CSE 373
Data Structures and Algorithms

Lecture 22: Graphs IV

Dijkstra's Algorithm

- **Dijkstra's algorithm**: finds shortest (minimum weight) path between a particular pair of vertices in a *weighted* directed graph with *nonnegative* edge weights
  - Solves the "one vertex, shortest path" problem
- Basic algorithm concept:
  - For each vertex, keep track of the currently known best way to reach it (distance, previous vertex)
  - Iterate until best way is found

Example Application

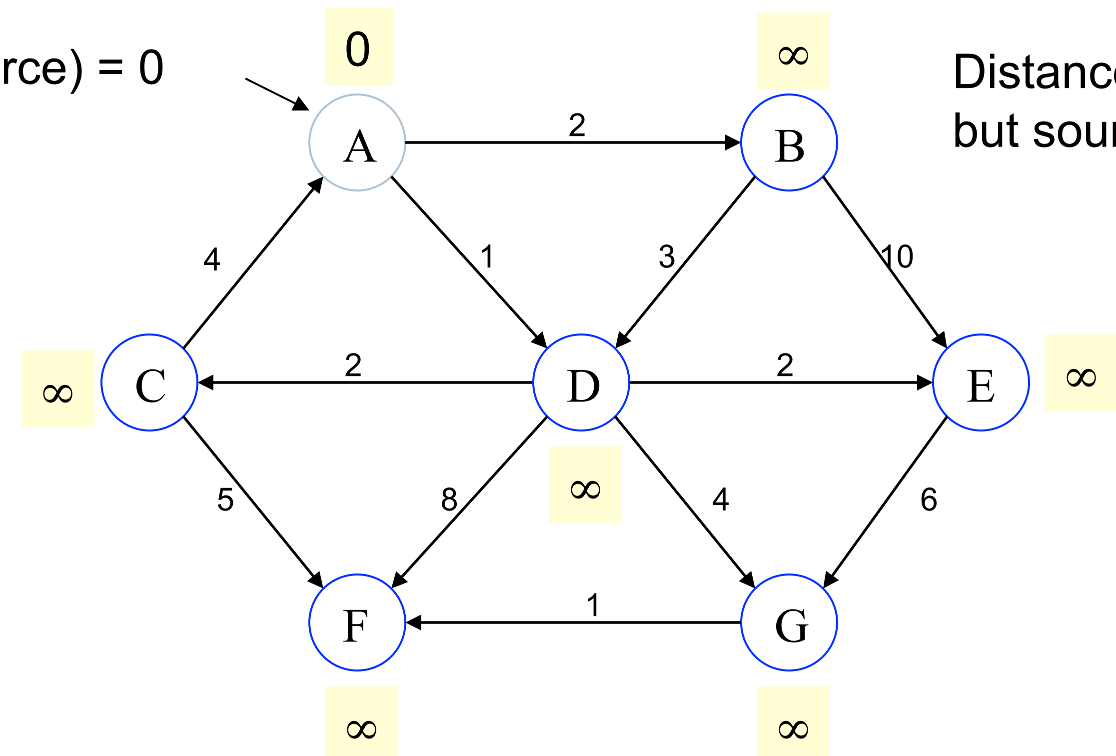
- Dijkstra's algorithm can be used to find the shortest route between one city and any other
  - vertices represent cities
  - edge weights represent driving distances between pairs of cities connected by a direct road

Dijkstra pseudocode

```
Dijkstra(v1, v2):
    for each vertex v:                        // Initialization
         v's distance := infinity.
         v's previous := none.
    v1's distance := 0.
    List := {all vertices}.

    while List is not empty:
        v := remove List vertex with minimum distance.
        mark v as known.
        for each unknown neighbor n of v:
             dist := v's distance + edge (v, n)'s weight.

             if dist is smaller than n's distance:
                 n's distance := dist.
                 n's previous := v.

    reconstruct path from v2 back to v1,
    following previous pointers.
```

