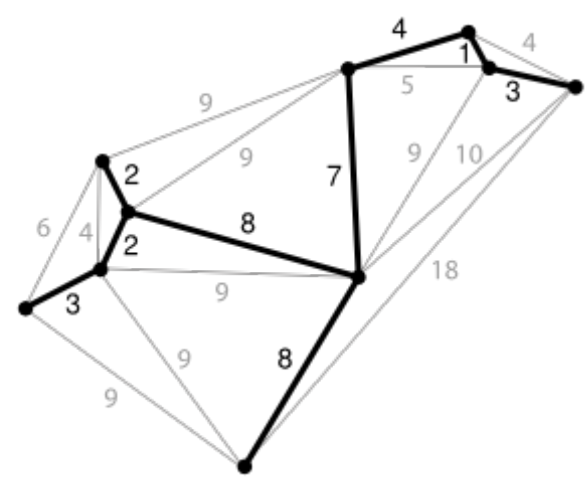
CSE 373: Data Structures and Algorithms

Lecture 22: Graphs VI

Minimum spanning tree

- **tree**: a connected, directed acyclic graph
- **spanning tree**: a subgraph of a graph, which meets the constraints to be a tree (connected, acyclic) and connects every vertex of the original graph
- **minimum spanning tree**: a spanning tree with weight less than or equal to any other spanning tree for the given graph



Min. span. tree applications

- Consider a cable TV company laying cable to a new neighborhood...
  - Can only bury the cable only along certain paths, then a graph could represent which points are connected by those paths.
  - Some of paths may be more expensive (i.e. longer, harder to install), so these paths could be represented by edges with larger weights.
  - A spanning tree for that graph would be a subset of those paths that has no cycles but still connects to every house.

- Similar situations: installing electrical wiring in a house, installing computer networks between cities, building roads between neighborhoods, etc.

Spanning Tree Problem

- Input: An undirected graph $G = (V, E)$. G is connected.
- Output: T subset of E such that
  - (V, T) is a connected graph
  - (V, T) has no cycles

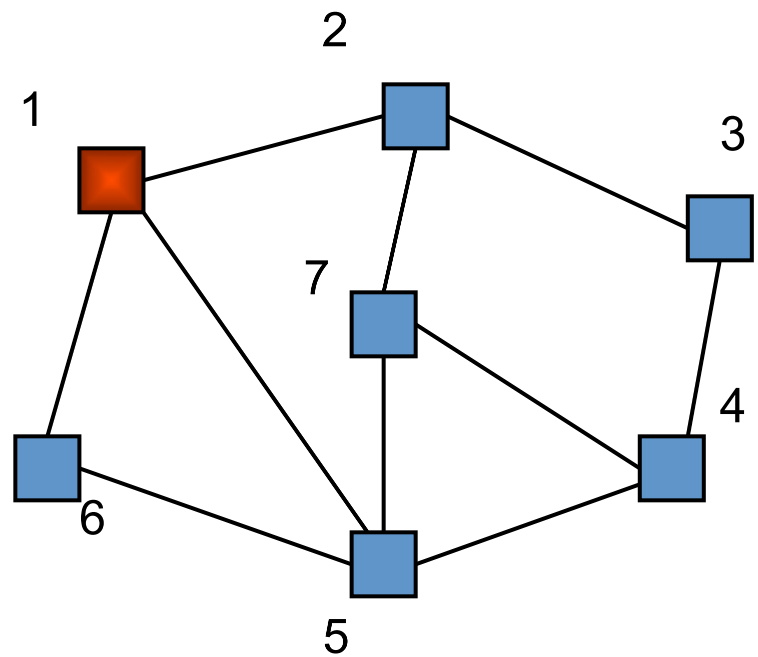
Spanning Tree Psuedocode

```
spanningTree():
    pick random vertex v.
    T := {}
    spanningTree(v, T)
    return T.

spanningTree(v, T):
    mark v as visited.
    for each neighbor v_i of v where there is an edge from v to v_i:
        if v_i is not visited
            add edge {v, v_i} to T.
            spanningTree(v_i, T)
    return T.
```

