CSE 373
Data Structures and Algorithms

Lecture 8: Sorting II

Sorting Classification

In memory sorting			External sorting
Comparison sorting $\Omega(N \log N)$		Specialized Sorting	
$O(N^2)$	$O(N \log N)$	$O(N)$	# of disk accesses
• Bubble Sort • Selection Sort • Insertion Sort • Shell Sort	• Merge Sort • Quick Sort • Heap Sort	• Bucket Sort • Radix Sort	• Simple External Merge Sort • Variations

O(n log n) Comparison Sorting

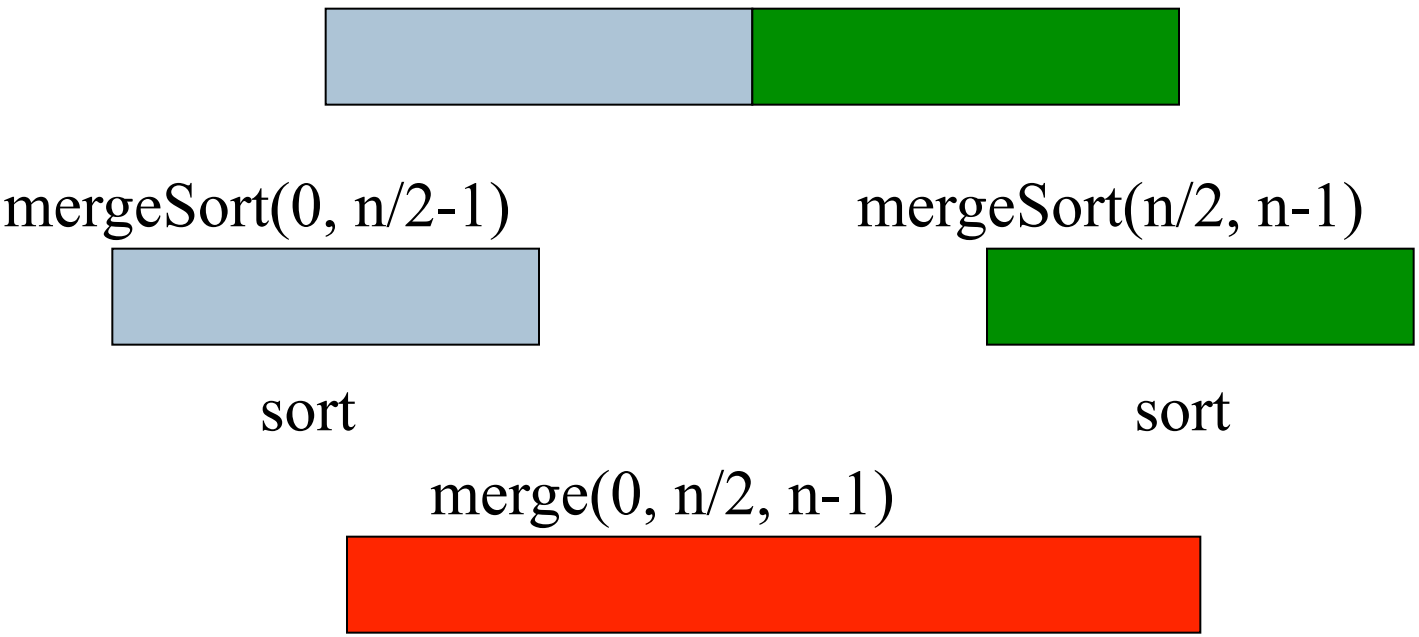
Merge sort

- **merge sort**: orders a list of values by recursively dividing the list in half until each sub-list has one element, then recombining
  - Invented by John von Neumann in 1945

- Another "divide and conquer" algorithm
  - *divide* the list into two roughly equal parts
  - *conquer* by sorting the two parts
    - recursively divide each part in half, continuing until a part contains only one element (one element is sorted)
  - *combine* the two parts into one sorted list

Merge sort idea

- Divide the array into two halves.
- Recursively sort the two halves (using merge sort).
- Use merge to combine the two arrays.