Example: Initialization

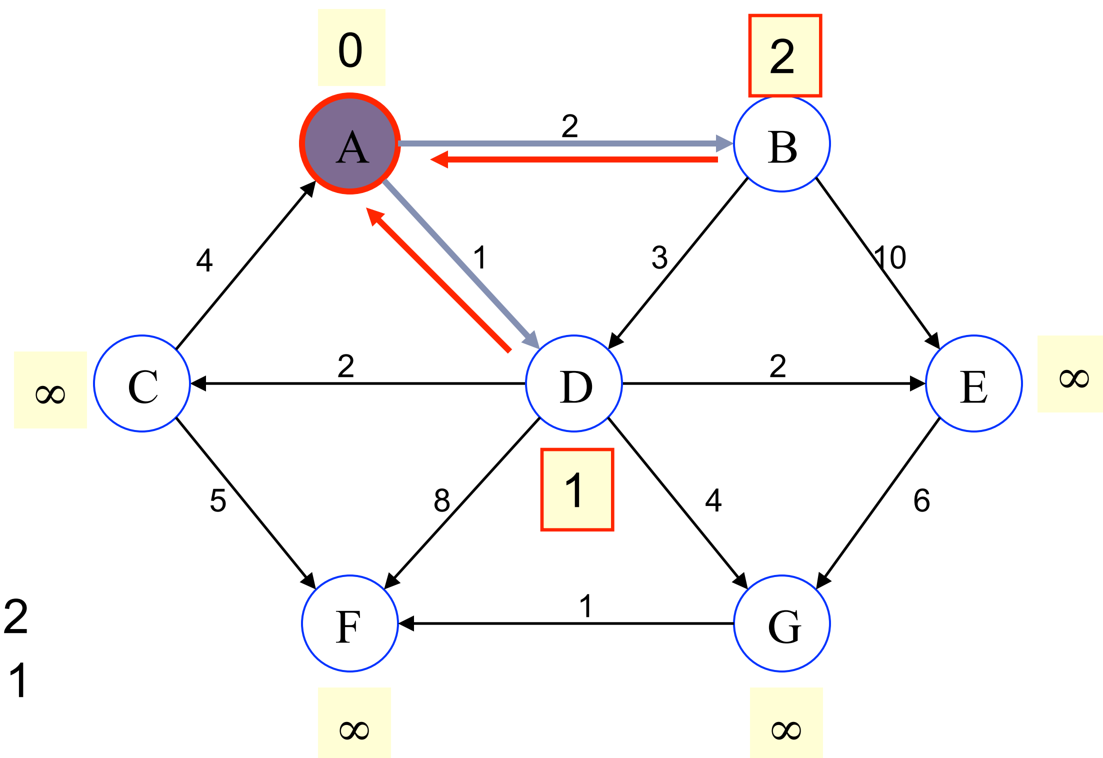
Distance(source) = 0



Distance (all vertices but source) = ∞

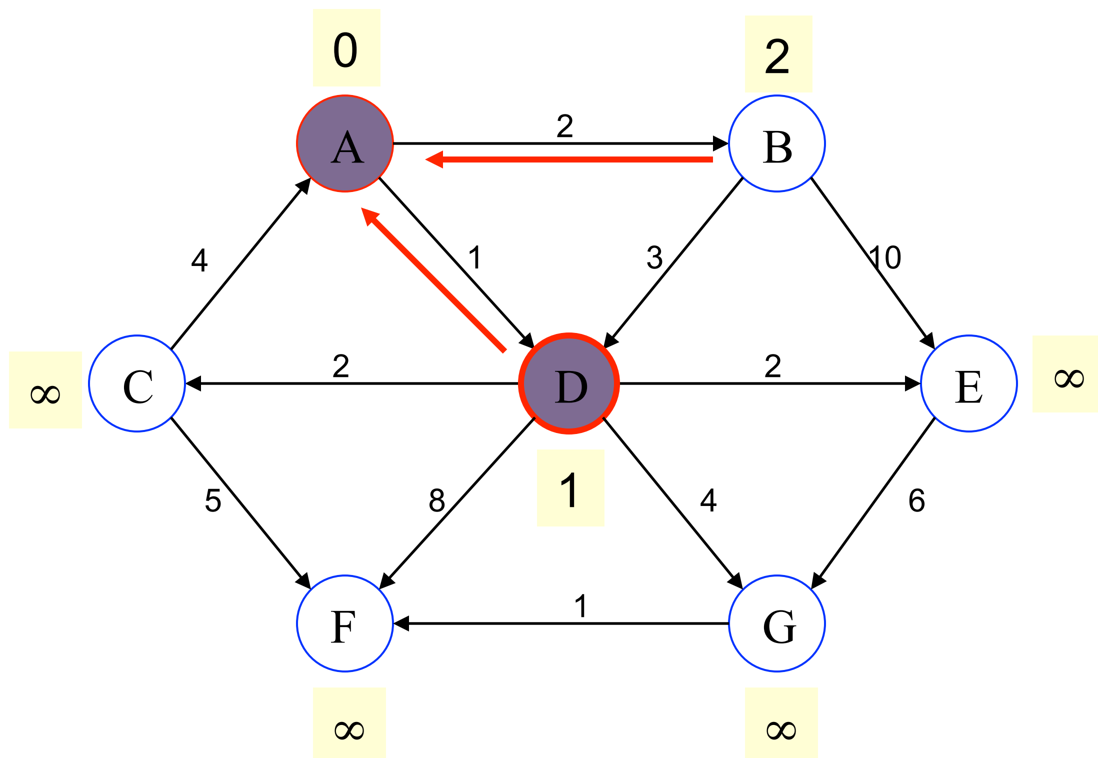
Pick vertex in List with minimum distance.

