n) is in $\theta($ **f(n)** $)$ if **both**: g(n) is in $O($ **f(n)** $)$ AND g(n) is in $\Omega($ **f(n)** $)$

Even More Definitions…

$O($ **f(n)** $)$ is the set of all functions asymptotically less than or equal to **f(n)**
- g(n) is in $O($ **f(n)** $)$ if there exist positive constants c and n_0 such that
  $g(n) \le c\, f(n)$ for all $n \ge n_0$

$o($ **f(n)** $)$ is the set of all functions asymptotically less than **f(n)**
- g(n) is in $o($**f(n)**$)$ if for any positive constant c, there exists a positive constant n_0 such that
  $g(n) < c\, f(n)$ for all $n \ge n_0$

$\Omega($ **f(n)** $)$ is the set of all functions asymptotically greater than or equal to **f(n)**
- g(n) is in $\Omega($ **f(n)** $)$ if there exist positive constants c and n_0 such that
  $g(n) \ge c\, f(n)$ for all $n \ge n_0$

$\omega($ **f(n)** $)$ is the set of all functions asymptotically greater than **f(n)**
- g(n) is in $\omega($**f(n)**$)$ if for any positive constant c, there exists a positive constant n_0 such that
  $g(n) > c\, f(n)$ for all $n \ge n_0$

Big-Omega et al. Intuitively

Asymptotic Notation	Mathematics Relation
O	≤
Ω	≥
Θ	=
o	<
ω	>

Types of Analysis

Two orthogonal axes:

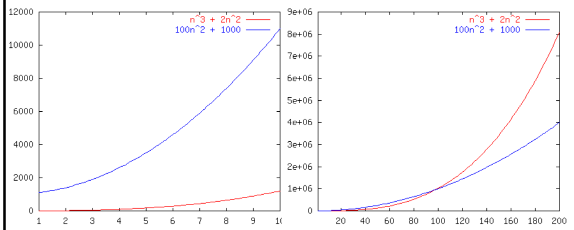
- bound flavor (usually we talk about upper or tight)
  - upper bound (O, o)
  - lower bound (Ω, ω)
  - asymptotically tight (Θ)
- analysis case (usually we talk about worst)
  - worst case (adversary)
  - average case
  - best case
  - "amortized"

Which Function Grows Faster?

$n^3 + 2n^2$ vs. $100n^2 + 1000$

Which Function Grows Faster?

$n^3 + 2n^2$ vs. $100n^2 + 1000$

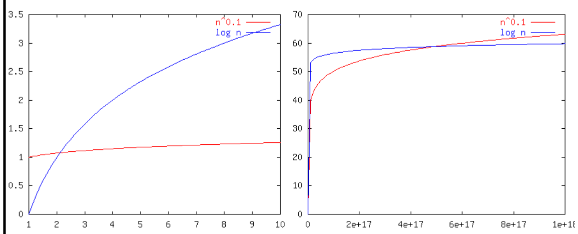


Which Function Grows Faster?

$n^{0.1}$ vs. $\log n$

Which Function Grows Faster?

$n^{0.1}$ vs. $\log n$

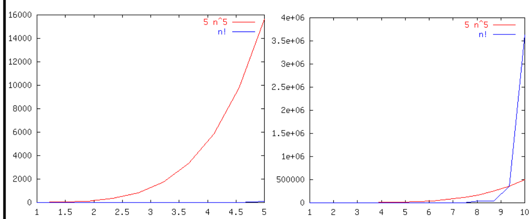


Which Function Grows Faster?

$5n^5$ vs. $n!$

Which Function Grows Faster?

$5n^5$ vs. $n!$



Nested Loops

```
for i = 1 to n do
  for j = 1 to n do
    sum = sum + 1

for i = 1 to n do
  for j = 1 to n do
    sum = sum + 1
```


More Nested Loops

```
for i = 1 to n do
  for j = 1 to n do
    if (cond) {
            do_stuff(sum)
      } else {
            for k  = 1 to n*n
                  sum += 1
```


Big-Oh Caveats

- Asymptotic complexity (Big-Oh) focuses on behavior for **large *n*** and is independent of any computer / coding trick
  - But you can “abuse” it to be misled about trade-offs
  - Example: $n^{1/10}$ vs. `log` n
    - Asymptotically $n^{1/10}$ grows more quickly
    - But the “cross-over” point is around $5 * 10^{17}$
    - So if you have input size less than 2^{58}, prefer $n^{1/10}$
- Comparing O() for ***small n*** values can be misleading
  - Quicksort: O(nlogn) (expected)
  - Insertion Sort: $O(n^2)$ (expected)
  - Yet in reality Insertion Sort is faster for small n’s
  - We’ll learn about these sorts later

Addendum: Timing vs. Big-Oh?

- At the core of CS is a backbone of theory & mathematics
  - Examine the algorithm itself, mathematically, not the implementation
  - Reason about performance as a function of n
  - Be able to mathematically prove things about performance
- Yet, timing has its place
  - In the real world, we do want to know whether implementation A runs faster than implementation B on data set C
  - Ex: Benchmarking graphics cards
  - We will do some timing in our homeworks
- Evaluating an algorithm? Use asymptotic analysis
- Evaluating an implementation of hardware/software? Timing can be useful