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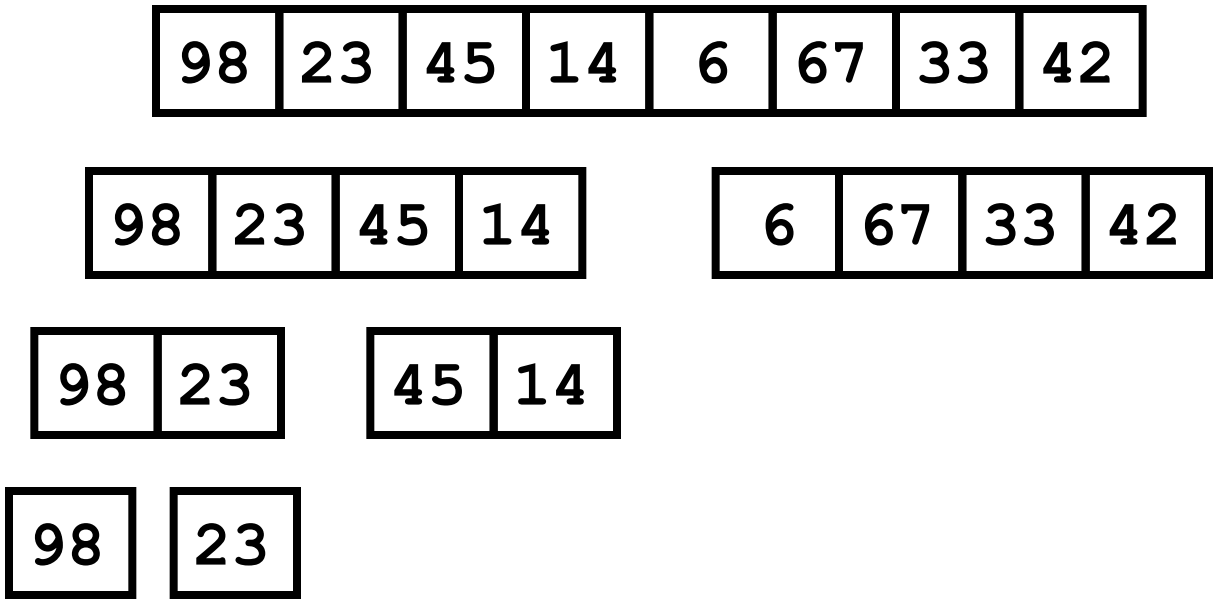
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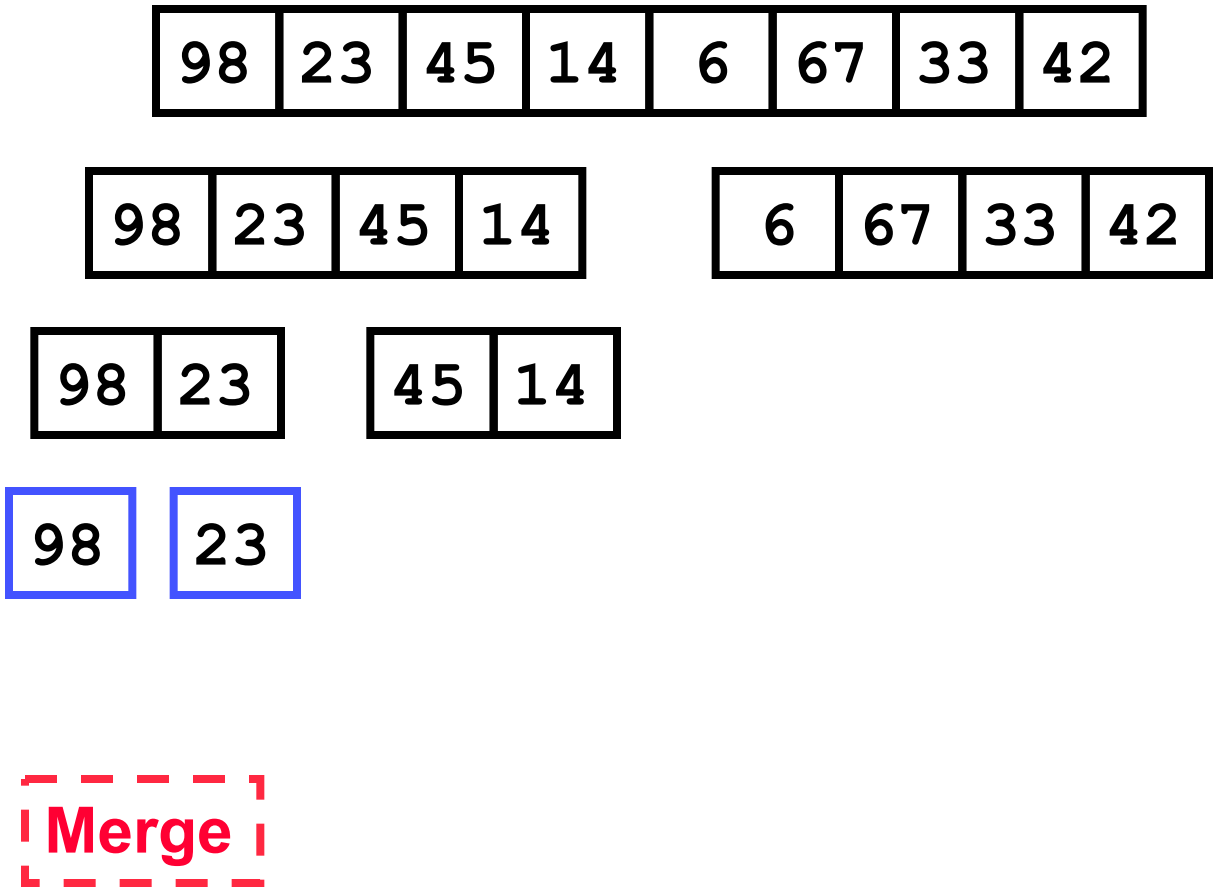
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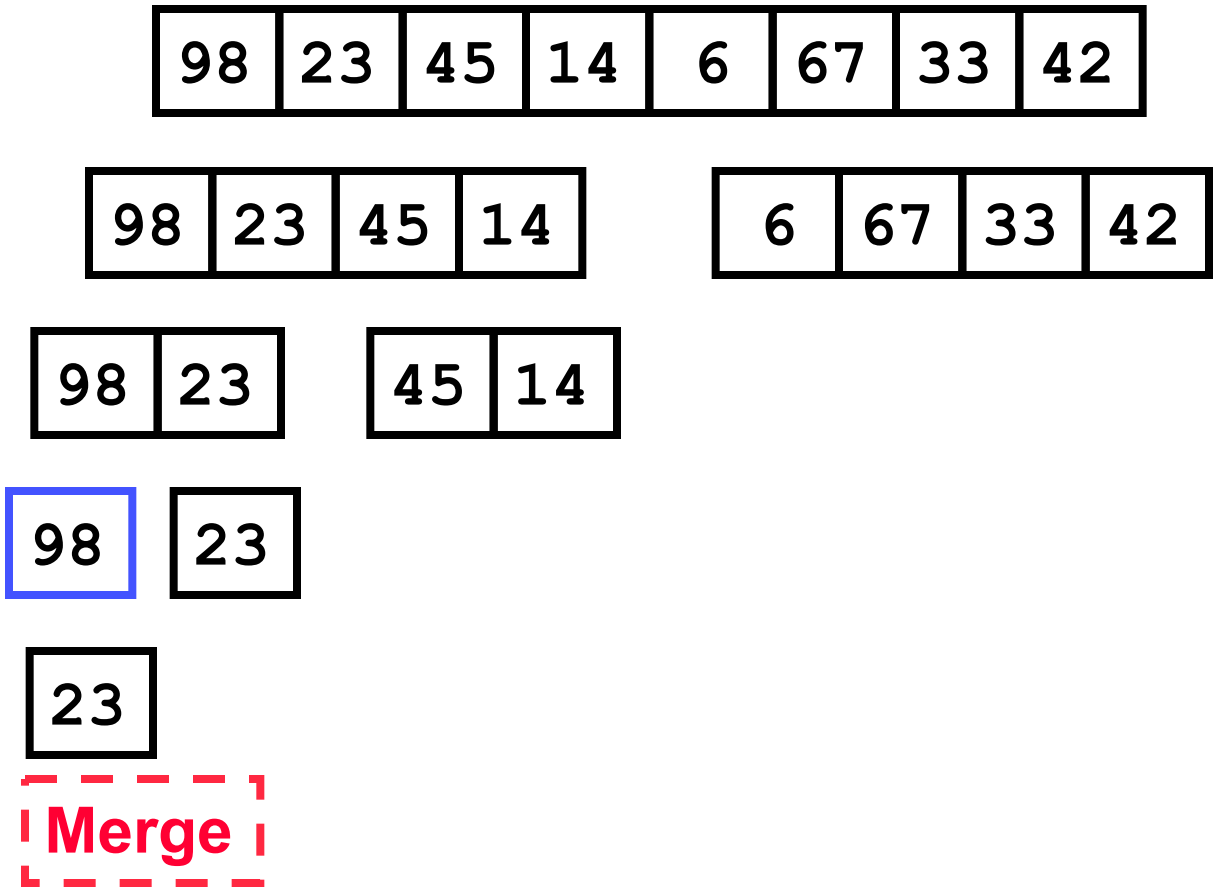
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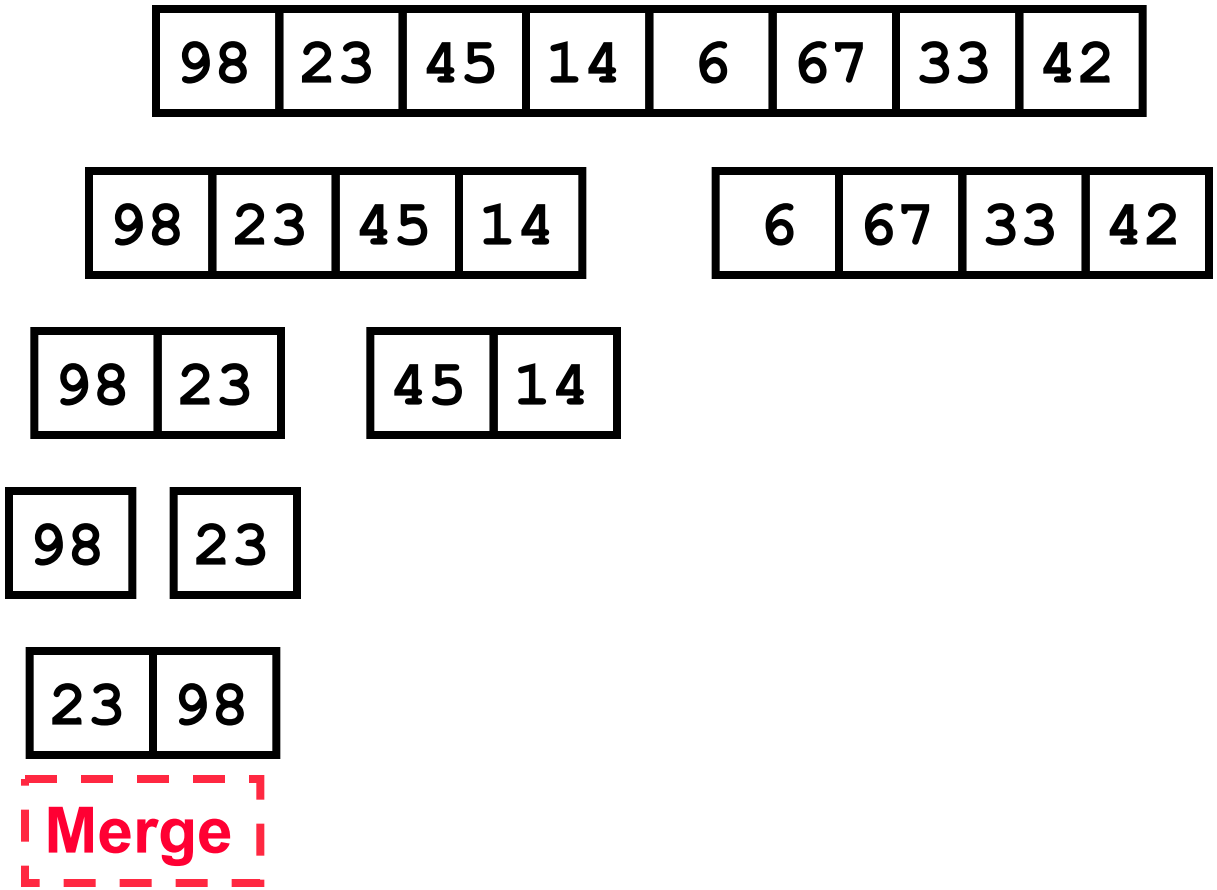
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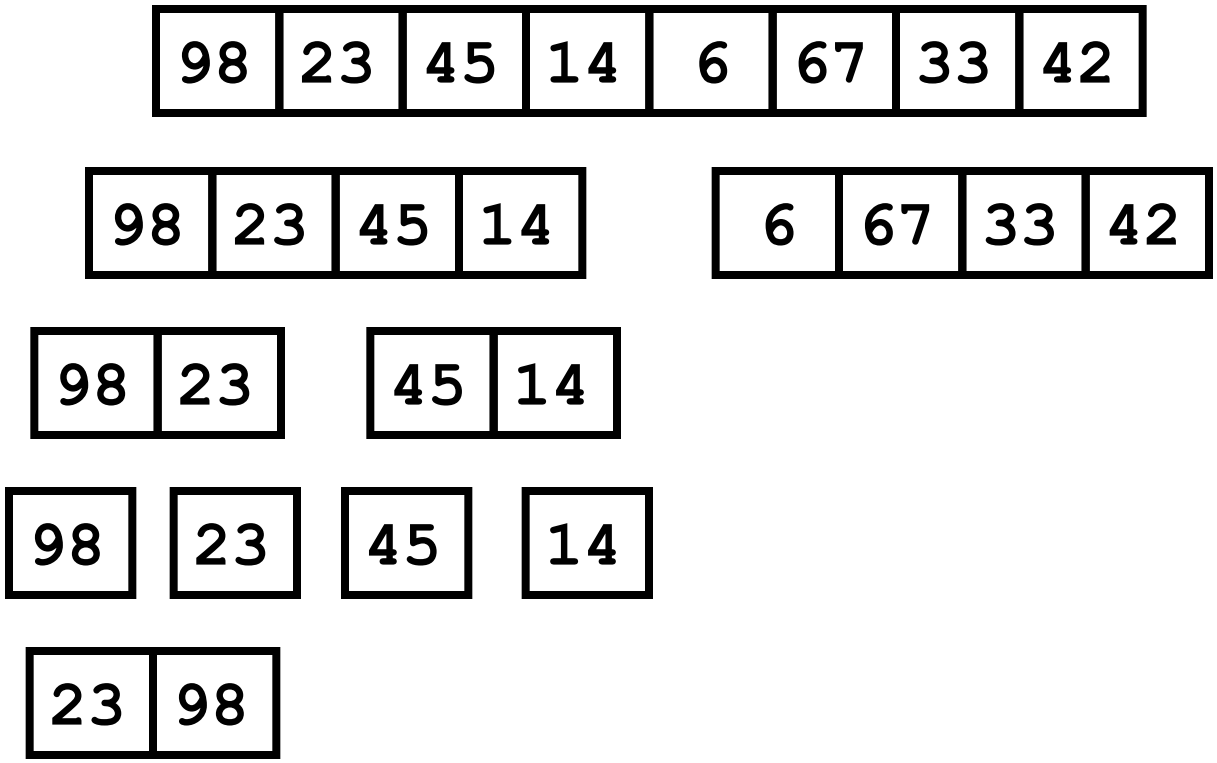
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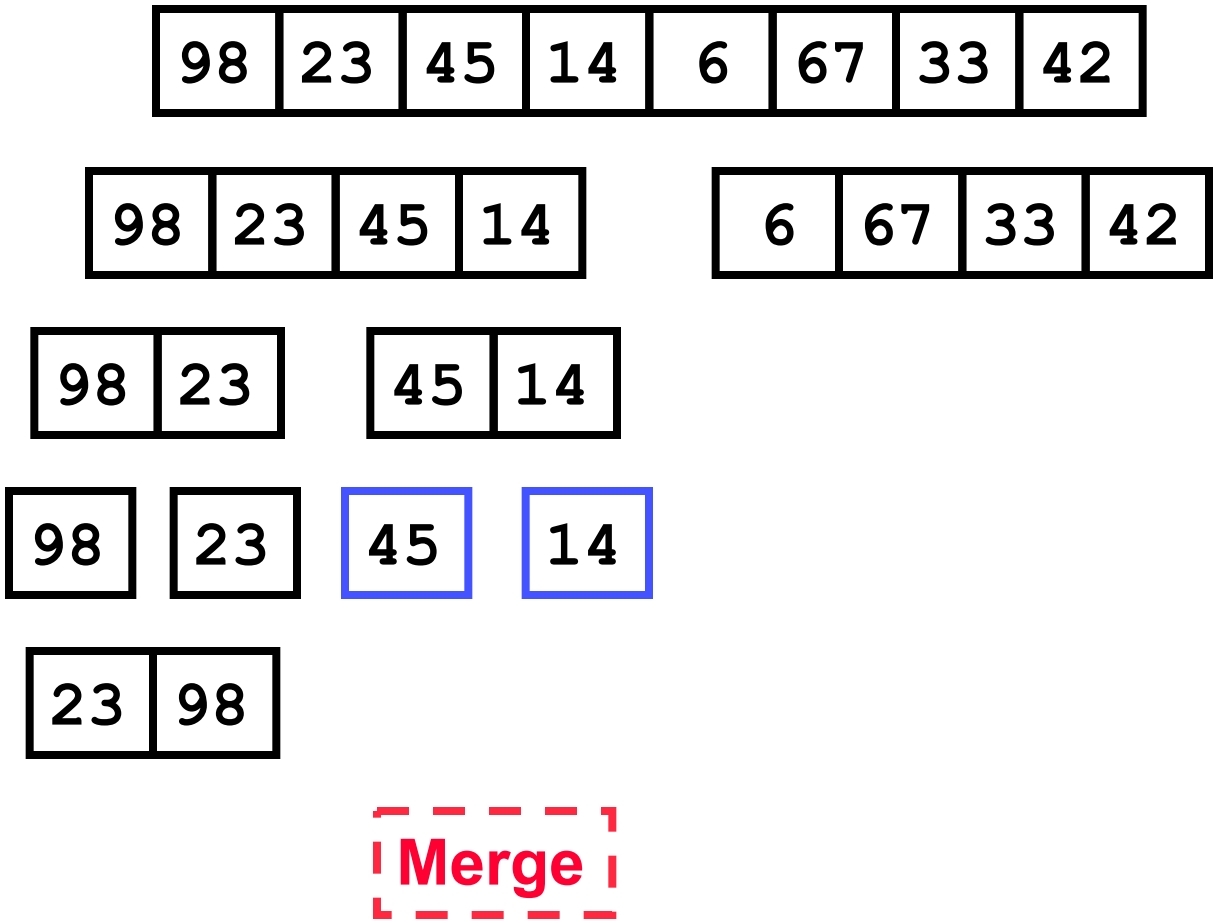