Example: Update neighbors' distance



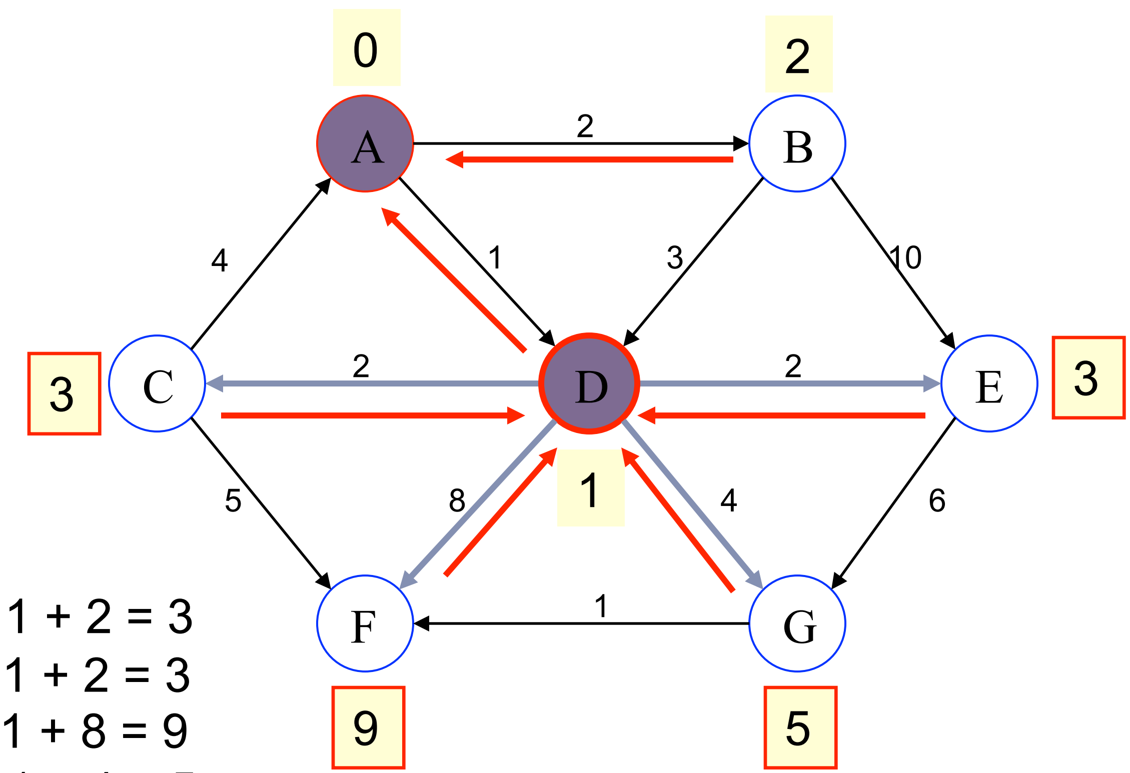
Distance(B) = 2
Distance(D) = 1

Example: Remove vertex with min. distance



Pick vertex in List with minimum distance, i.e., D

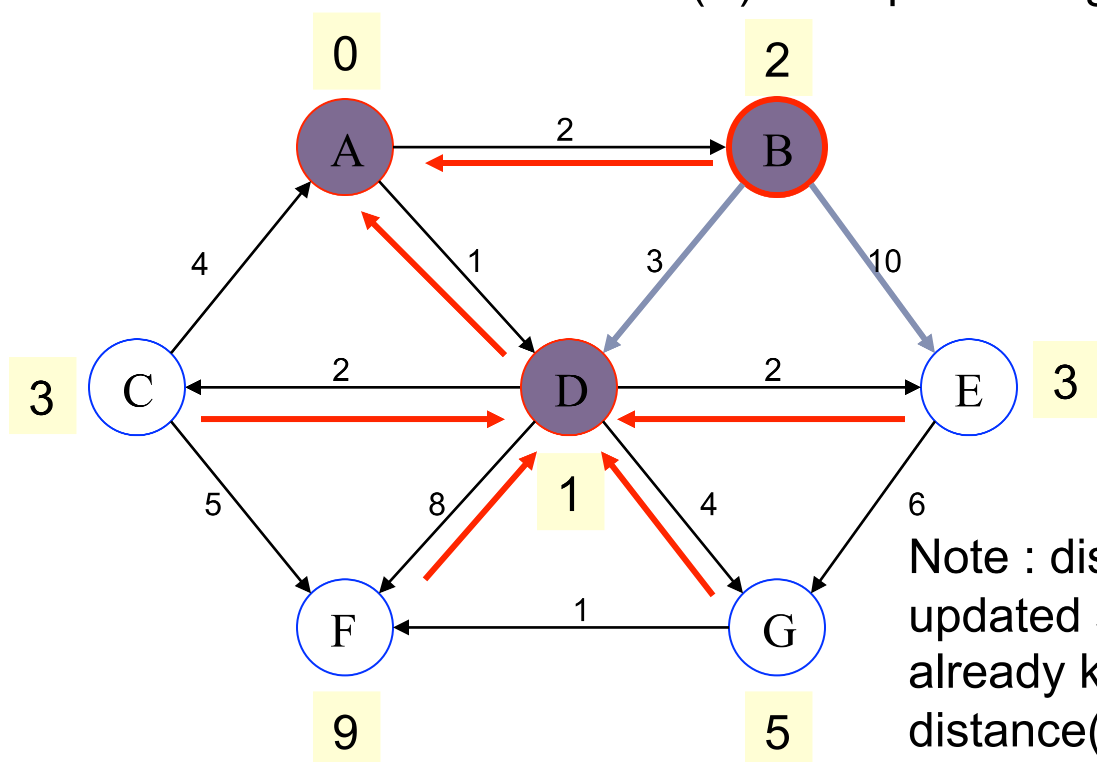
Example: Update neighbors



Distance(C) = 1 + 2 = 3
Distance(E) = 1 + 2 = 3
Distance(F) = 1 + 8 = 9
Distance(G) = 1 + 4 = 5

Example: Continued...