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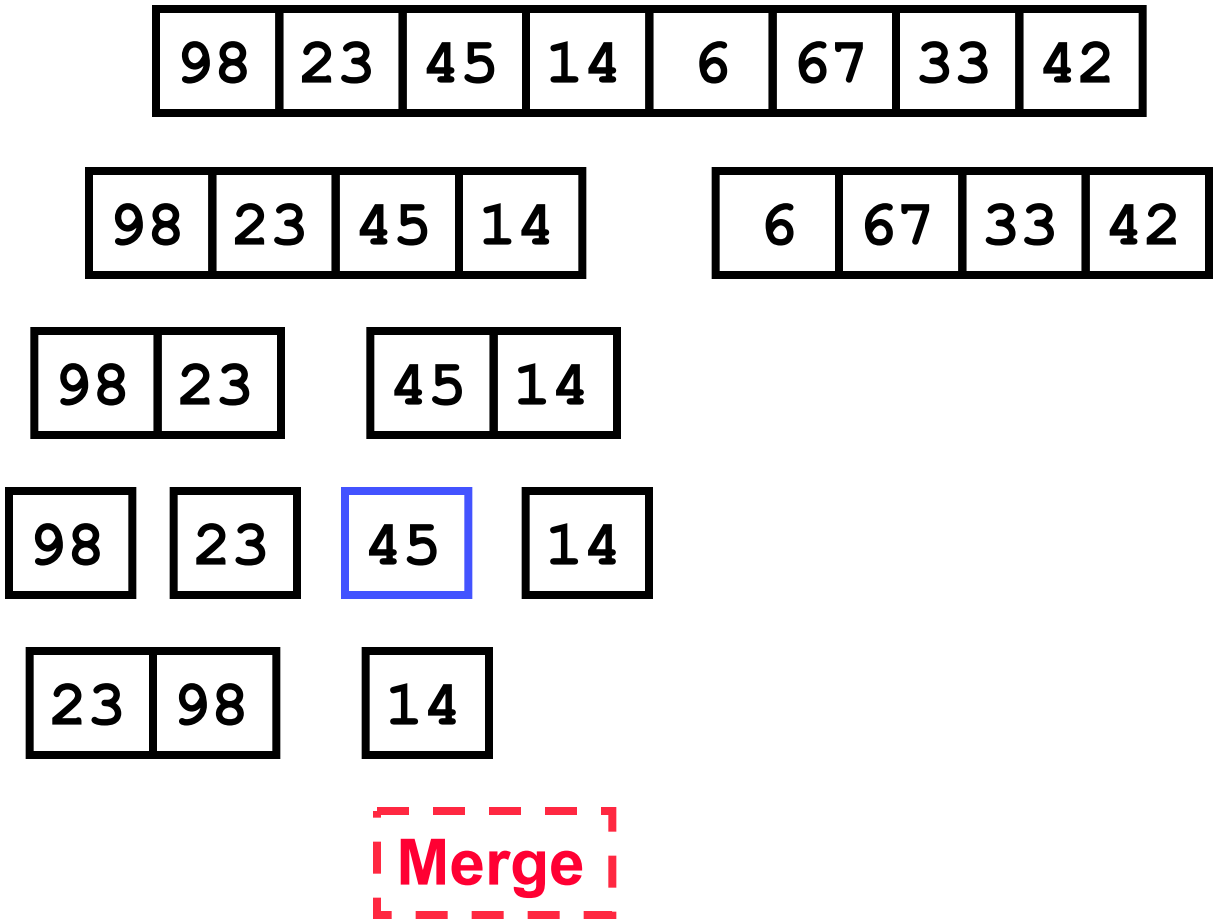
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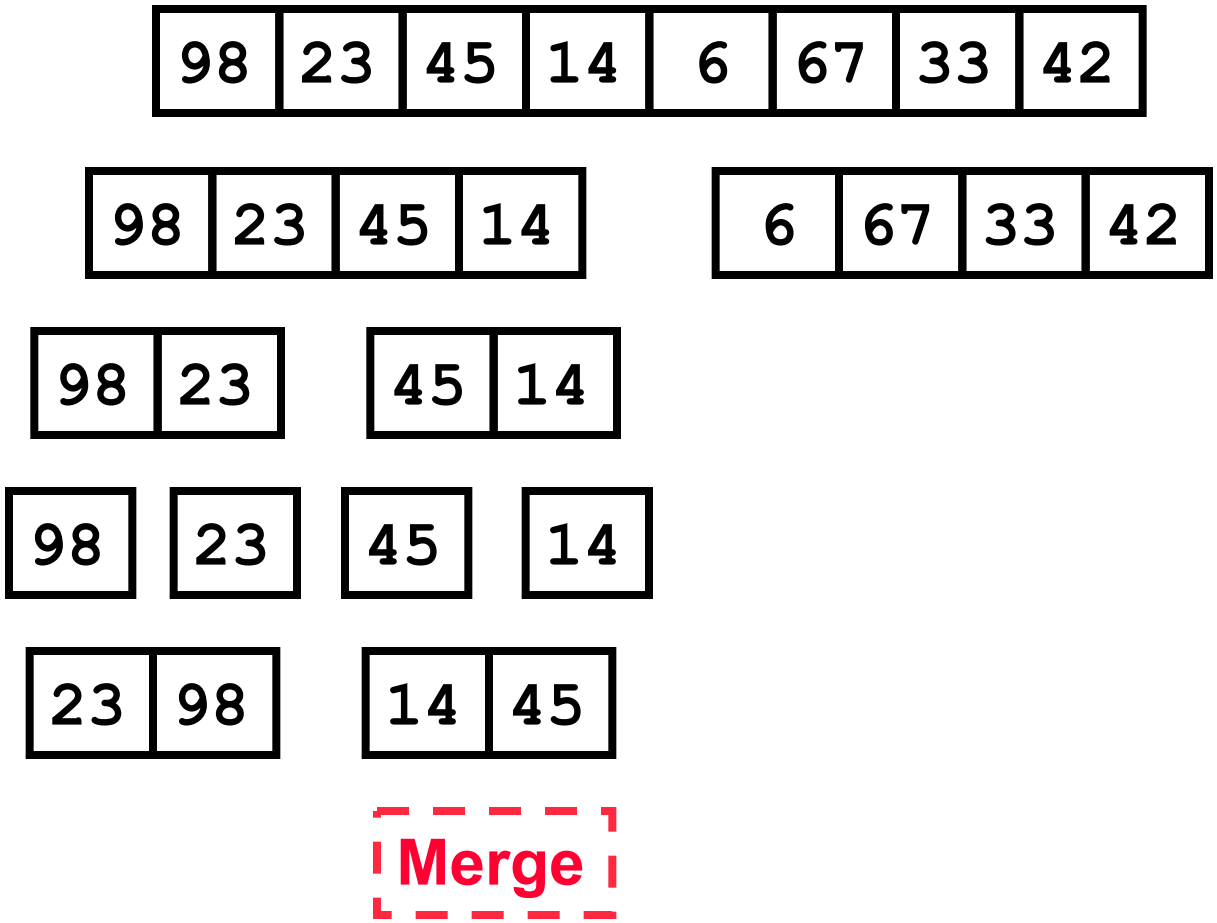
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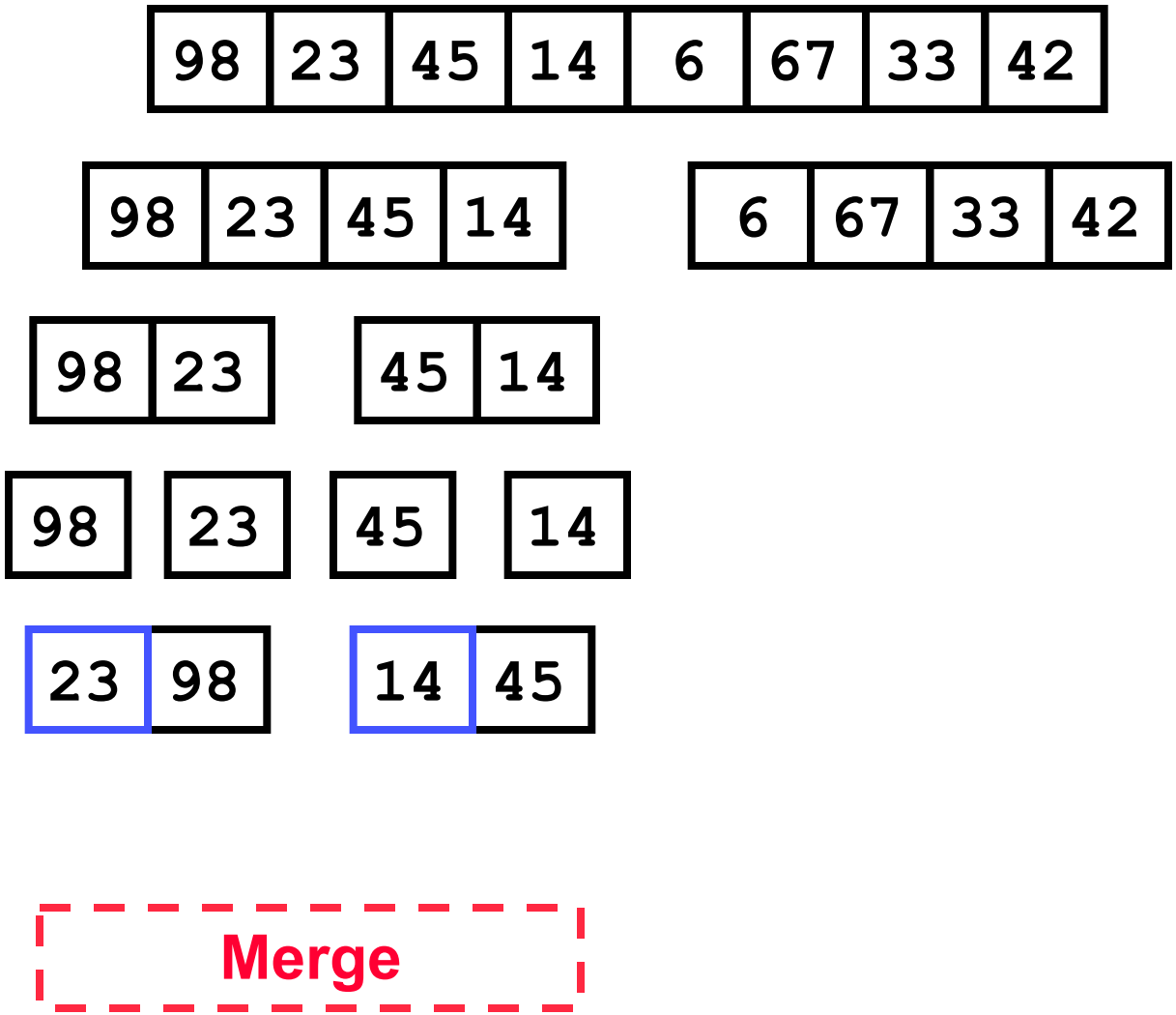
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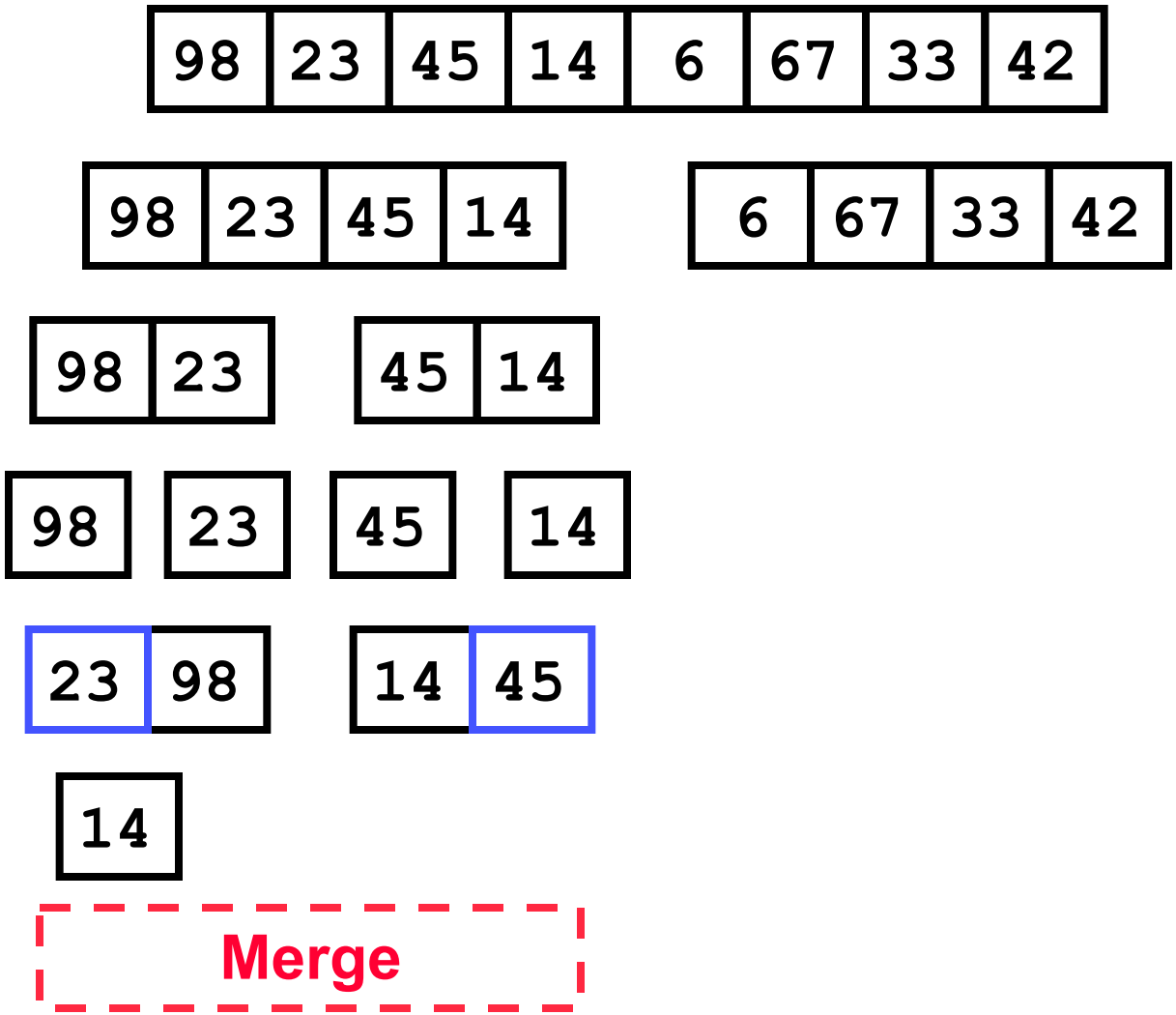
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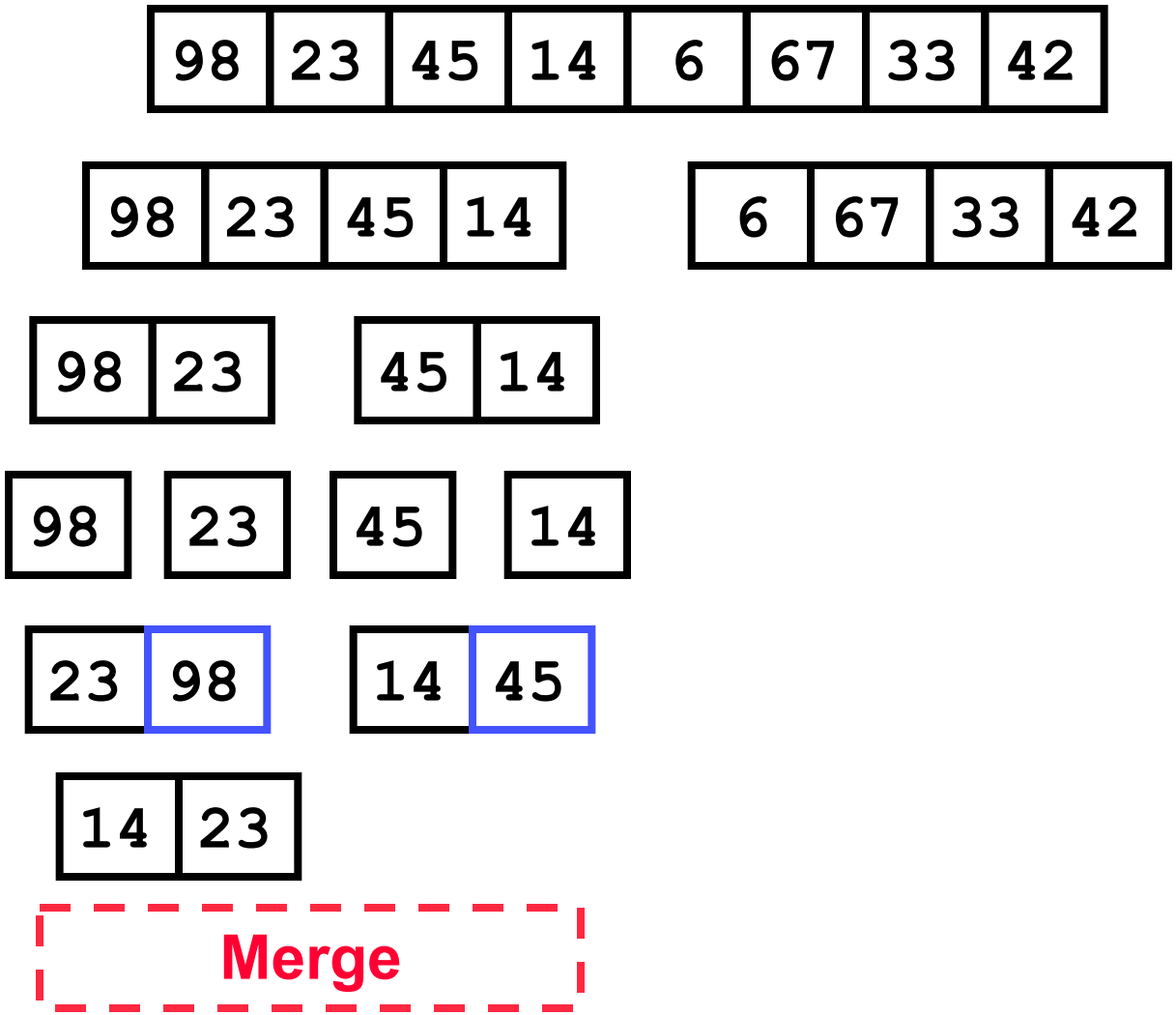
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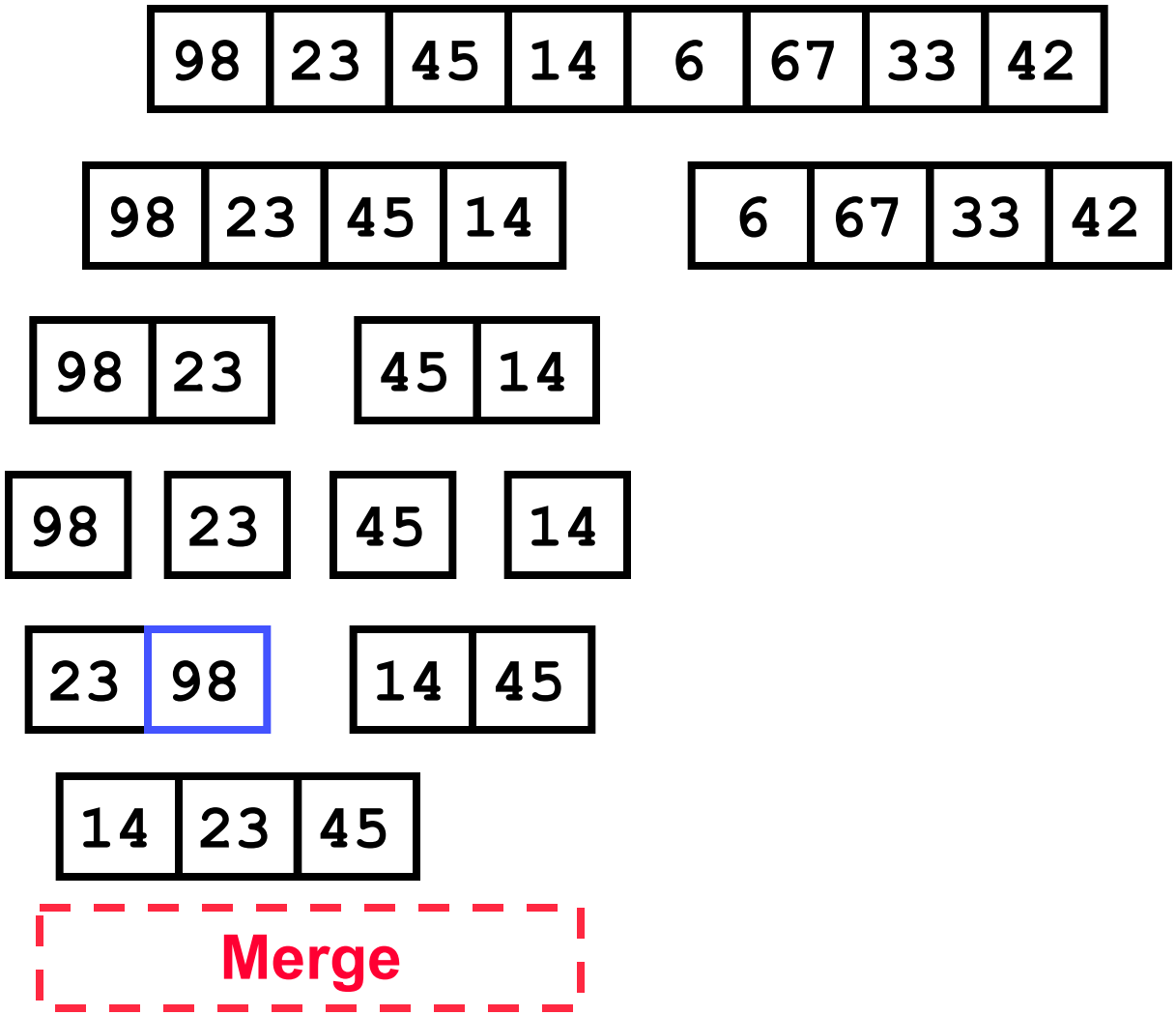
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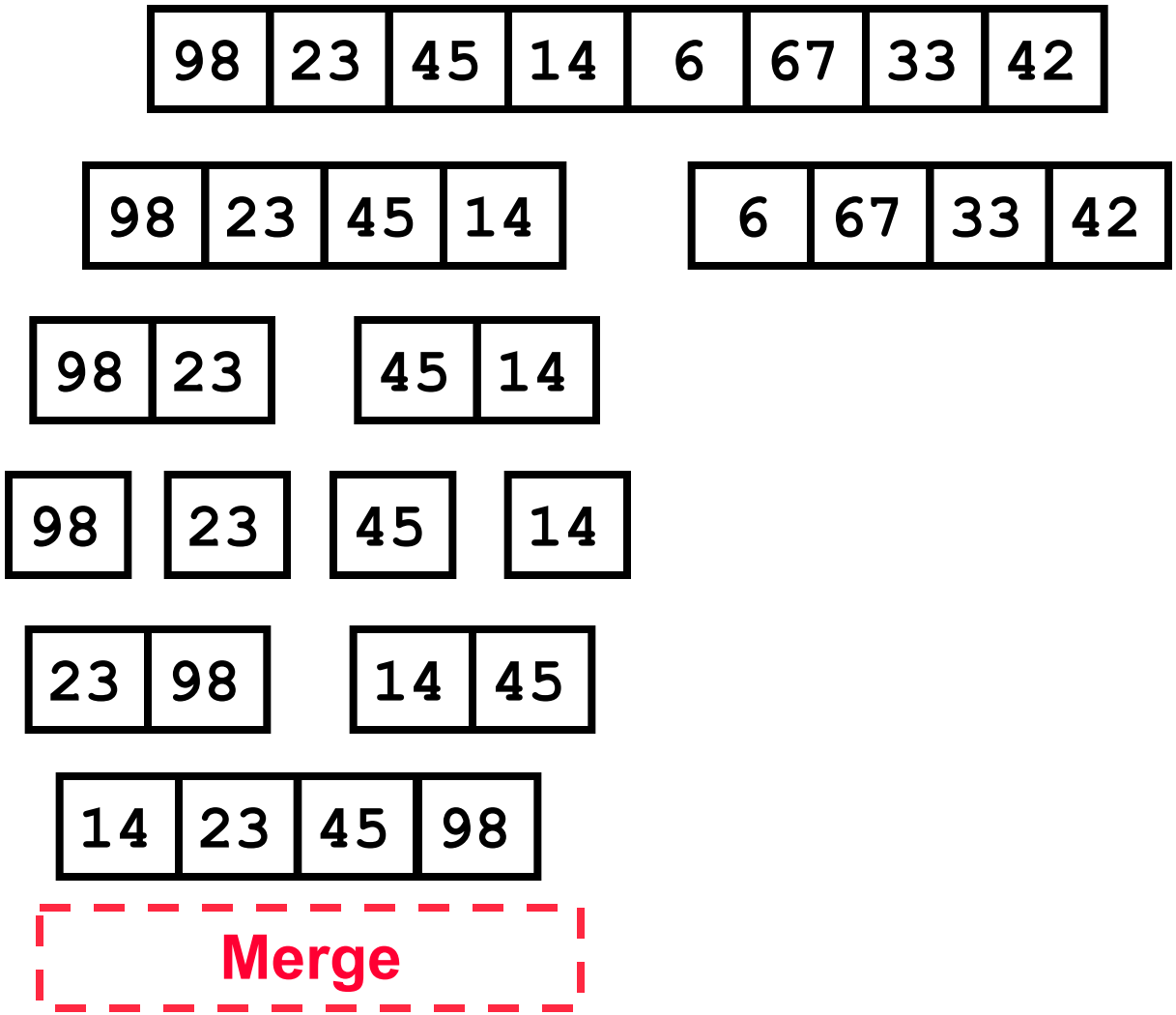
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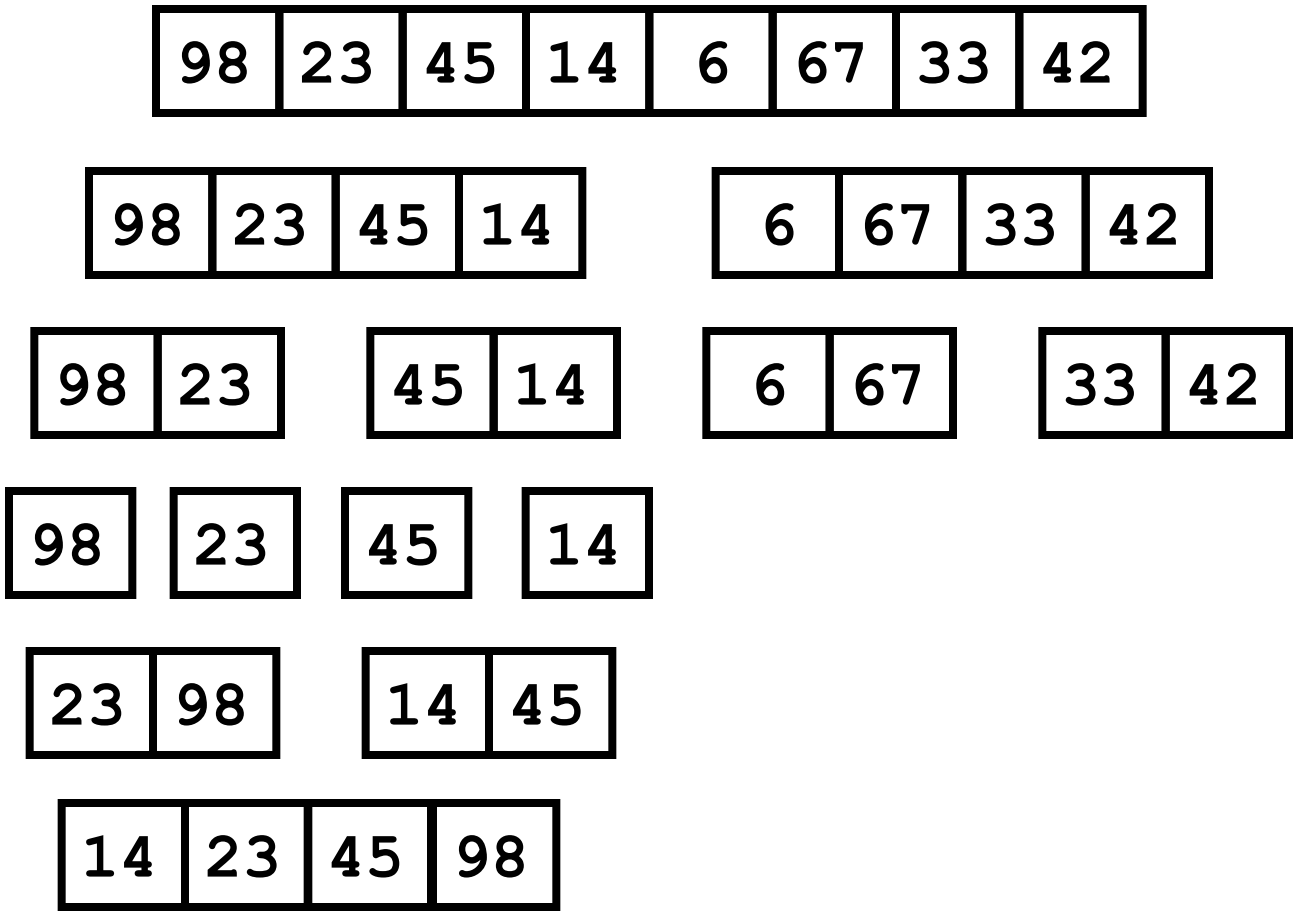
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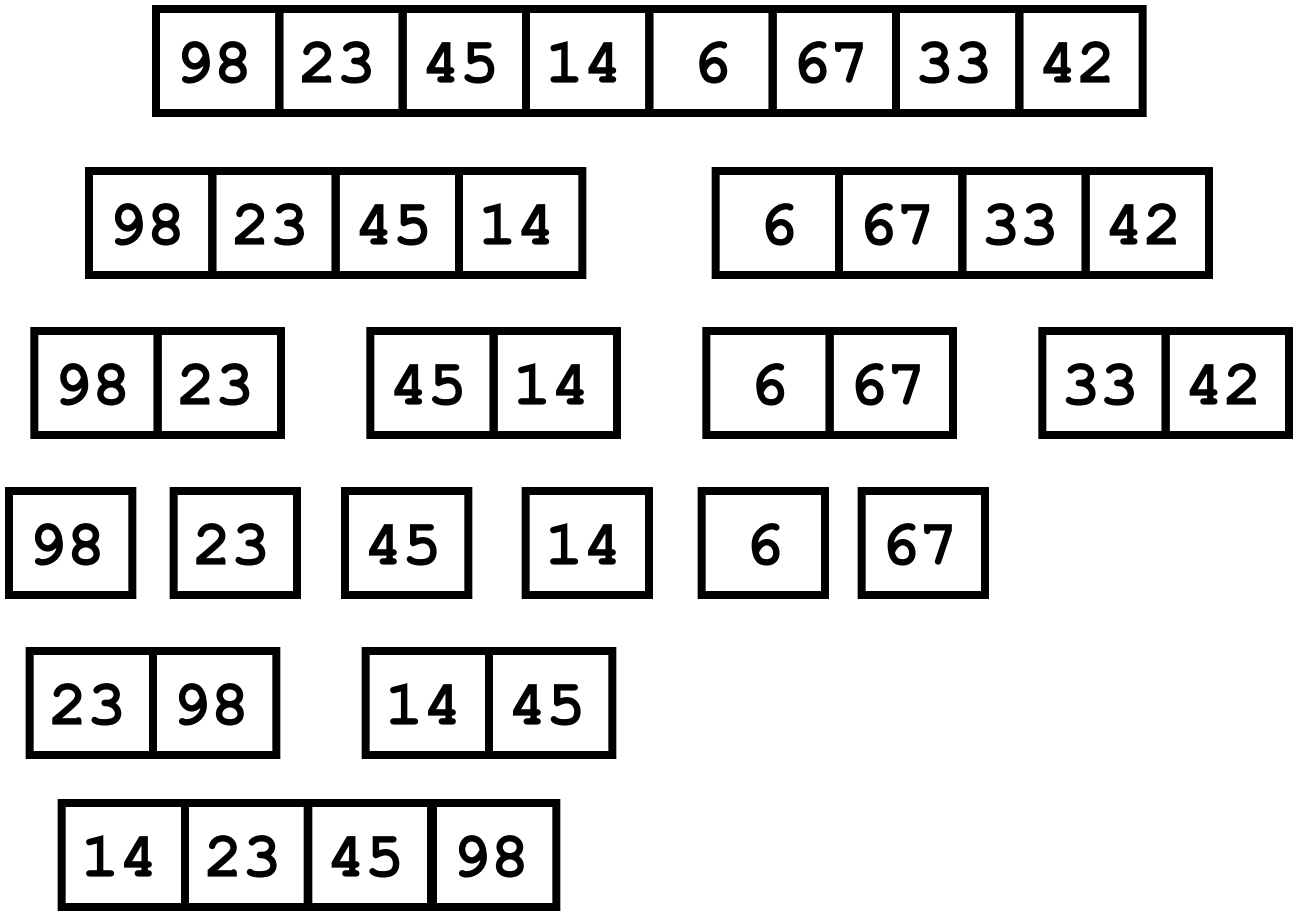
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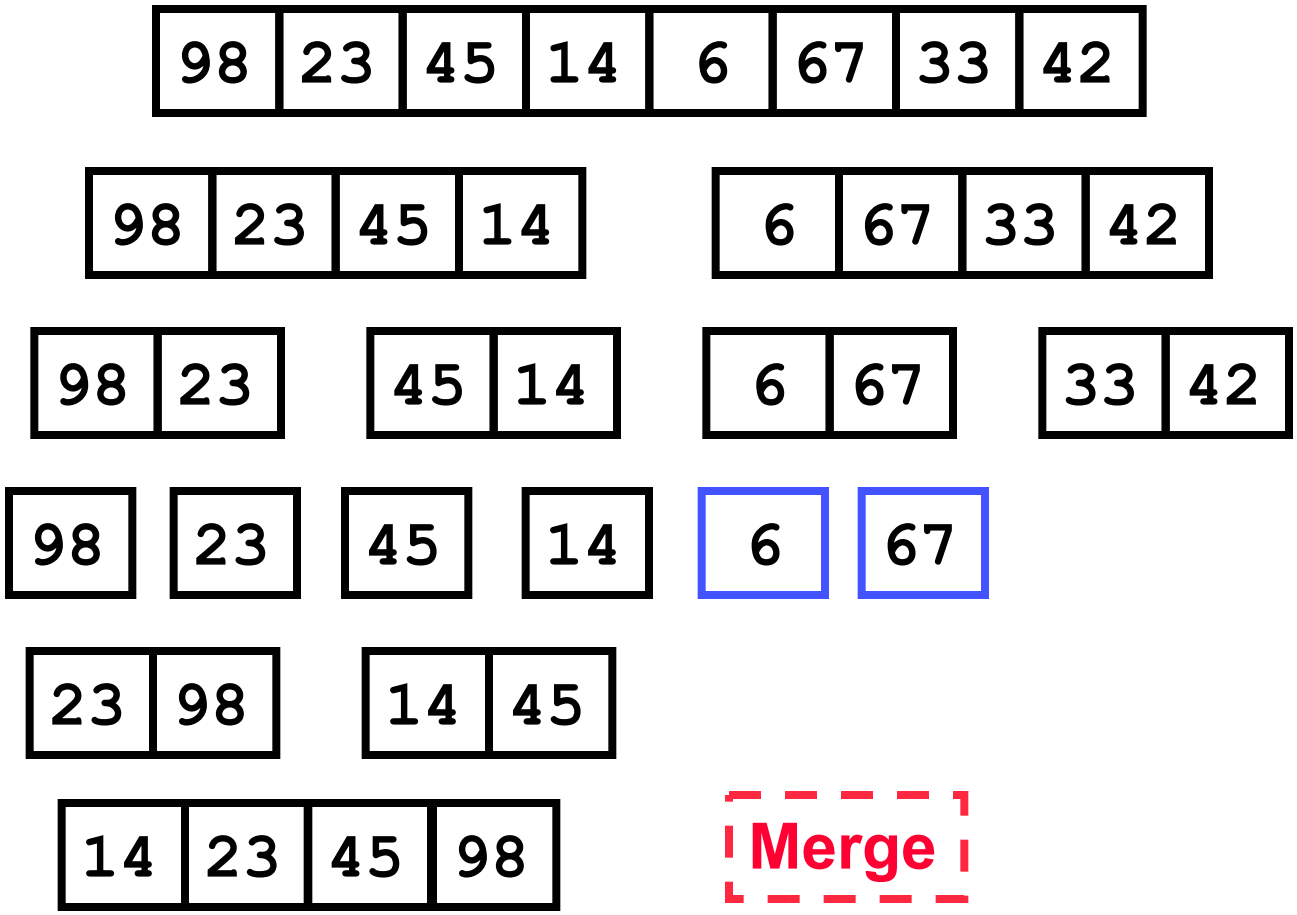
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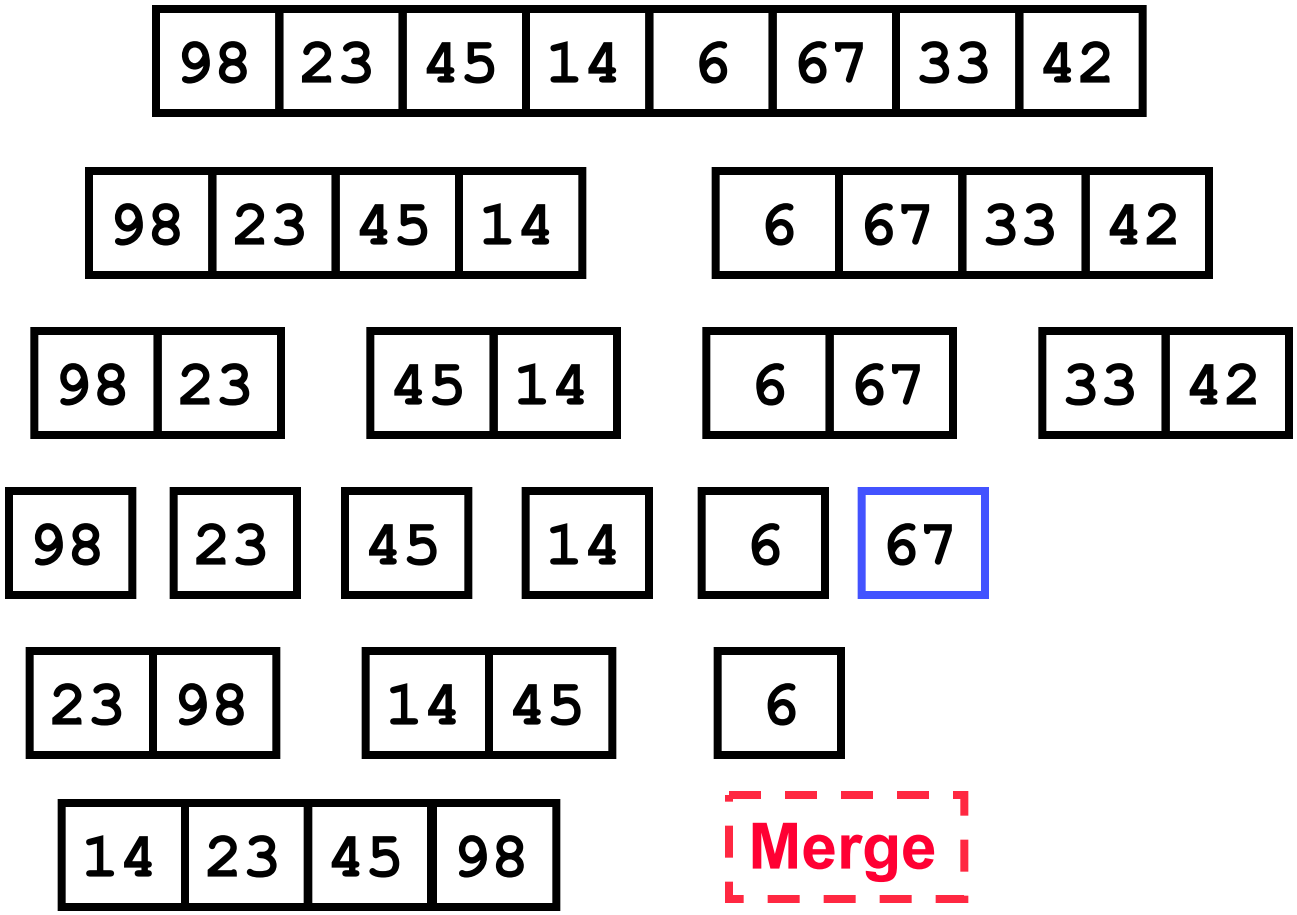
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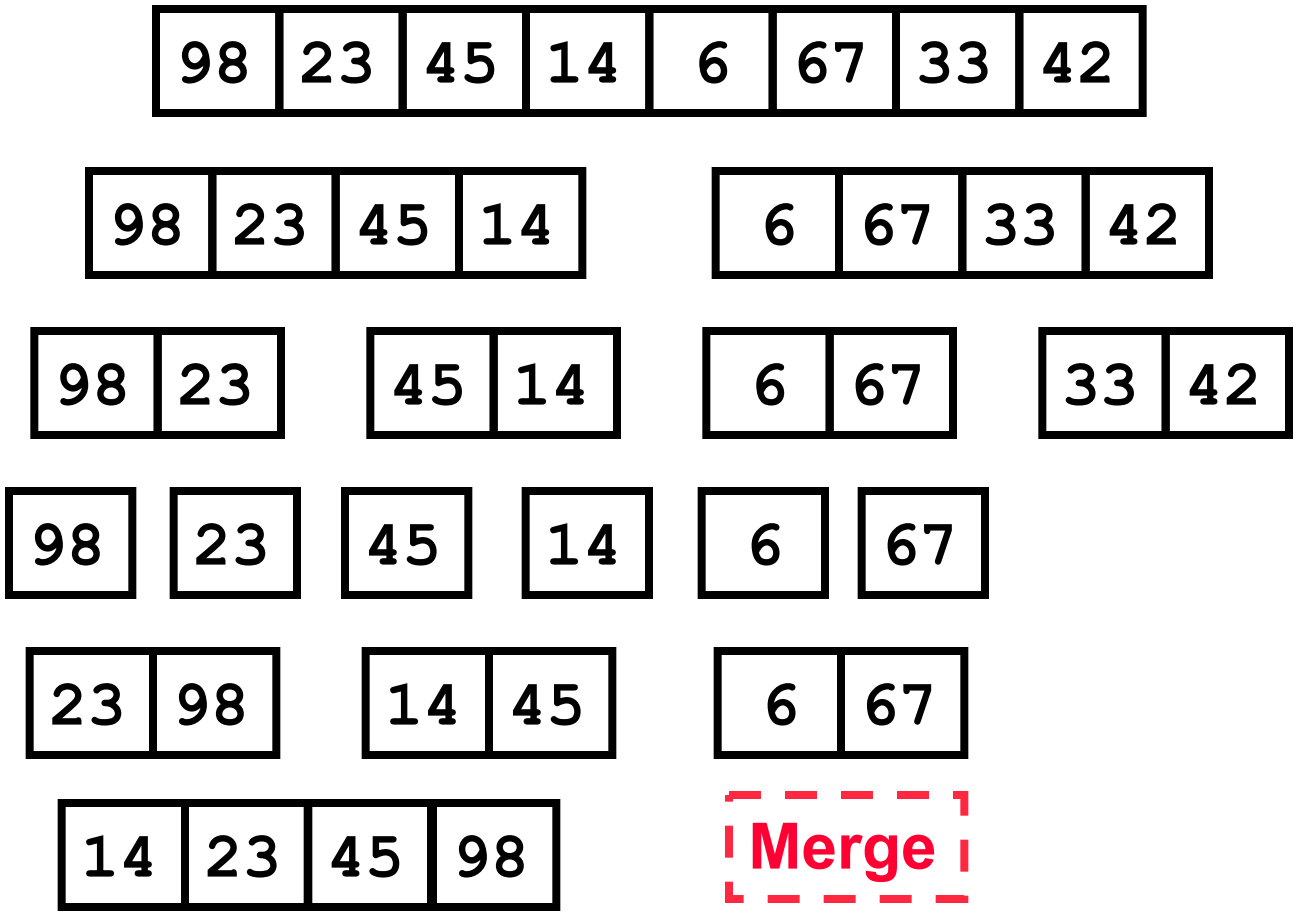
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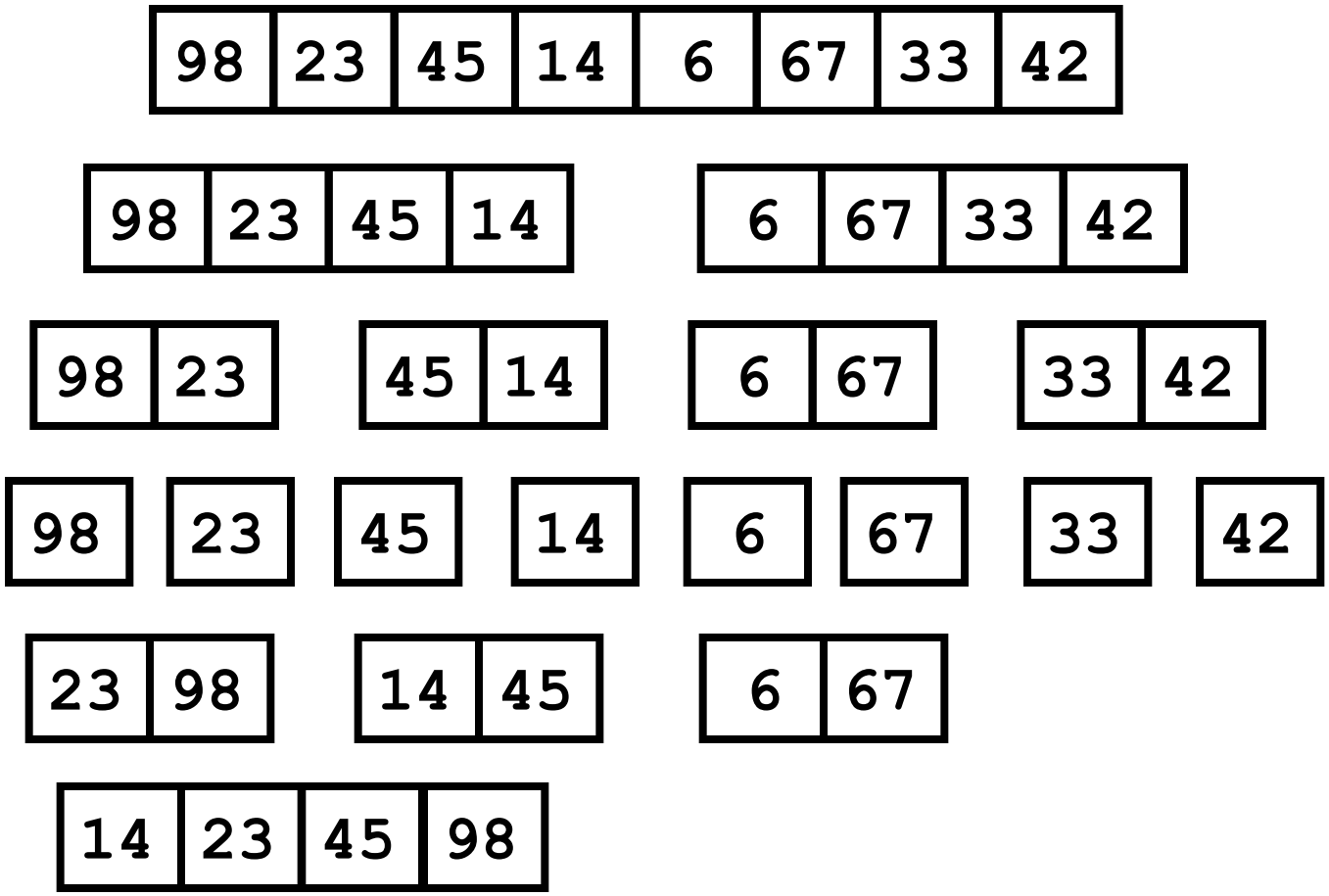