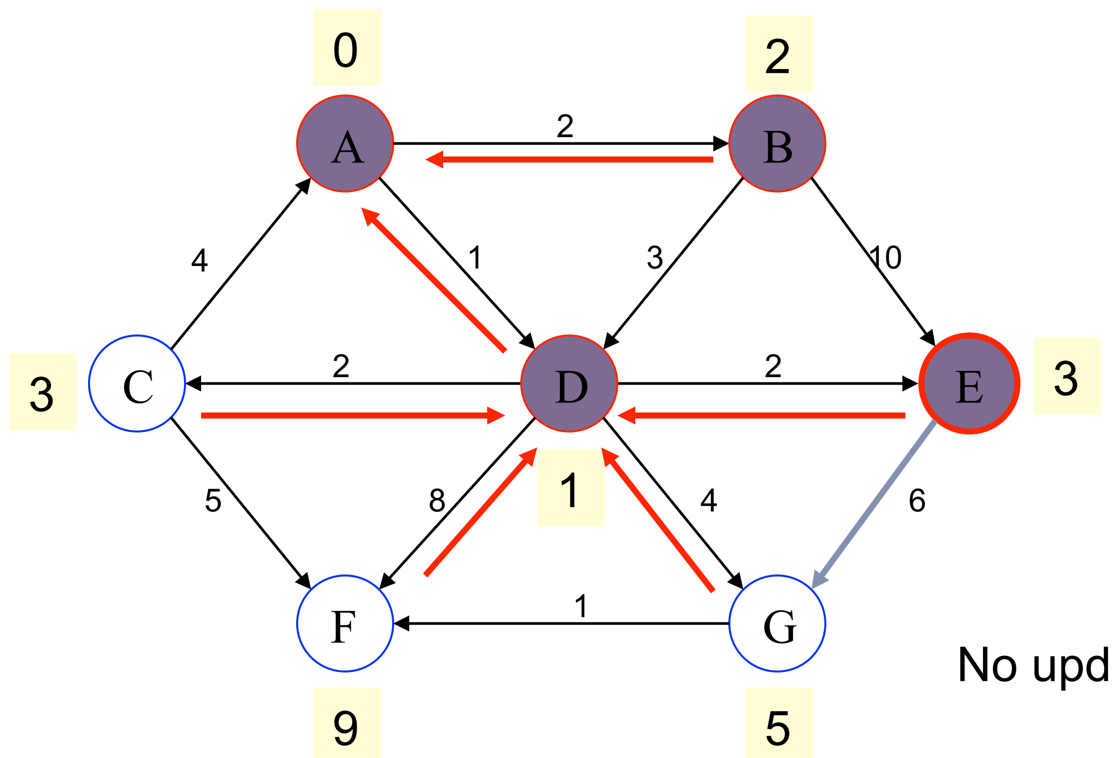
Pick vertex in List with minimum distance (B) and update neighbors



Note : distance(D) not updated since D is already known and distance(E) not updated since it is larger than previously computed

Example: Continued...

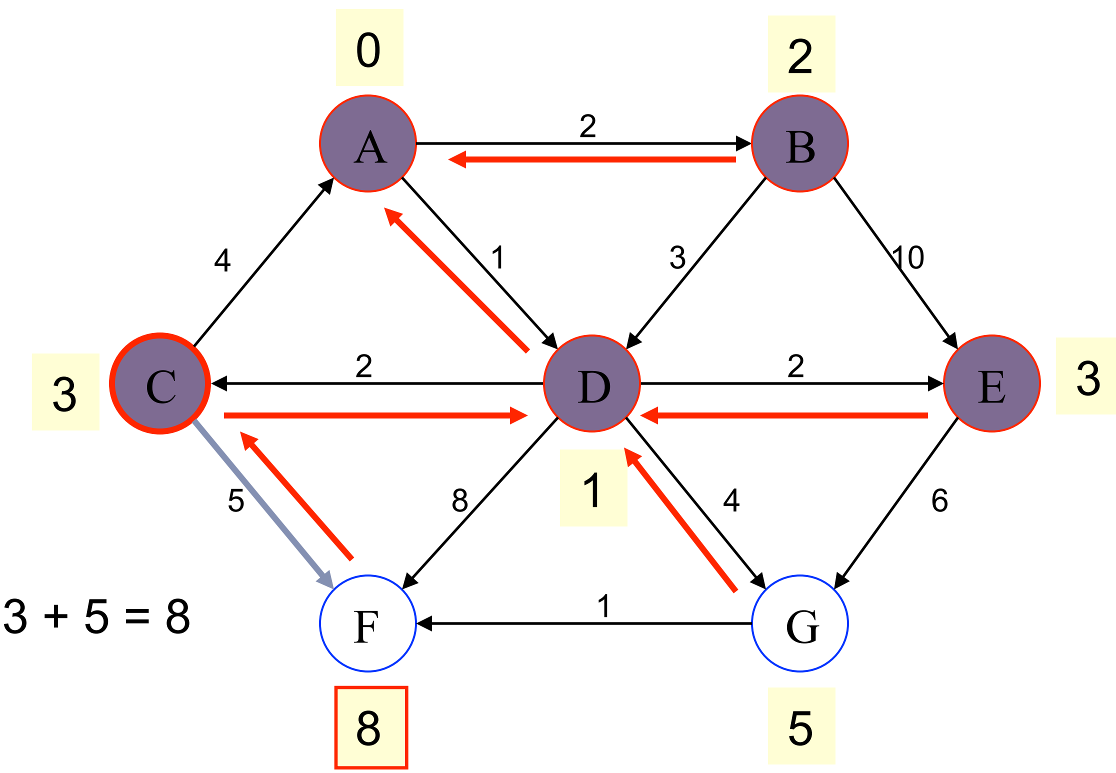
Pick vertex List with minimum distance (E) and update neighbors