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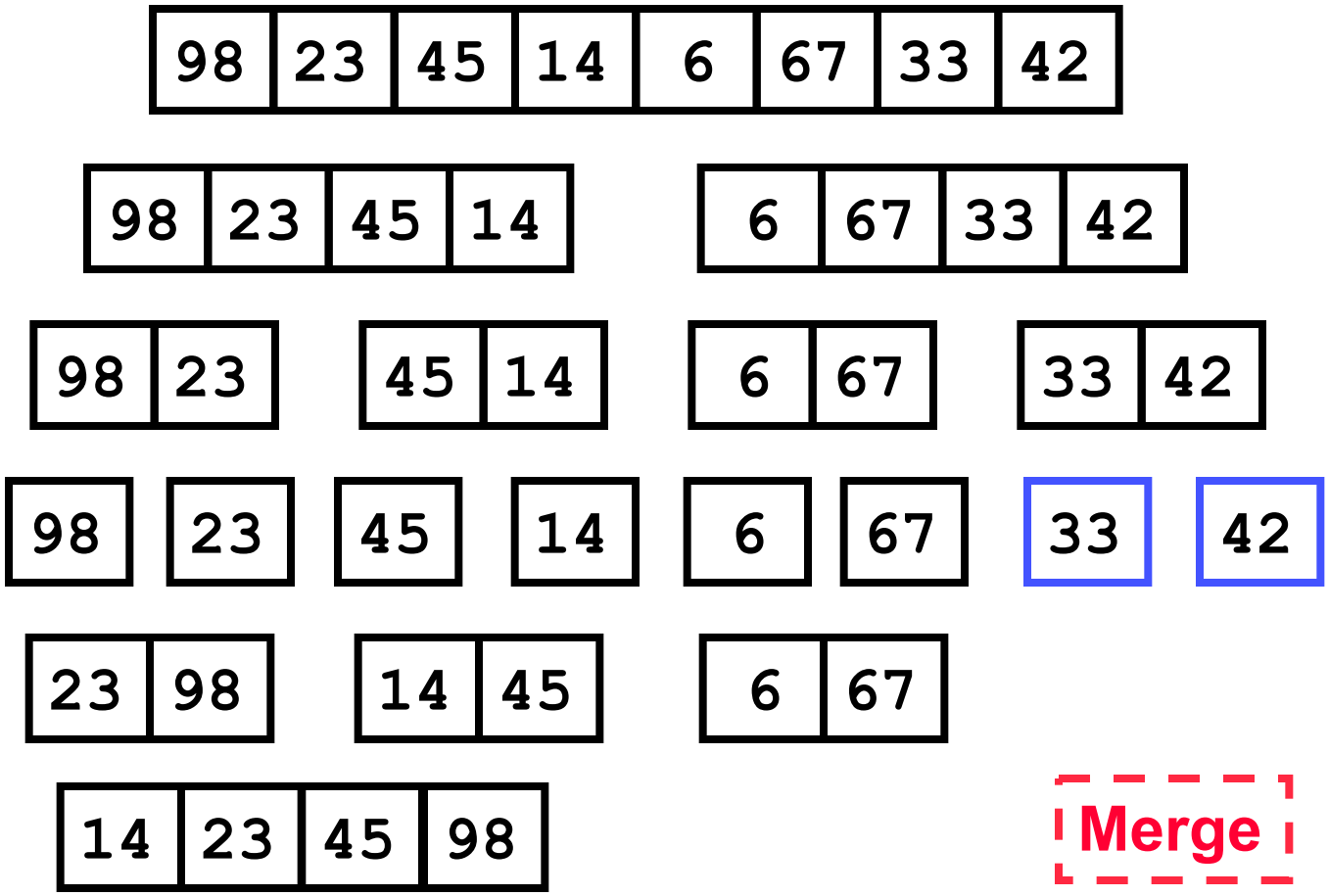
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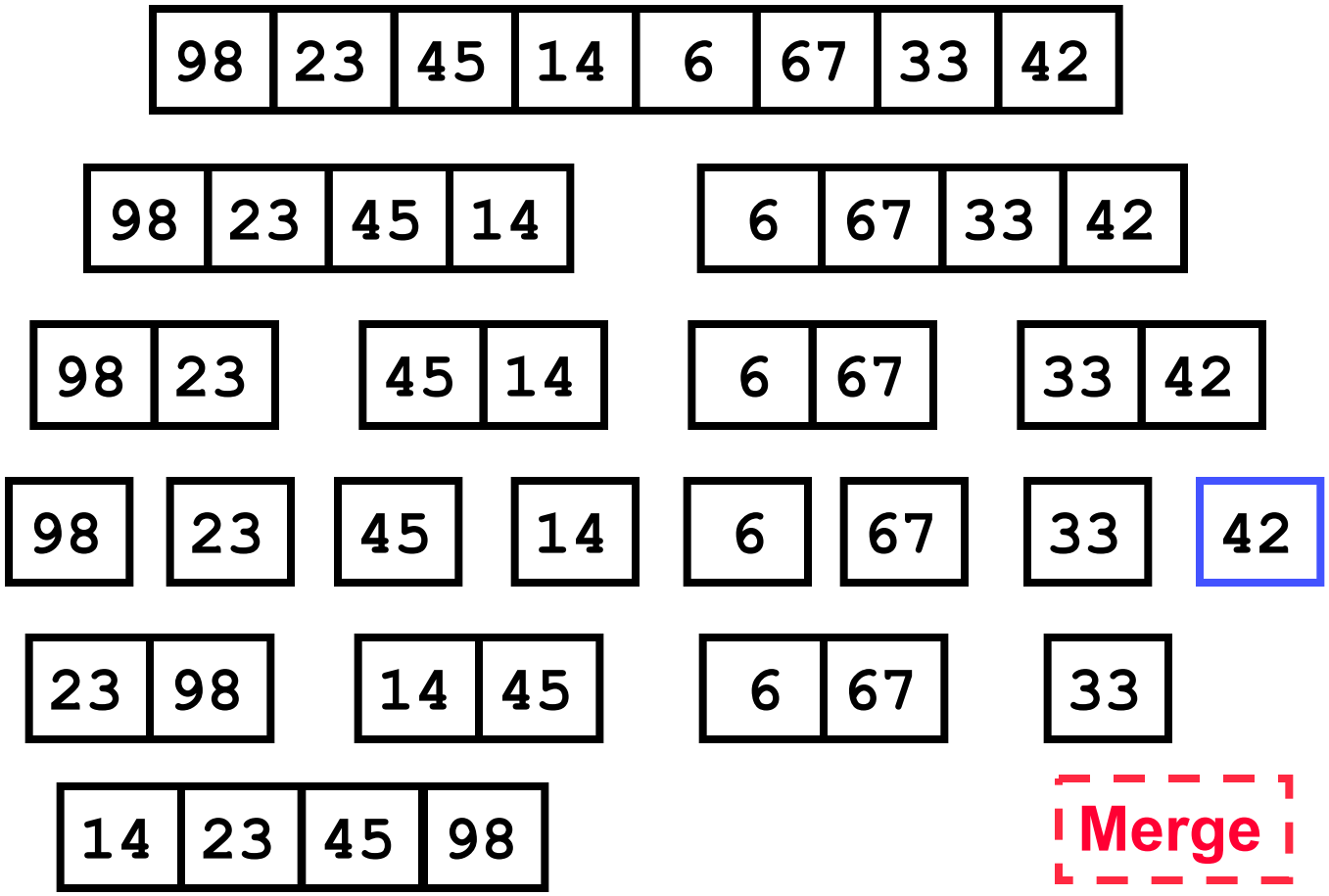
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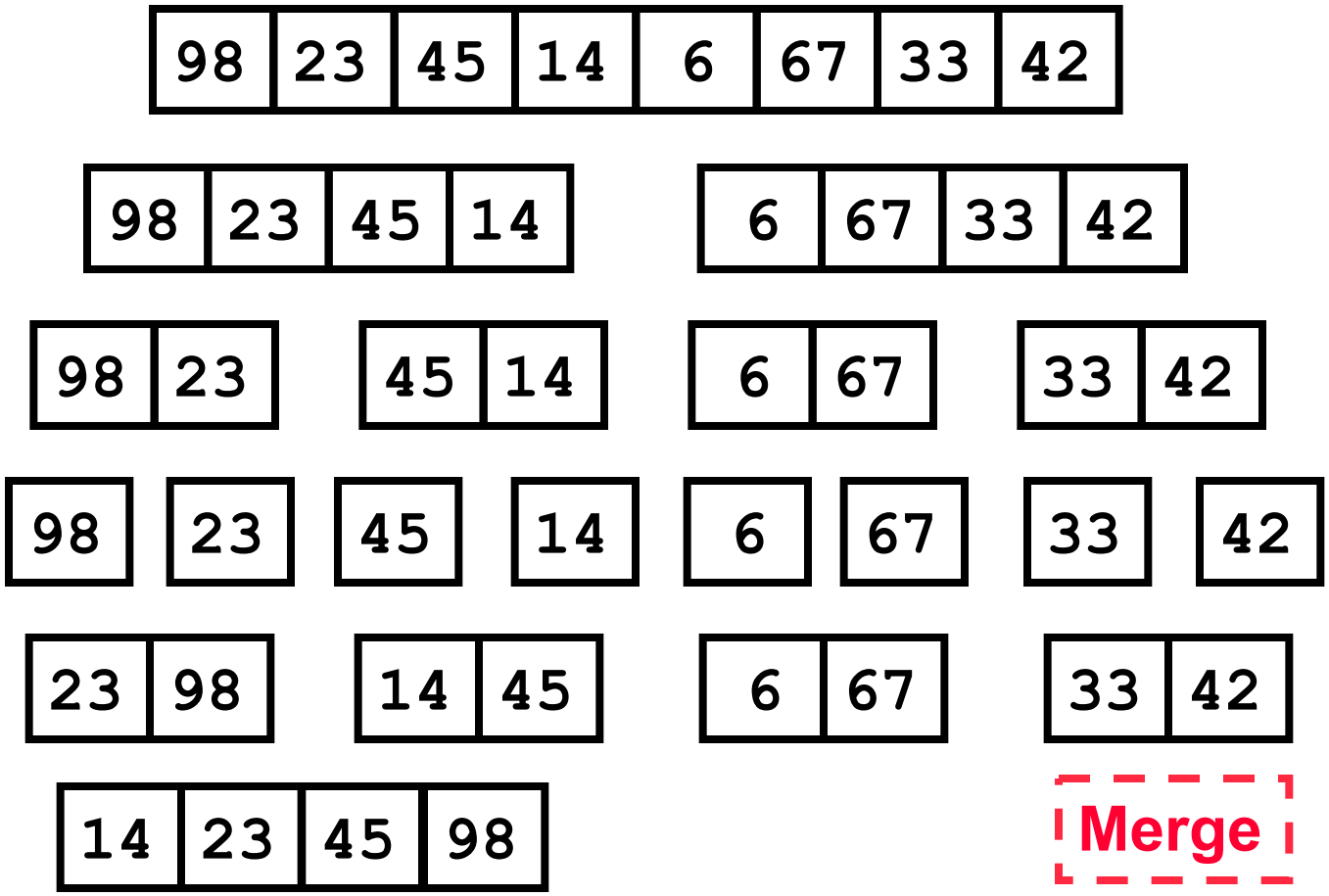
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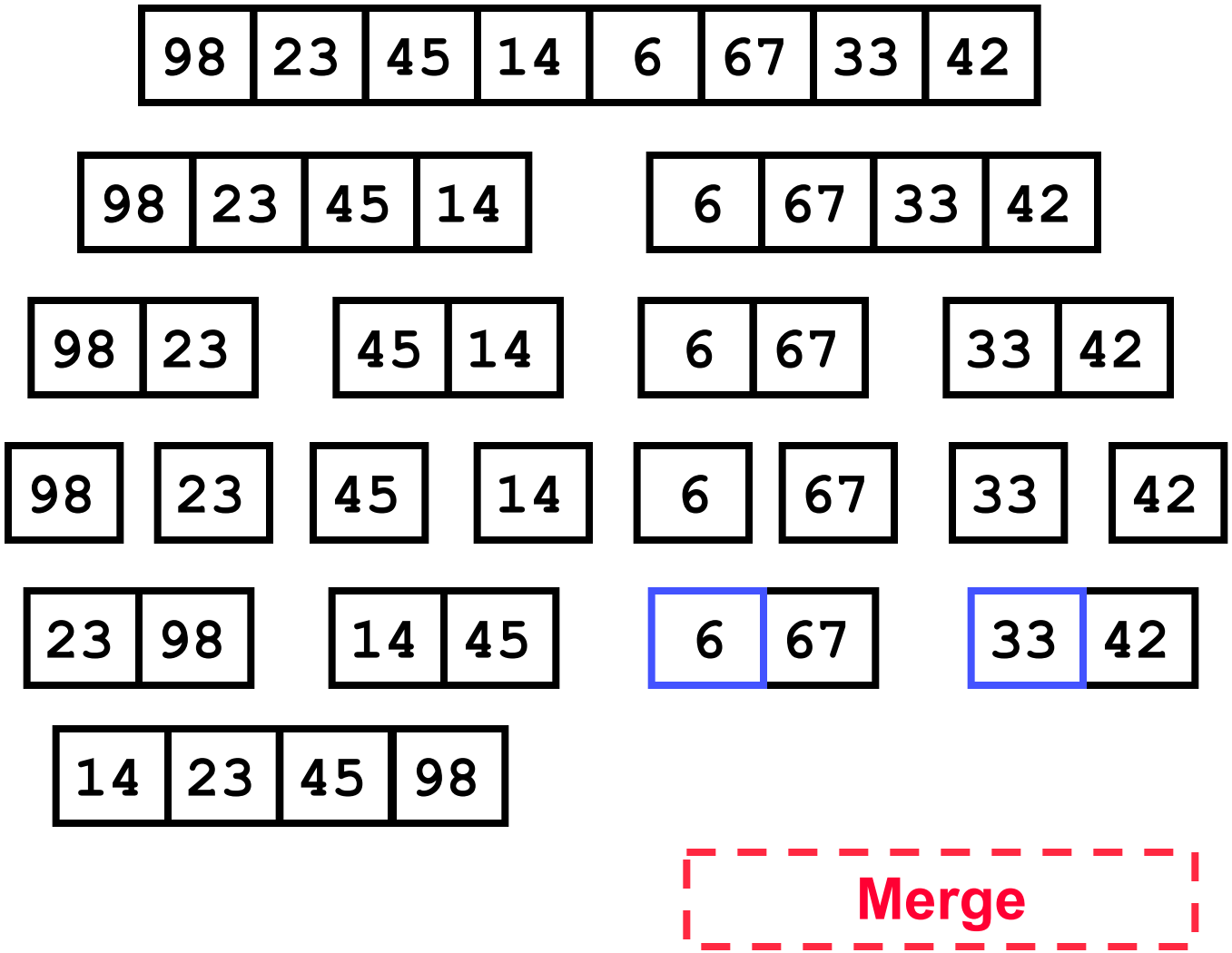
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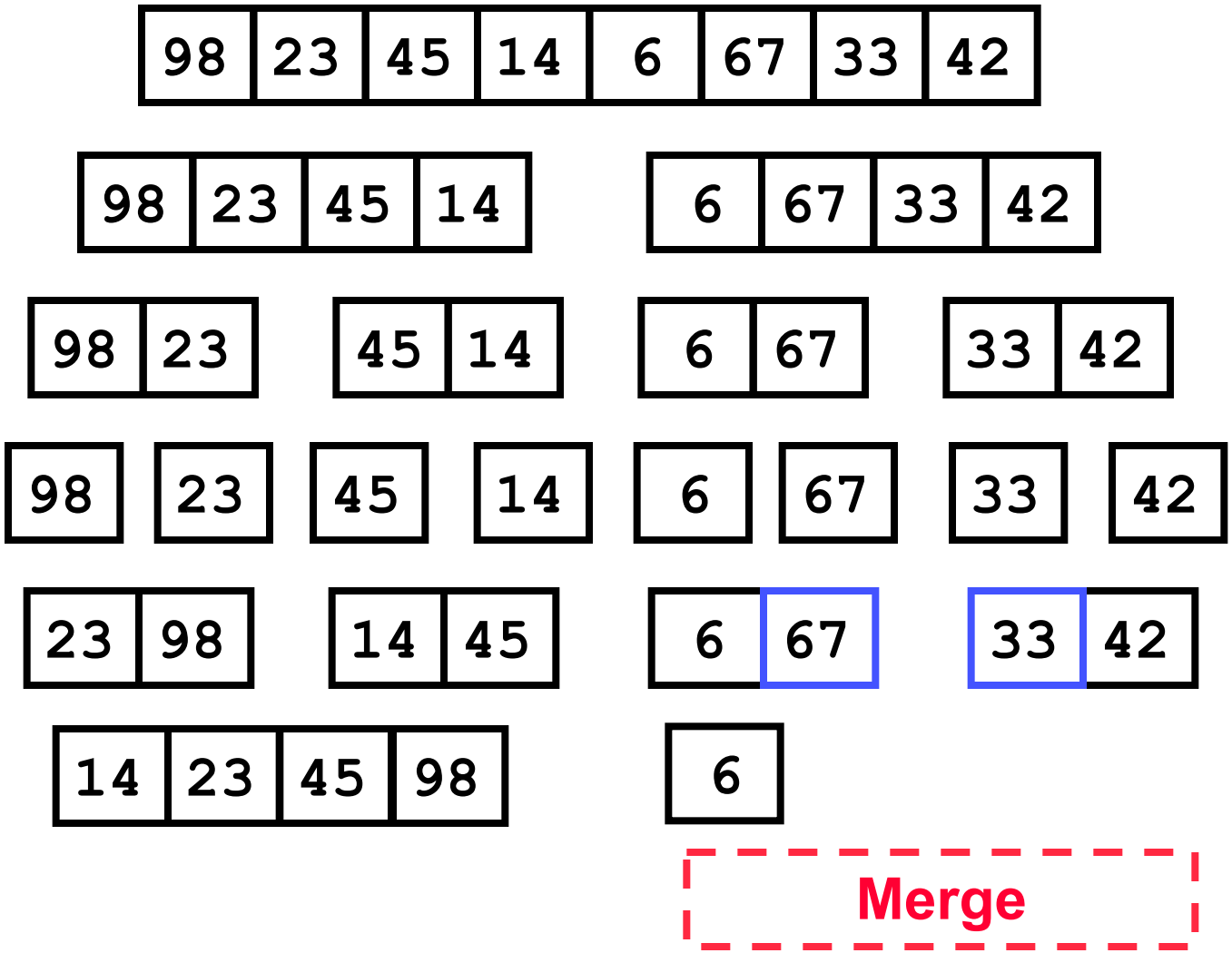
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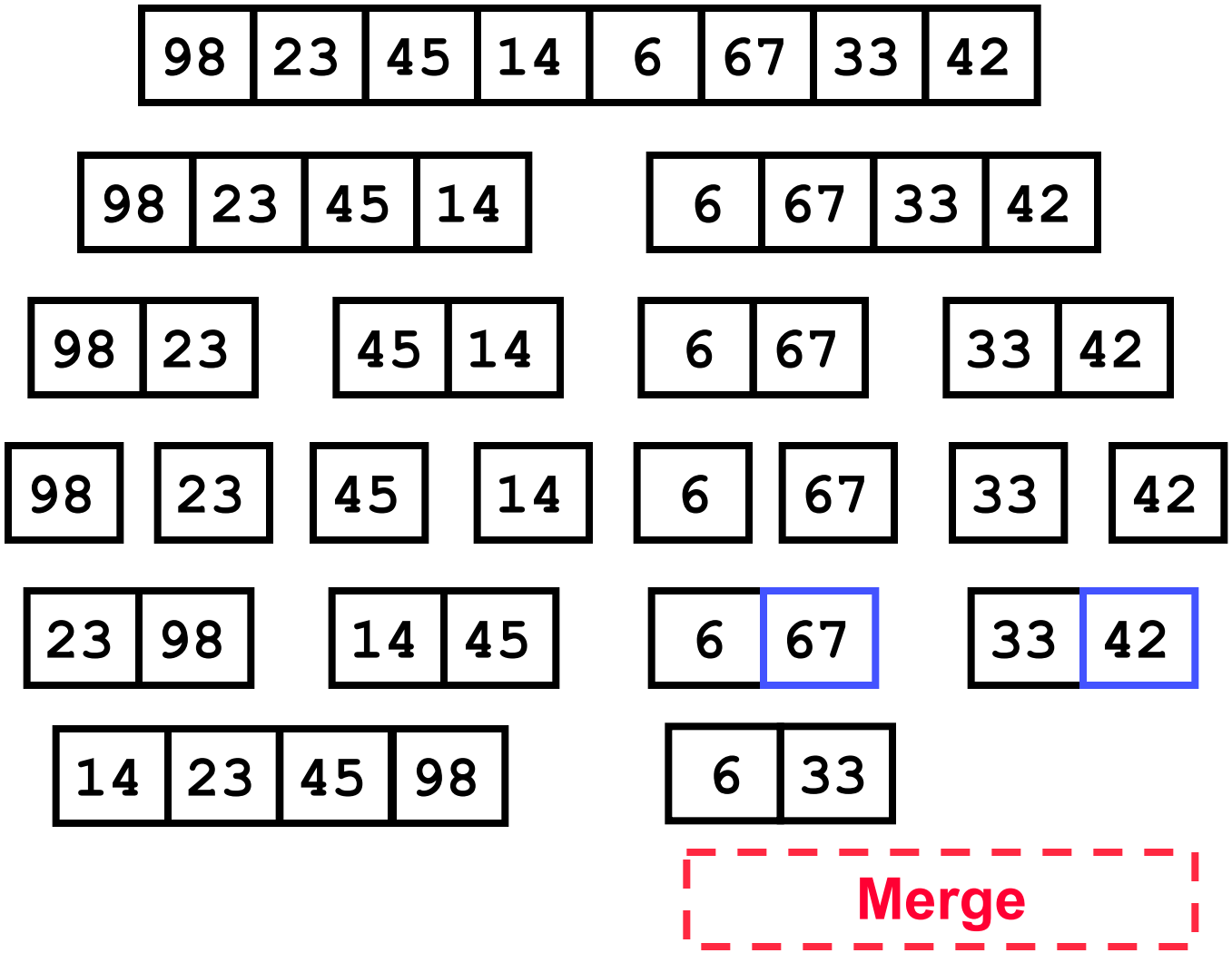
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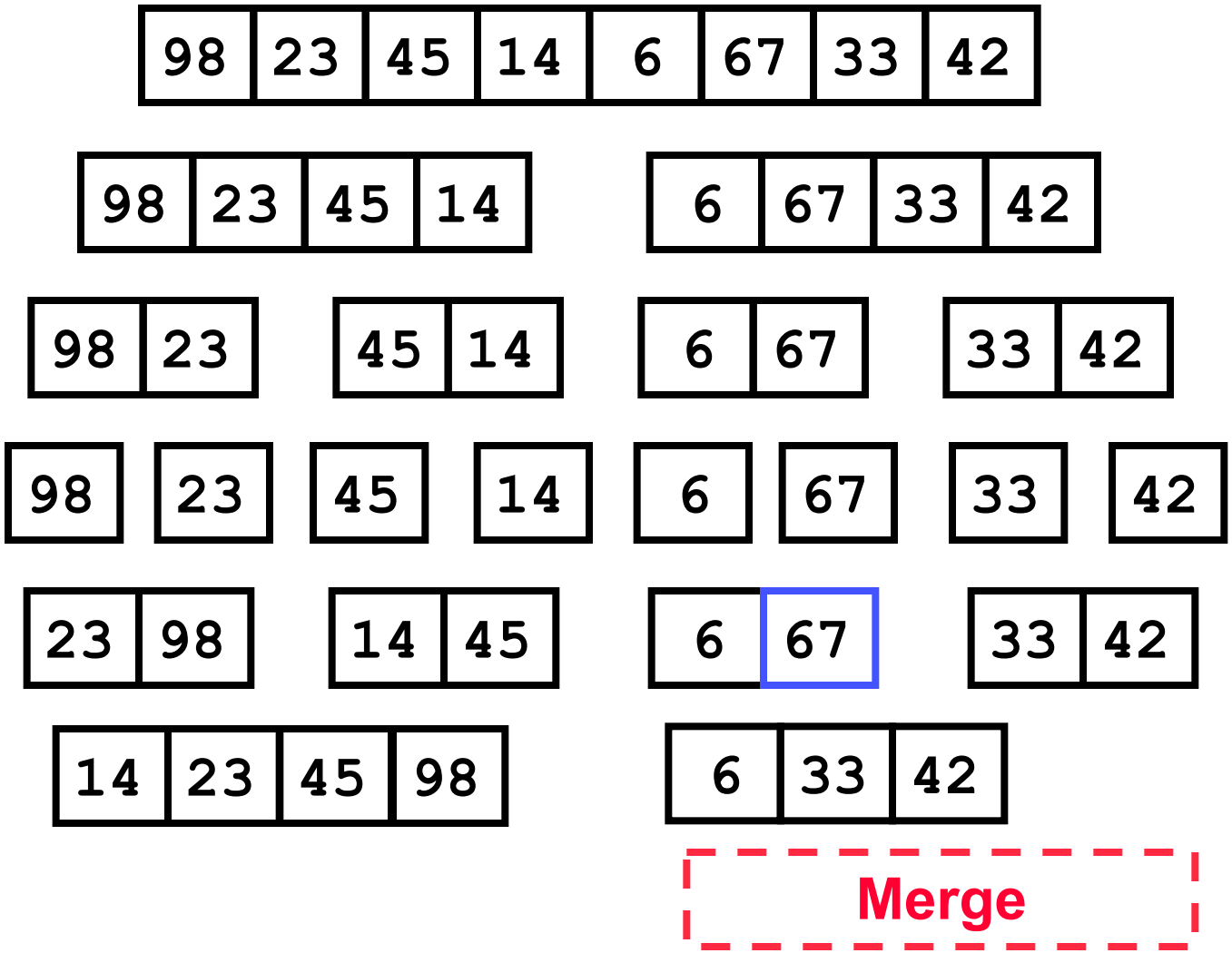
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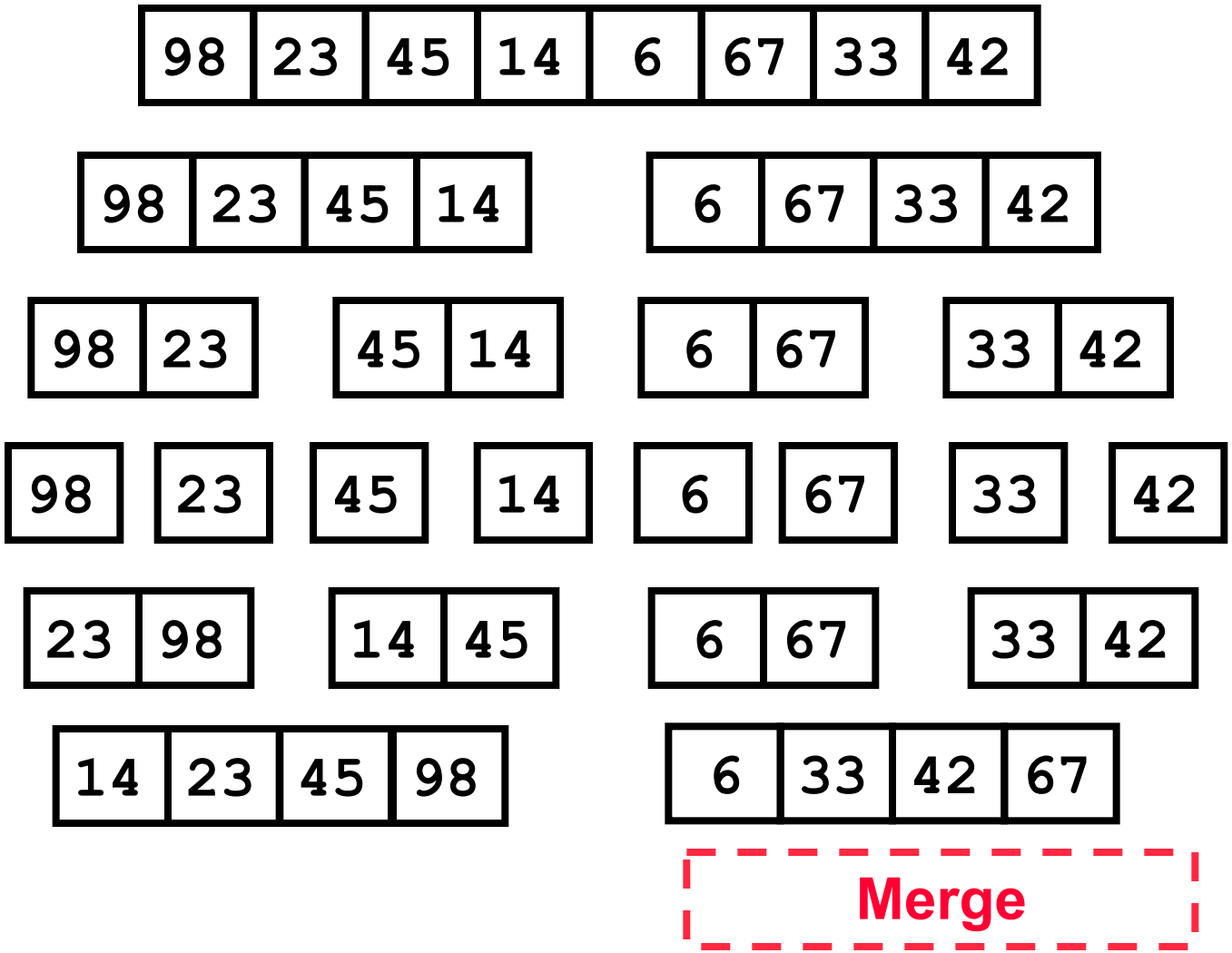
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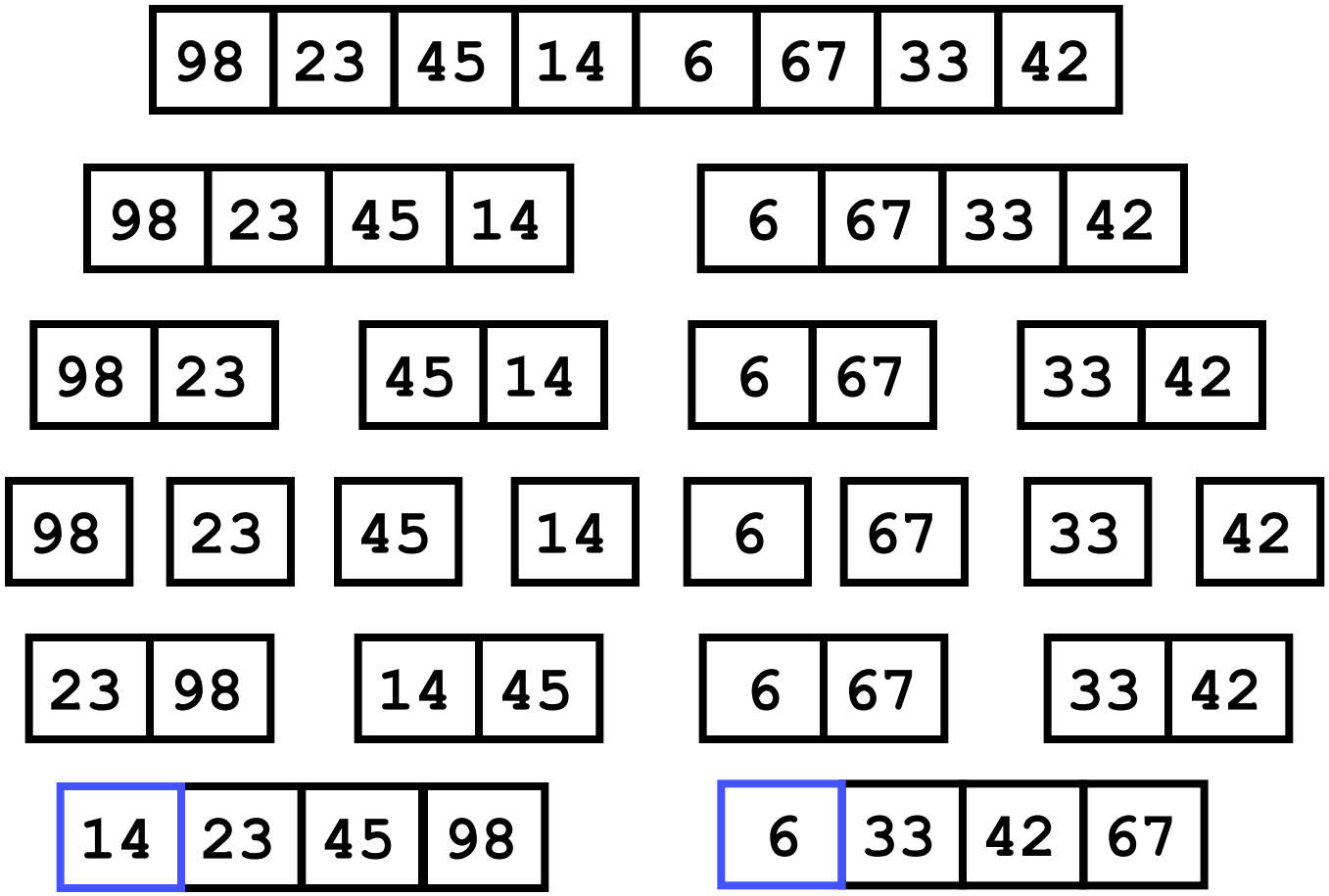
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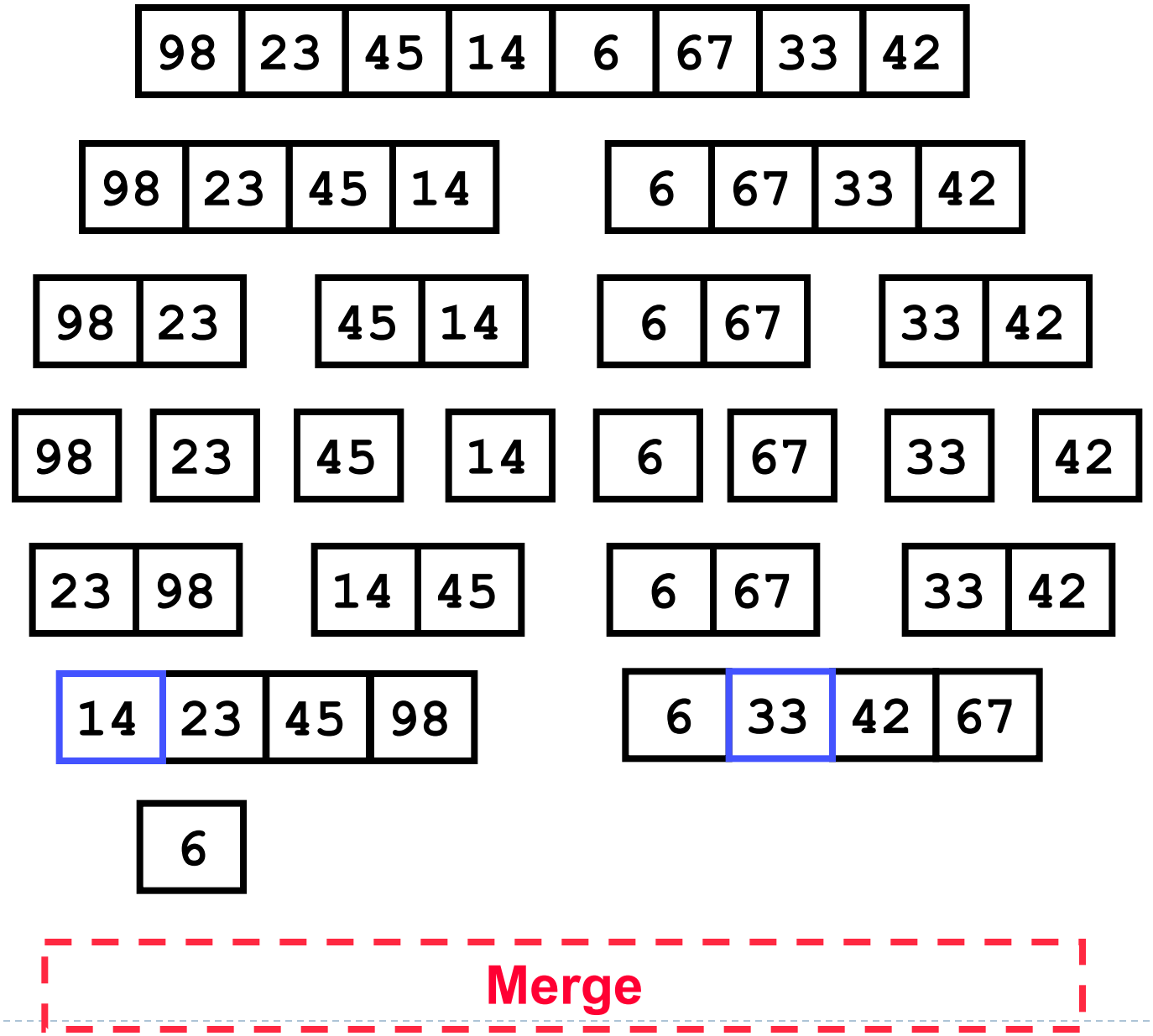
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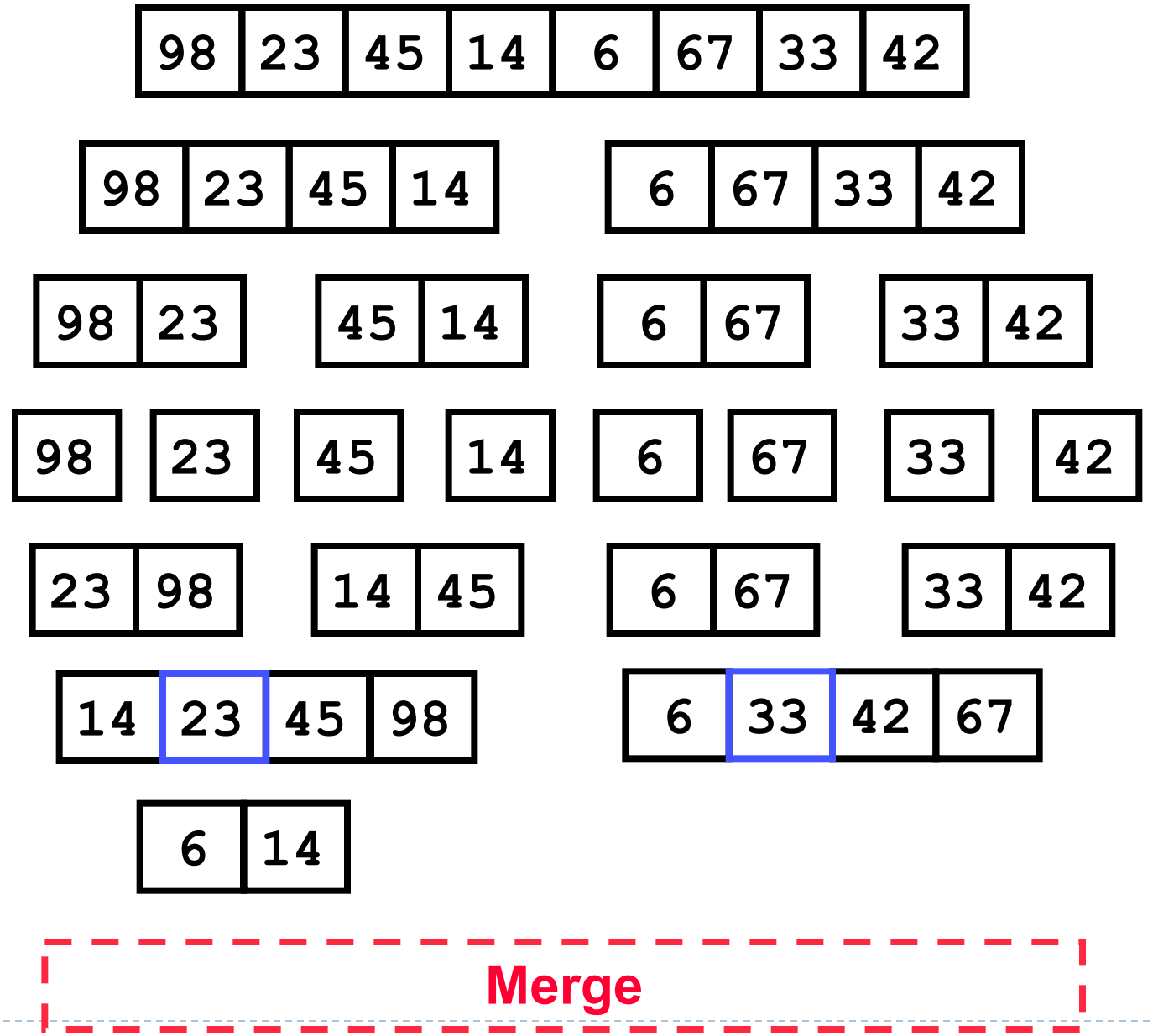
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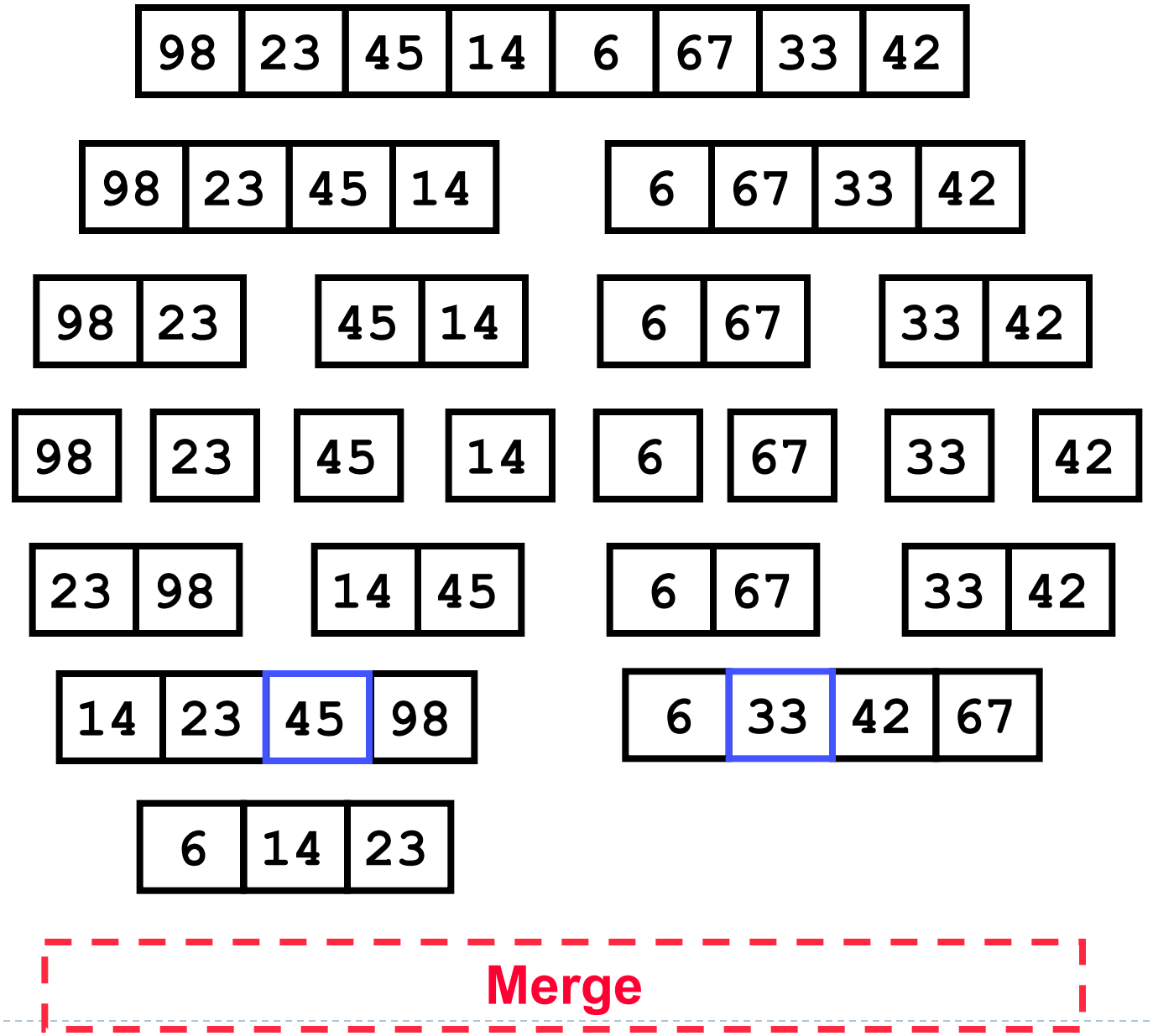
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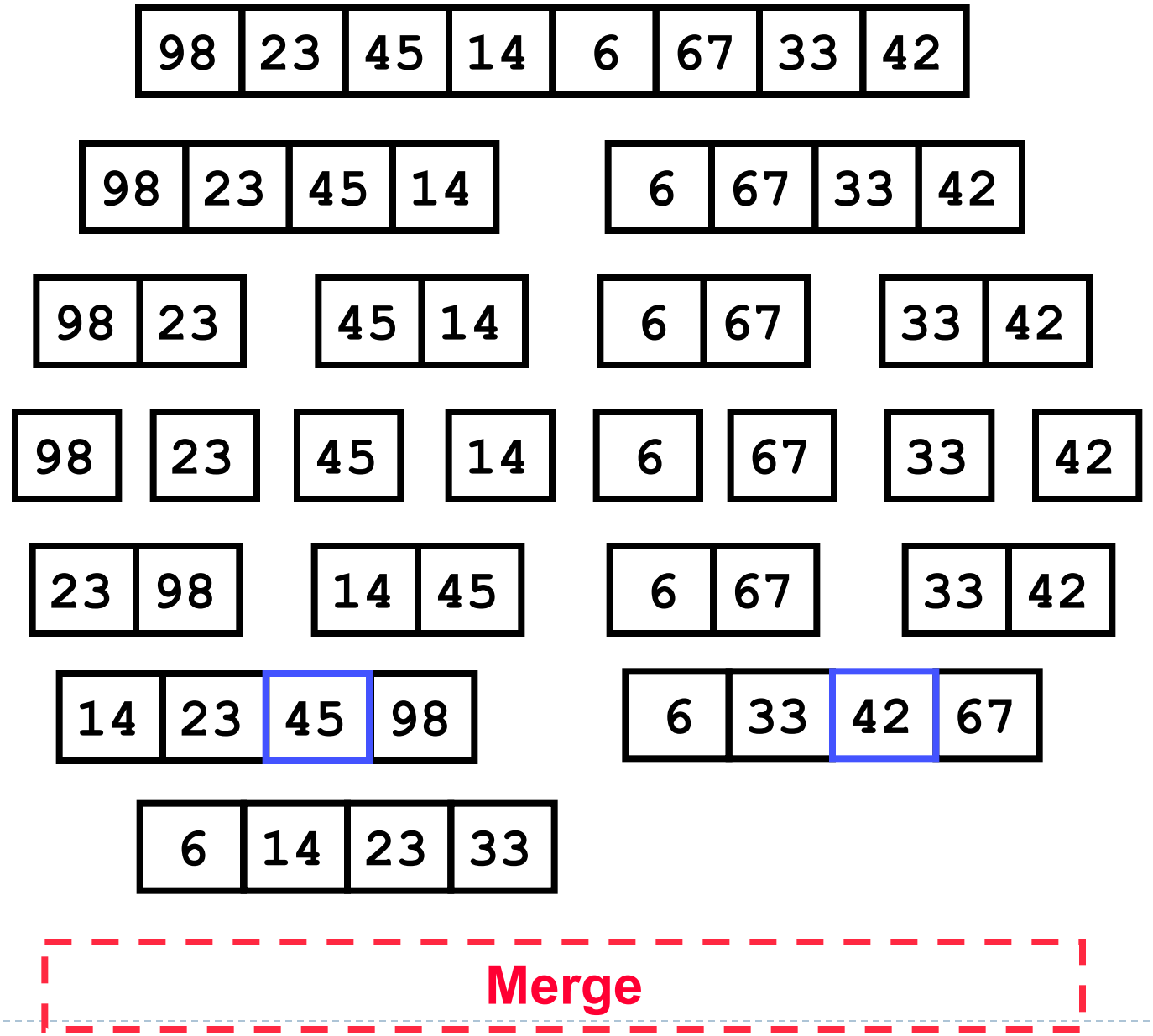