No updating

Example: Continued...

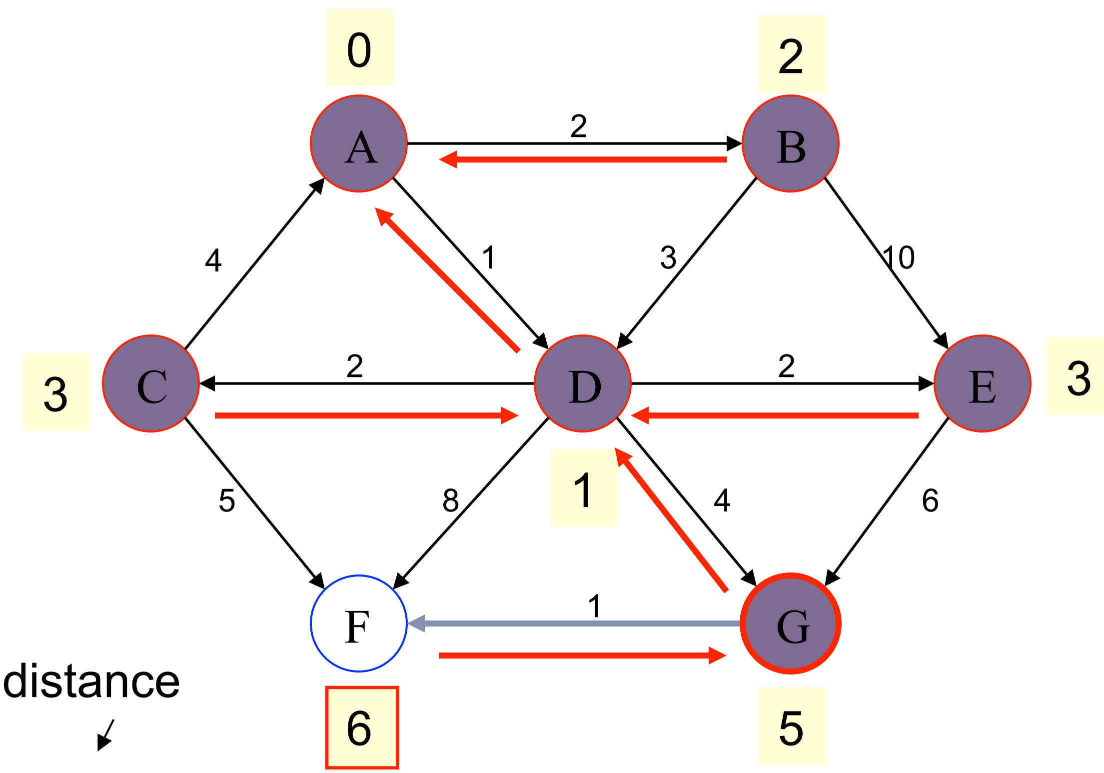
Pick vertex List with minimum distance (C) and update neighbors



Distance(F) = 3 + 5 = 8

Example: Continued...