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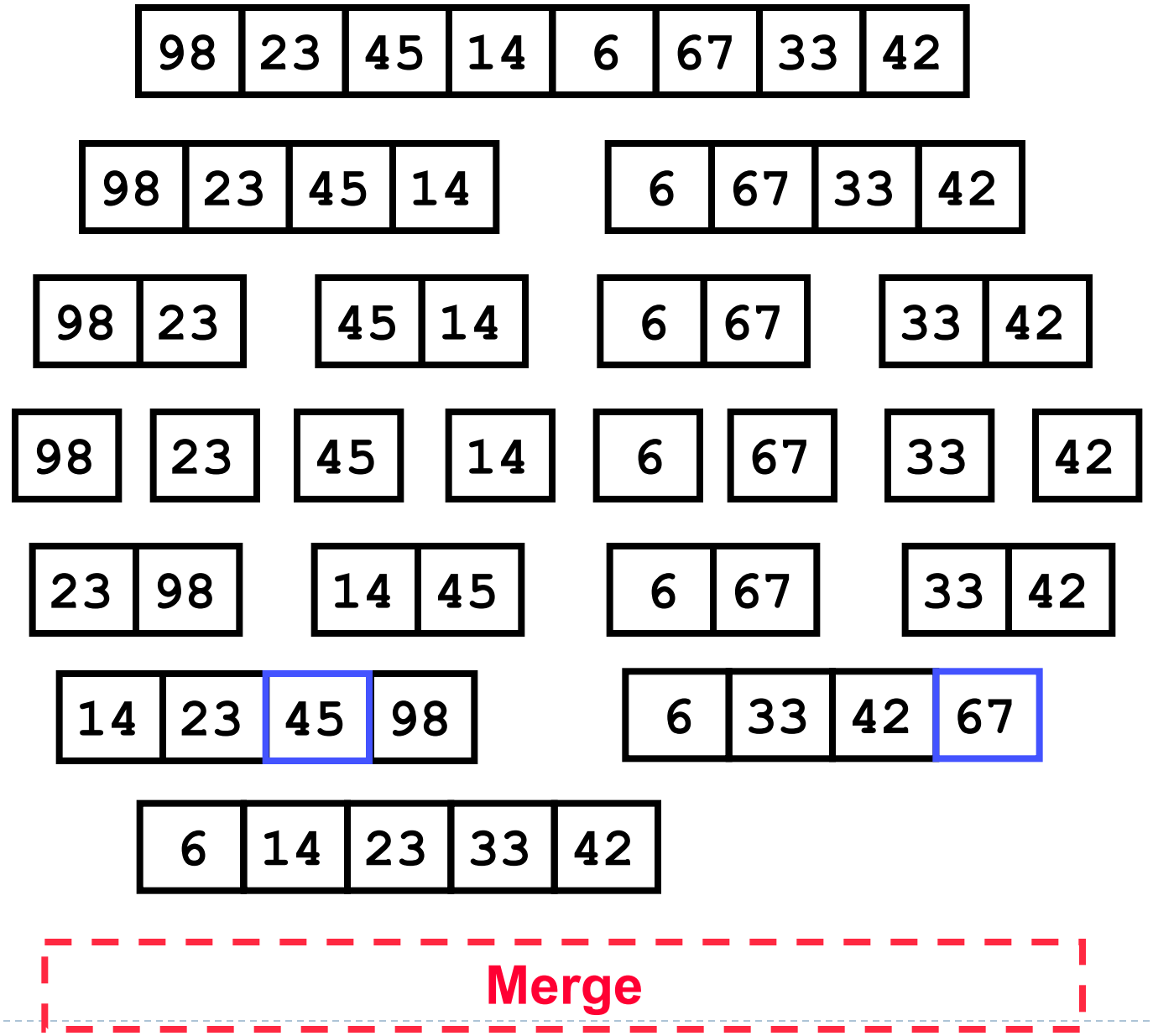
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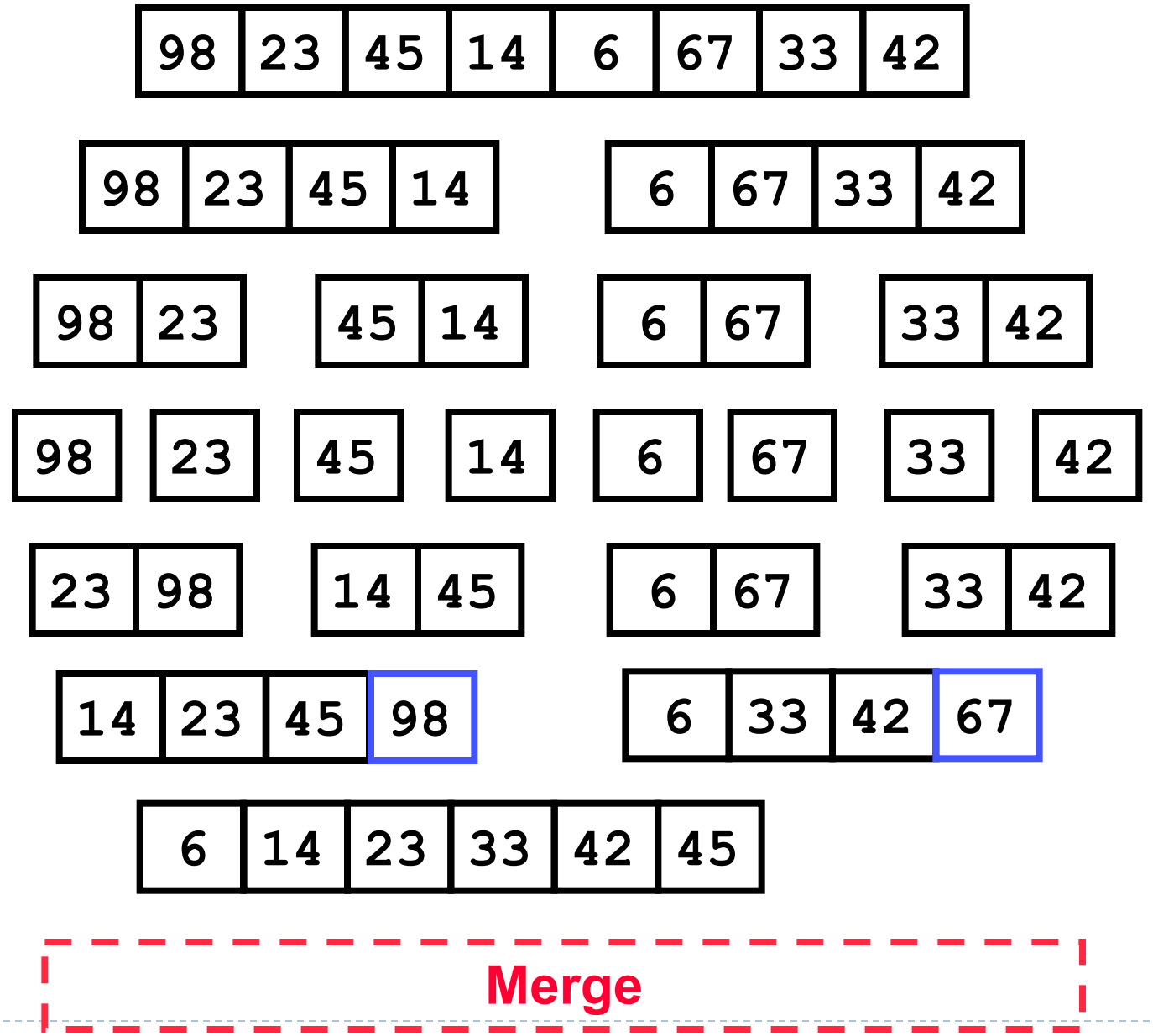
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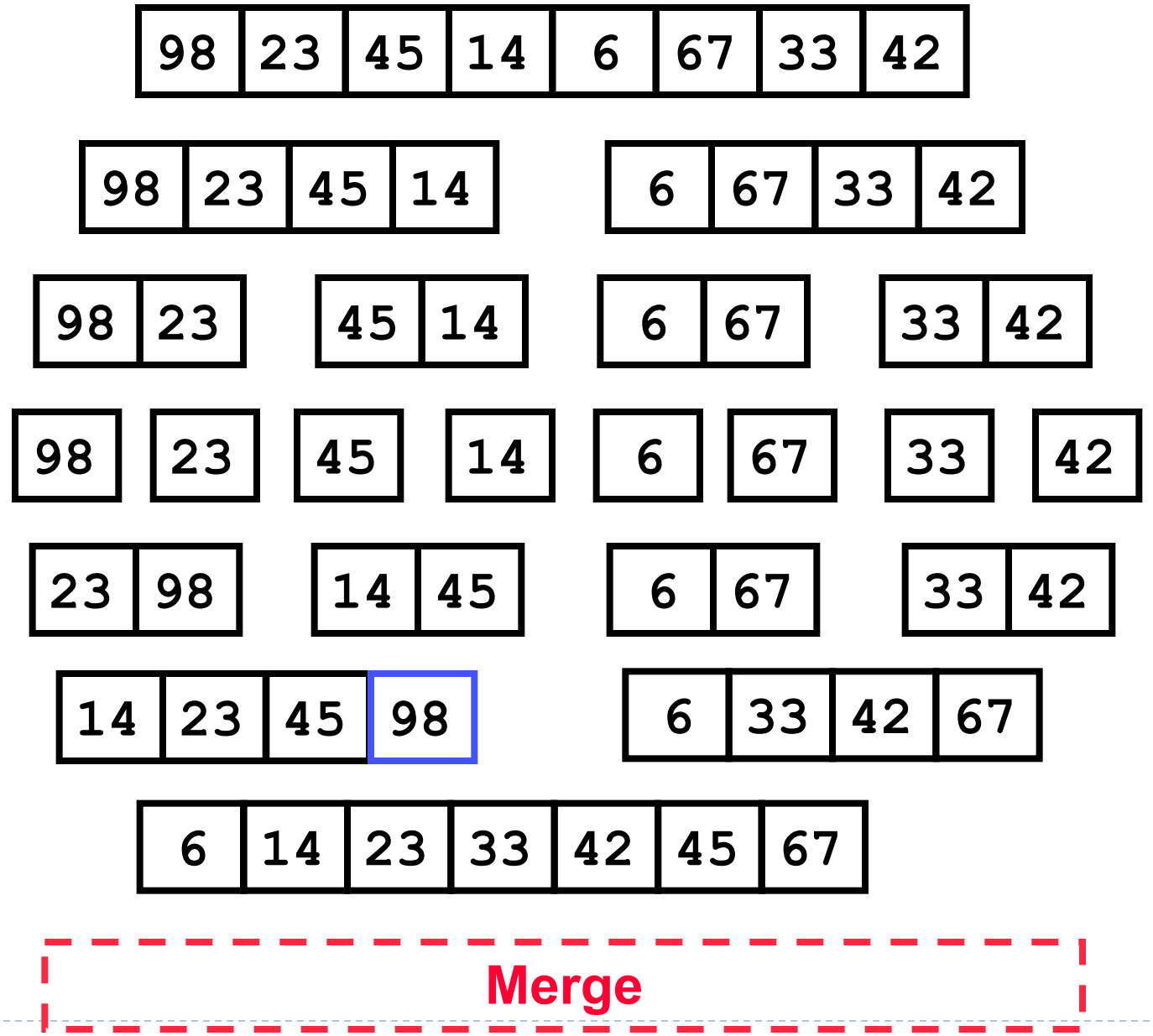
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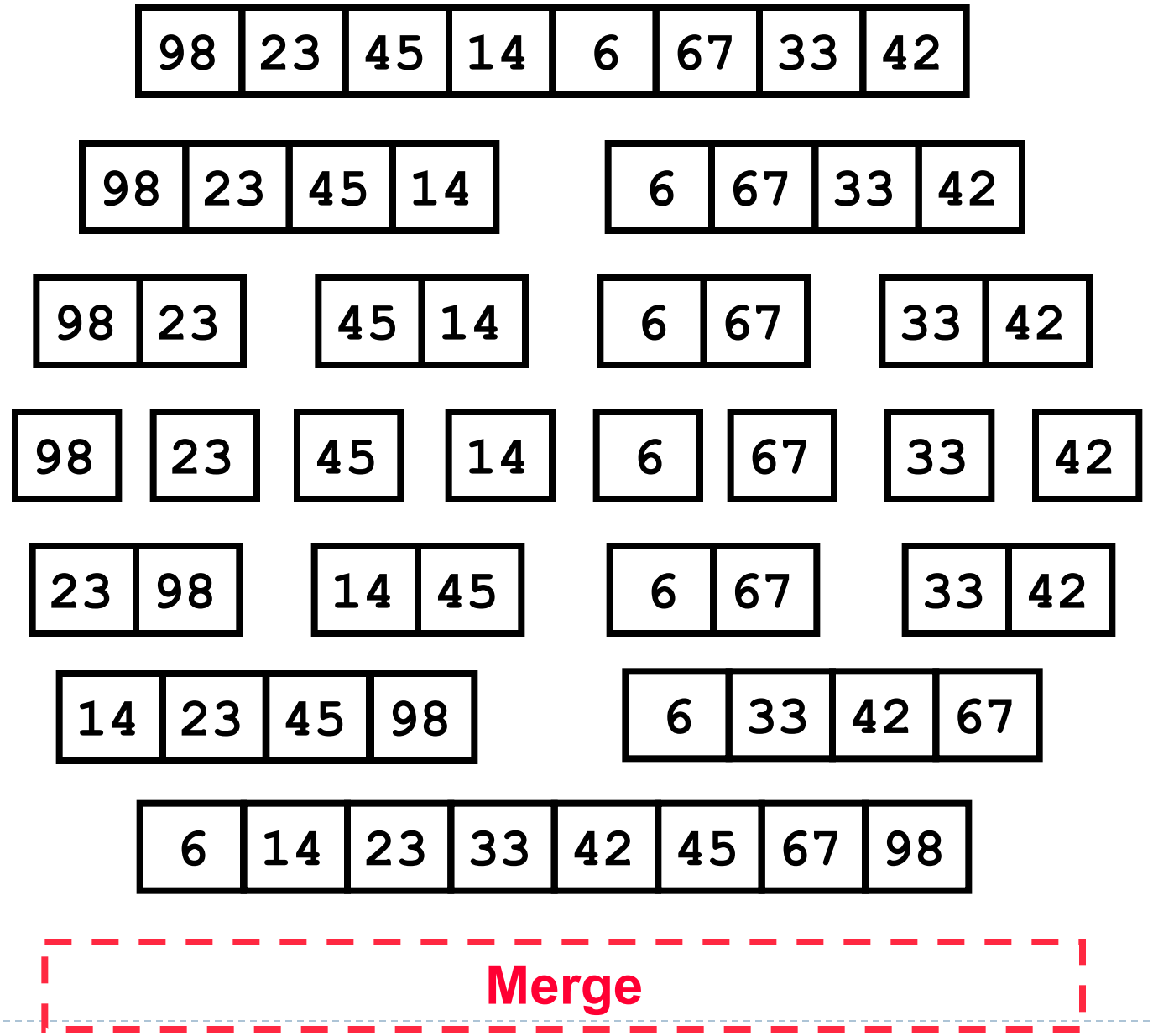
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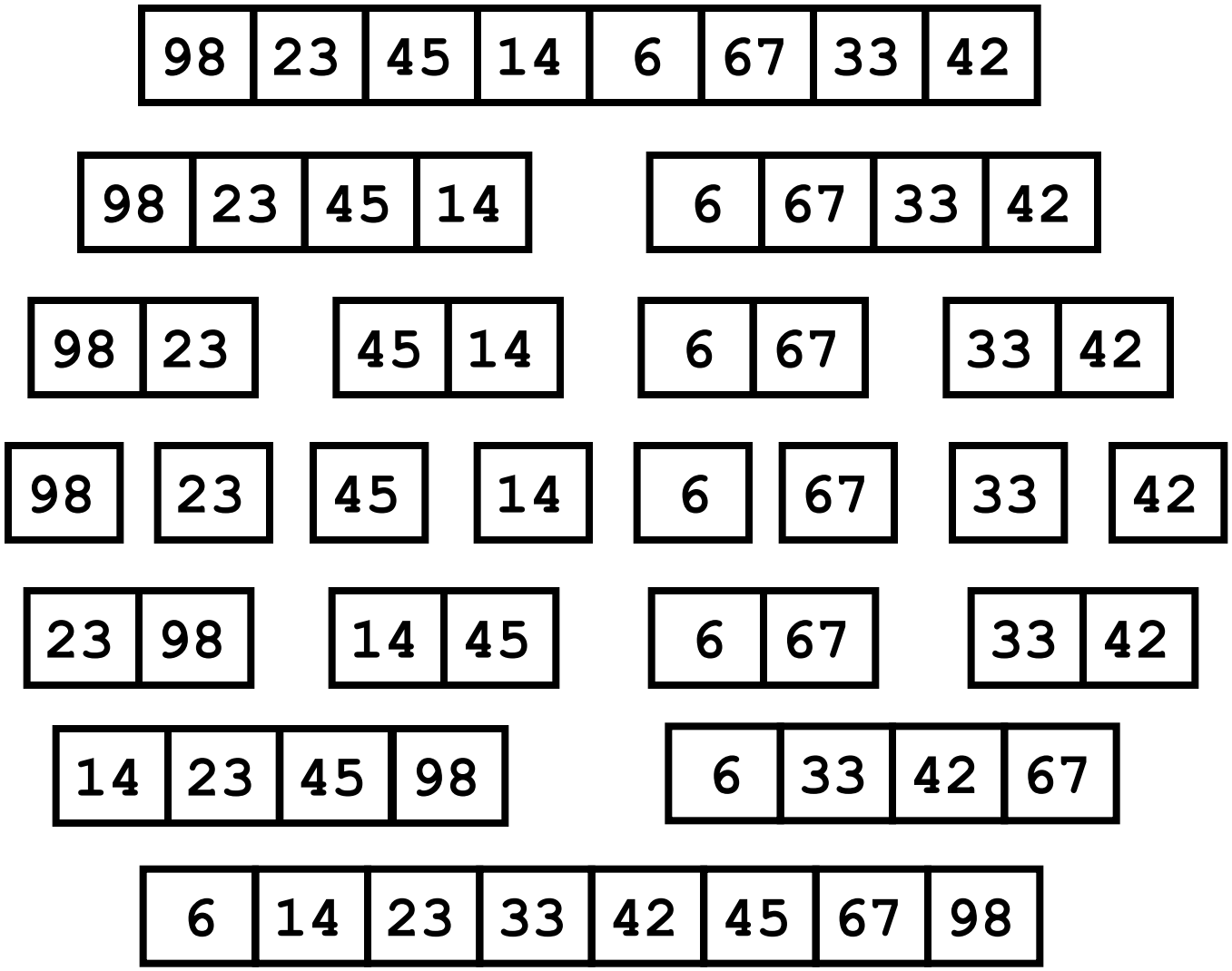
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Merging two sorted arrays

- *merge* operation:
  - Given two sorted arrays, *merge* operation produces a sorted array with all the elements of the two arrays

A	6	13	18	21

B	4	8	9	20

C	4	6	8	9	13	18	20	21

Running time of *merge*: O(n), where n is the number of elements in the merged array.

when merging two sorted parts of the same array, we'll need a *temporary array* to store the merged whole

Merge sort code

```
public static void mergeSort(int[] a) {
    int[] temp = new int[a.length];
    mergeSort(a, temp, 0, a.length - 1);
}

private static void mergeSort(int[] a, int[] temp,
                              int left, int right) {
    if (left >= right) {  // base case
        return;
    }

    // sort the two halves
    int mid = (left + right) / 2;
    mergeSort(a, temp, left, mid);
    mergeSort(a, temp, mid + 1, right);

    // merge the sorted halves into a sorted whole
    merge(a, temp, left, right);
}
```

Merge code

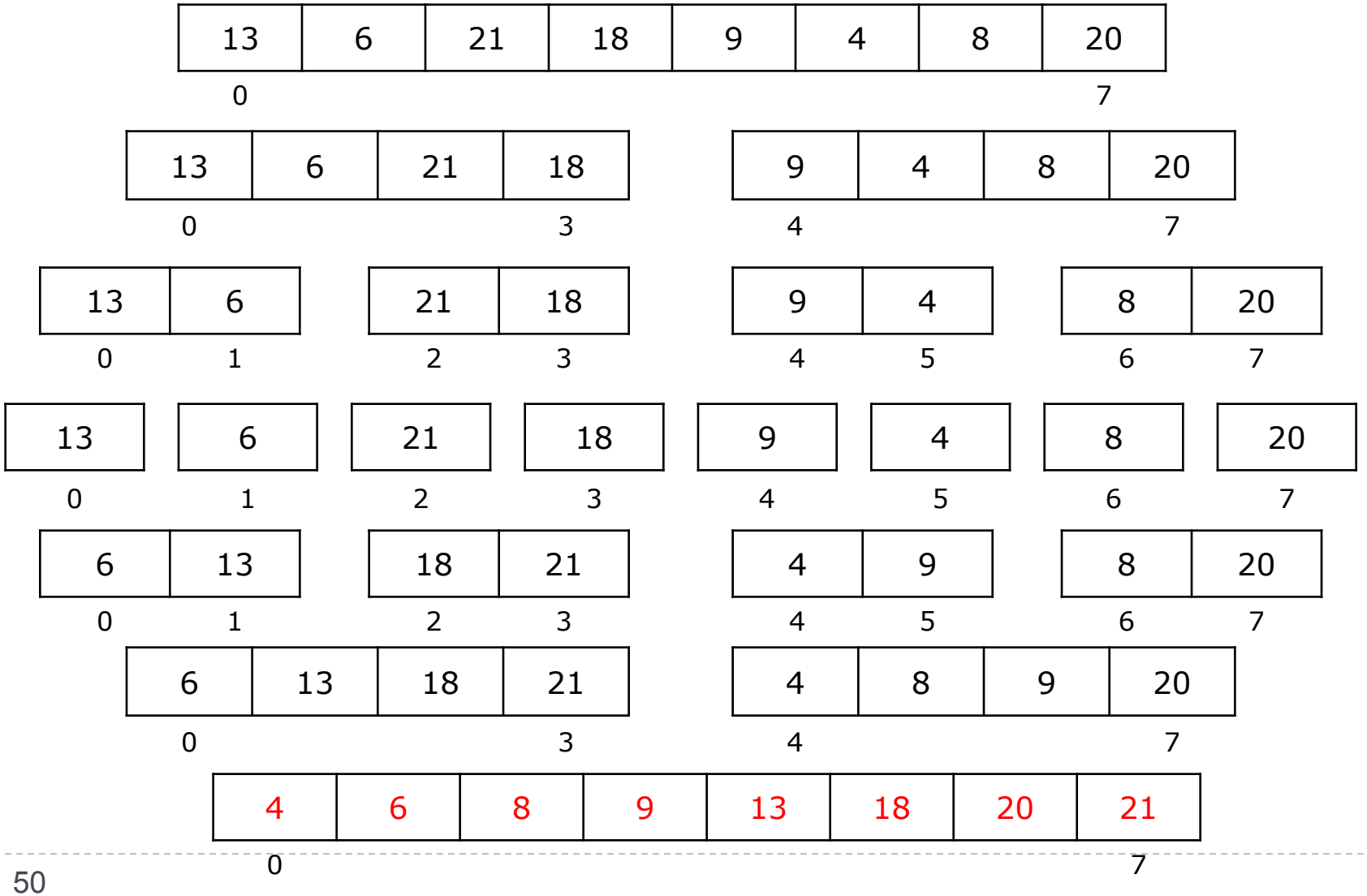
```
private static void merge(int[] a, int[] temp,
                          int left, int right) {
    int mid = (left + right) / 2;
    int count = right - left + 1;

    int l = left;                    // counter indexes for L, R
    int r = mid + 1;

    // main loop to copy the halves into the temp array
    for (int i = 0; i < count; i++)
        if (r > right) {             // finished right; use left
            temp[i] = a[l++];
        } else if (l > mid) {        // finished left; use right
            temp[i] = a[r++];
        } else if (a[l] < a[r]) {    // left is smaller (better)
            temp[i] = a[l++];
        } else {                     // right is smaller (better)
            temp[i] = a[r++];
        }

    // copy sorted temp array back into main array
    for (int i = 0; i < count; i++) {
        a[left + i] = temp[i];
    }
}
```

Merge sort example 2



Merge sort runtime

- Let $T(n)$ be runtime of merge sort on n items
  - $T(0) = 1$
  - $T(1) = 2*T(0) + 1$
  - $T(2) = 2*T(1) + 2$
  - $T(4) = 2*T(2) + 4$
  - $T(8) = 2*T(4) + 8$
  - ...
  - $T(n/2) = 2*T(n/4) + n/2$
  - $T(n) \quad = 2*T(n/2) + n$

  - Substitute to solve for $T(n)$

Repeated Substitution Method

$T(n) = 2*T(n/2) + n$
$T(n/2) = 2*T(n/4) + n/2$

$T(n) = 2*(2*T(n/4) + n/2) + n$
$T(n) = 4*T(n/4) + 2n$
$T(n) = 8*T(n/8) + 3n$
...
$T(n) = 2^k T(n/2^k) + kn$

What is *k*? How many times can you cut *n* in half?

Setting $k = \log_2 n$.

$T(n) = 2^{\log n} T(n/2^{\log n}) + (\log n)\, n$
$T(n) = n * T(n/n) + n \log n$
$T(n) = n * T(1) + n \log n$
$T(n) = n * 1 + n \log n$
$T(n) = n + n \log n$
$T(n) = O(n \log n)$

Sorting practice problem

- Consider the following array of int values.

```
[22, 11, 34, -5,  3, 40,  9, 16,  6]
```

Write the contents of the array after all the recursive calls of merge sort have finished (before merging).

Quick sort

- **quick sort**: orders a list of values by partitioning the list around one element called a pivot, then sorting each partition
  - invented by British computer scientist C.A.R. Hoare in 1960

- another divide and conquer algorithm:
  - choose one element in the list to be the pivot (partition element)
  - divide the elements so that all elements less than the pivot are to its left and all greater are to its right
  - conquer by applying the quick sort algorithm (recursively) to both partitions

Quick sort, continued

- For correctness, it's okay to choose any pivot.

- For efficiency, one of following is best case, the other worst case:
  - pivot partitions the list roughly in half
  - pivot is greatest or least element in list

- Which case above is best?

- We will divide the work into two methods:
  - `quickSort` – performs the recursive algorithm
  - `partition` – rearranges the elements into two partitions

Quick sort pseudo-code

Let S be the input set.

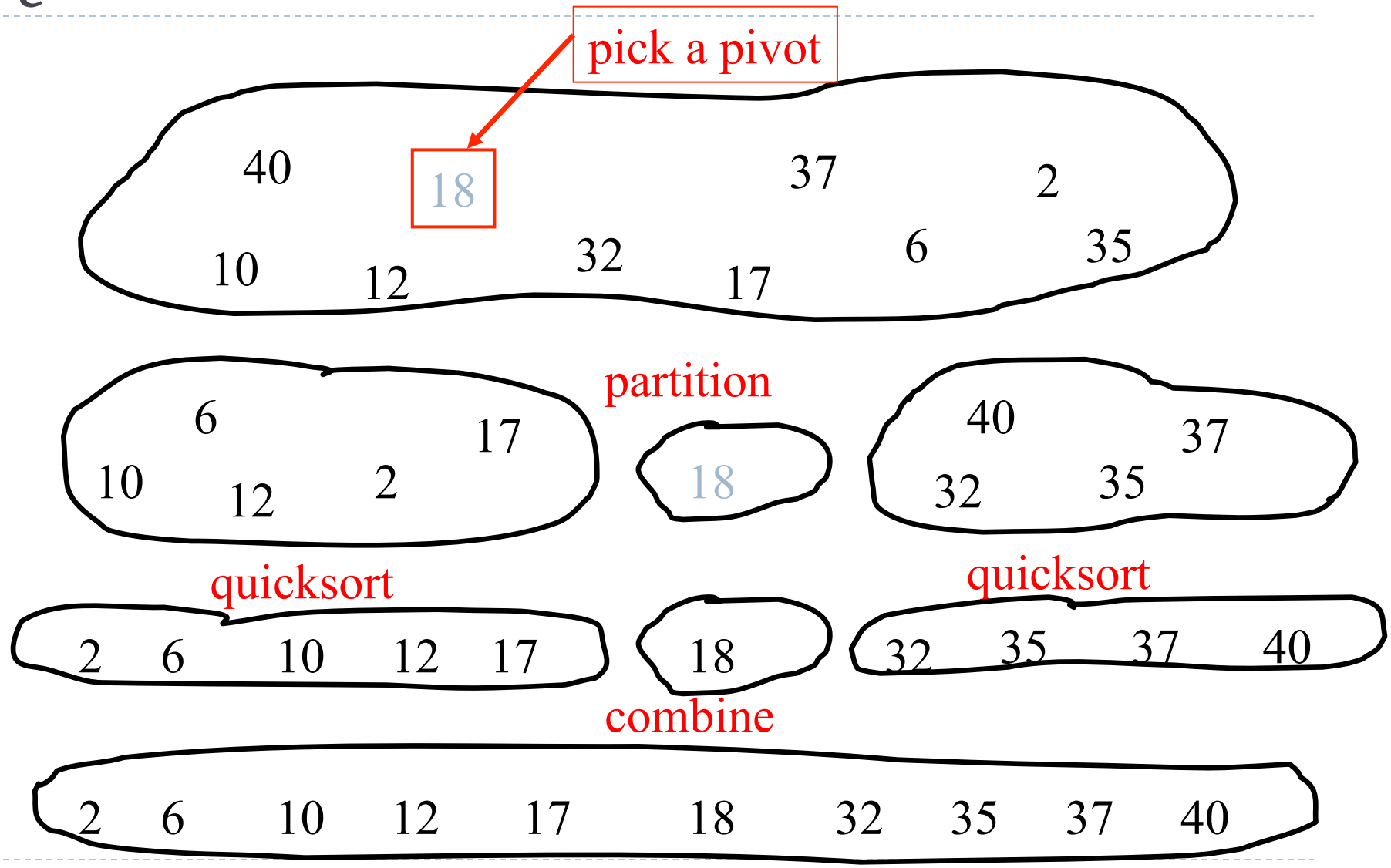
1. If number of elements in S is 0 or 1, then return.

2. Pick an element *v* in S. Call *v* the pivot.

3. Partition remaining elements of S into two groups:
   - S_1 = {elements ≤ *v*}
   - S_2 = {elements ≥ *v*}

4. Return { quicksort(S_1), *v*, quicksort(S_2) }

Quick sort illustrated



How to choose a pivot

- first element
  - bad if input is sorted or in reverse sorted order
  - bad if input is nearly sorted
  - variation: particular element (e.g. middle element)
- random element
  - even a malicious agent cannot arrange a bad input
- median of three elements
  - choose the median of the left, right, and center elements

Partitioning algorithm

The basic idea:

1. Move the pivot to the rightmost position.
2. Starting from the left, find an element ≥ pivot. Call the position i.
3. Starting from the right, find an element ≤ pivot. Call the position j.
4. Swap S[i] and S[j].

8	1	4	9	0	3	5	2	7	6
0									9

"Median of three" pivot

9	17	3	12	8	7	21	1
0							7

pick pivot

1	17	3	12	8	7	21	9
0	i				j		7

1	7	3	12	8	17	21	9
0			i	j			7

1	7	3	8	12	17	21	9
0			j	i			7

swap S[i]
with S[right]

1	7	3	8	9	17	21	12
0							7

Quick sort code

```
public static void quickSort(int[] a) {
    quickSort(a, 0, a.length - 1);
}

private static void quickSort(int[] a, int min, int max) {
    if (min >= max) {  // base case; no need to sort
        return;
    }

    // choose pivot -- we'll use the first element (might be bad!)
    int pivot = a[min];
    swap(a, min, max);  // move pivot to end

    // partition the two sides of the array
    int middle = partition(a, min, max - 1, pivot);

    // restore the pivot to its proper location
    swap(a, middle, max);

    // recursively sort the left and right partitions
    quickSort(a, min, middle - 1);
    quickSort(a, middle + 1, max);
}
```

Quick sort code, cont'd.

```
// partitions a with elements < pivot on left and
// elements > pivot on right;
// returns index of element that should be swapped with pivot
private static int partition(int[] a, int i, int j, int pivot) {
    i--;  j++;  // kludge because the loops pre-increment
    while (true) {
        // move index markers i,j toward center
        // until we find a pair of mis-partitioned elements
        do { i++; } while (i < j && a[i] < pivot);
        do { j--; } while (i < j && a[j] > pivot);

        if (i >= j) {
            break;
        } else {
            swap(a, i, j);
        }
    }

    return i;
}
```

Quick sort runtime

- Worst case: pivot is the smallest (or largest) element all the time (recurrence solution technique: telescoping)

  $T(n) = T(n-1) + cn$

  $T(n-1) = T(n-2) + c(n-1)$

  $T(n-2) = T(n-3) + c(n-2)$

  …

  $T(2) = T(1) + 2c$

$$T(N) = T(1) + c\sum_{i=2}^{N} i = O(N^2)$$

- Best case: pivot is the median (recurrence solution technique: Master's Theorem)

  $T(n) = 2T(n/2) + cn$

  $T(n) = cn \log n + n = O(n \log n)$

Runtime summary

	comparisons
merge	O(n log n)
quick	average: O(n log n) worst: $O(n^2)$

Sorting practice problem

- Consider the following array of int values.

  [22, 11, 34, -5, 3, 40, 9, 16, 6]

  - Write the contents of the array after all the partitioning of quick sort has finished (before any recursive calls).
  - Assume that the median of three elements (first, middle, and last) is chosen as the pivot.

Sorting practice problem

- Consider the following array:

  [7, 17, 22, -1, 9, 6, 11, 35, -3]

- Each of the following is a view of a sort-in-progress on the elements. Which sort is which?
  - (If the algorithm is a multiple-loop algorithm, the array is shown after a few of these loops have completed. If the algorithm is recursive, the array is shown after the recursive calls have finished on each sub-part of the array.)
  - Assume that the quick sort algorithm chooses the first element as its pivot at each pass.

    (a) [-3, -1, 6, 17, 9, 22, 11, 35, 7]

    (b) [-1, 7, 17, 22, -3, 6, 9, 11, 35]

    (c) [-1, 7, 6, 9, 11, -3, 17, 22, 35]

    (d) [-3, 6, -1, 7, 9, 17, 11, 35, 22]

    (e) [-1, 7, 17, 22, 9, 6, 11, 35, -3]

Sorting practice problem

- For the following questions, indicate which of the five sorting algorithms will successfully sort the elements in the least amount of time.
  - The algorithm chosen should be the one that completes fastest, without crashing.
  - Assume that the quick sort algorithm chooses the first element as its pivot at each pass.
  - Assume stack overflow occurs on 5000+ stacked method calls.

(a) array size 2000, random order

(b) array size 500000, ascending order

(c) array size 100000, descending order

- special constraint: no extra memory may be allocated! (O(1) storage)

(d) array size 1000000, random order

(e) array size 10000, ascending order

- special constraint: no extra memory may be allocated! (O(1) storage)

External Sorting

Simple External Merge Sort

- Divide and conquer: divide the file into smaller, sorted subfiles (called runs) and merge runs

- Initialize:
  - Load chunk of data from file into RAM
  - Sort internally
  - Write sorted data (run) back to disk (in separate files)
- While we still have runs to sort:
  - Merge runs from previous pass into runs of twice the size (think merge() method from mergesort)
  - Repeat until you only have one run