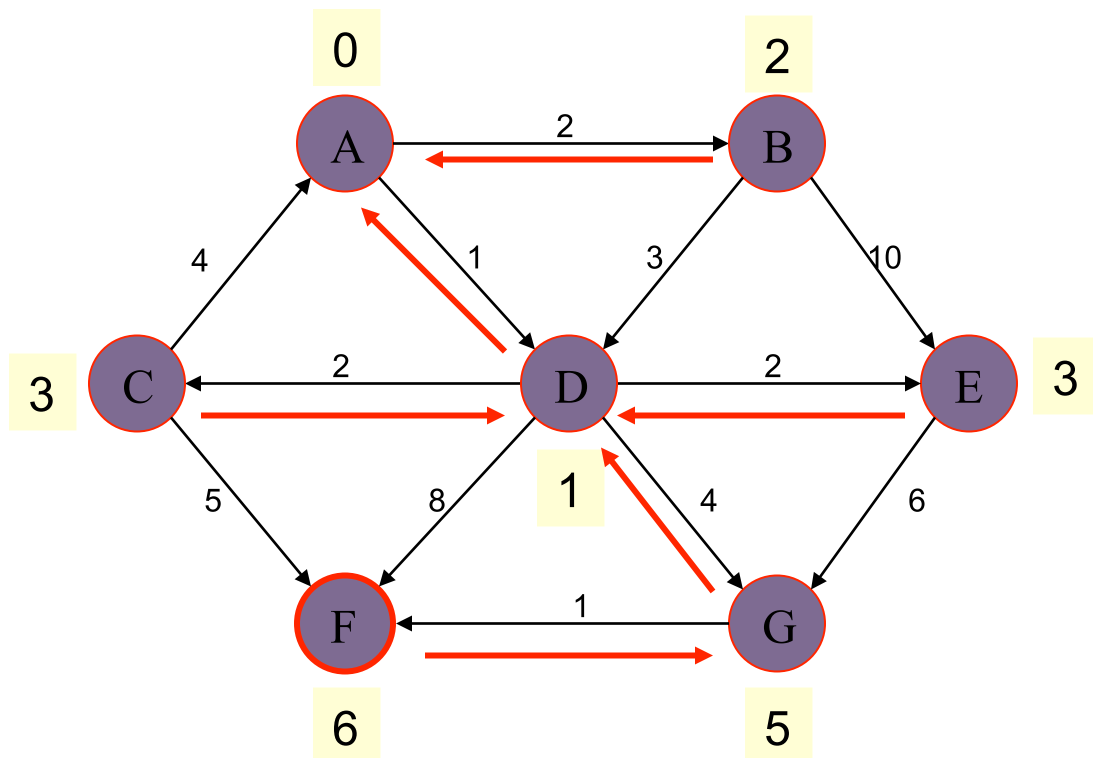
Pick vertex List with minimum distance (G) and update neighbors



Previous distance

Distance(F) = min (8, 5+1) = 6

Example (end)



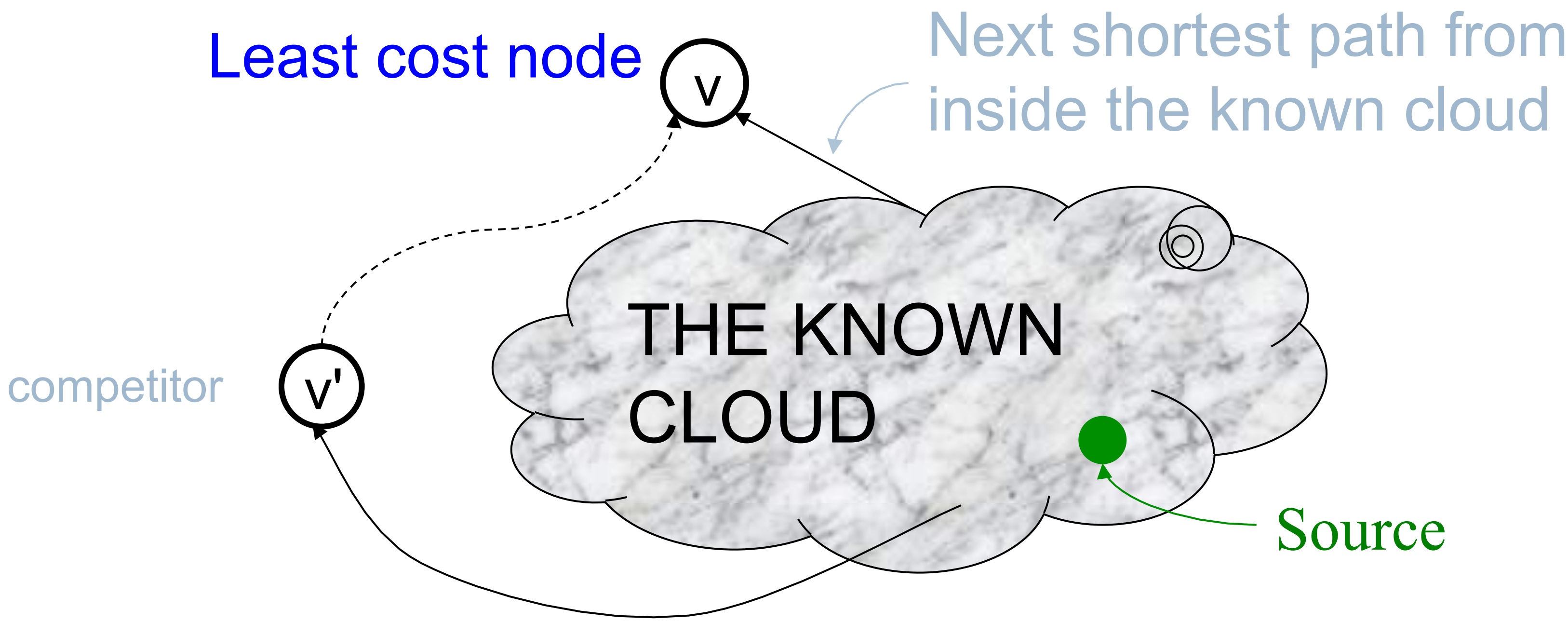
Pick vertex not in S with lowest cost (F) and update neighbors

Correctness

- Dijkstra's algorithm is a greedy algorithm
  - Makes choices that currently seem the best
  - In general, locally optimal does not always mean globally optimal (think hill-climbing), but in this case, it is.

- Correct because maintains following two properties:
  - For every known vertex, recorded distance is shortest distance to that vertex from source vertex
  - For every unknown vertex *v*, its recorded distance is shortest path distance to *v* from source vertex, considering only currently known vertices and *v*

“Cloudy” Proof: The Idea



- If the path to v is the next shortest path, the path to v' must be at least as long (if it were shorter, it would be picked over v). Therefore, any path through v' to v cannot be shorter!

Time Complexity: List

- The simplest implementation of the Dijkstra's algorithm stores vertices in an ordinary linked list or array
  - Good if the graph is dense (lots of edges: $|E| \sim O(|V^2|)$)

- Initialization (setting to infinity, unknown) $O(|V|)$
- While loop $O(|V|)$
  - Find and remove min distance vertex $O(|V|)$
- Potentially $O(|E|)$ distance updates
  - Update costs $O(1)$
- Reconstruct path $O(|E|)$

- Total time $O(|V^2| + |E|) = O(|V^2|)$

Time Complexity: Priority Queue

- For sparse graphs (i.e. $|E| \sim O(|V|)$), Dijkstra's implemented more efficiently by *priority queue*

- Initialization $O(|V|)$ using $O(|V|)$ buildHeap
- While loop $O(|V|)$
  - Find and remove min distance vertex $O(\log |V|)$ using deleteMin
- Potentially $O(|E|)$ distance updates
  - Update costs $O(\log |V|)$ using decreaseKey
- Reconstruct path $O(|E|)$

- Total time $O(|V|\log|V| + |E|\log|V|) = O(|E|\log|V|)$
- $|V| = O(|E|)$ assuming a connected graph

Exercise

- Use Dijkstra's algorithm to determine the lowest cost path from vertex *A* to all of the other vertices in the graph. Keep track of previous vertices so that you can reconstruct the path later.