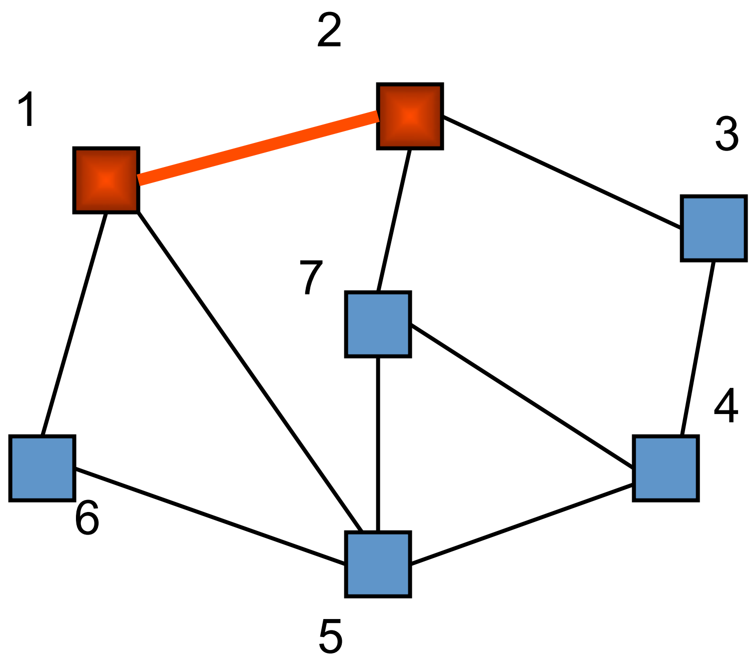
Example of Depth First Search

ST(1)



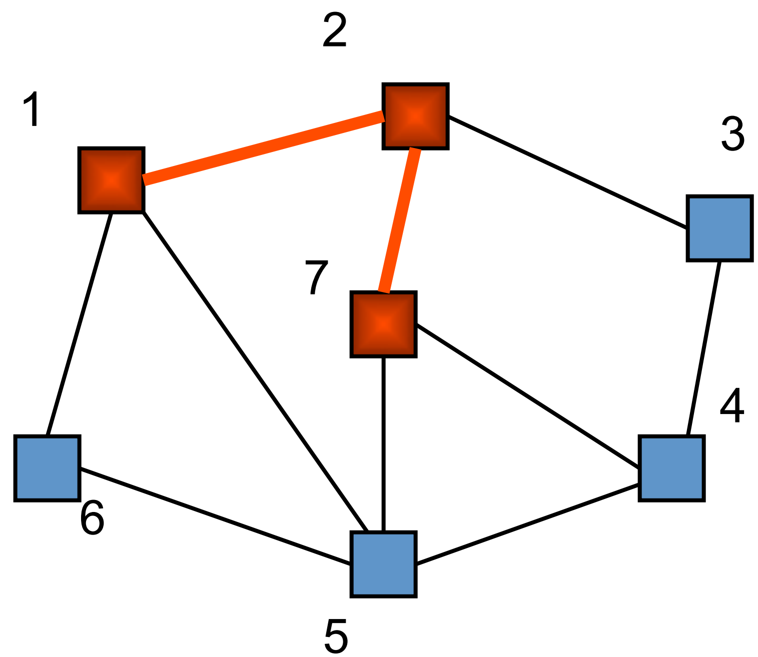
Example Step 2

ST(1)
ST(2)



{1,2}

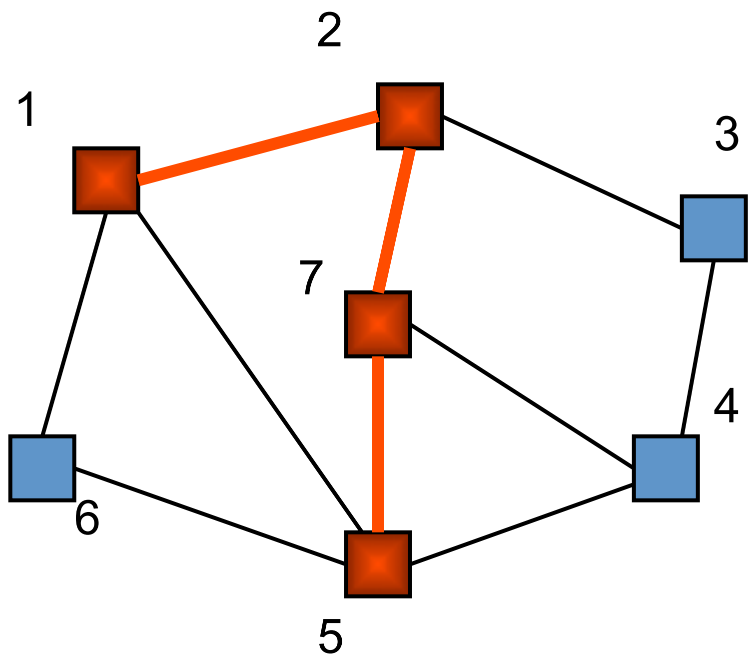
Example Step 3



ST(1)
ST(2)
ST(7)

{1,2} {2,7}

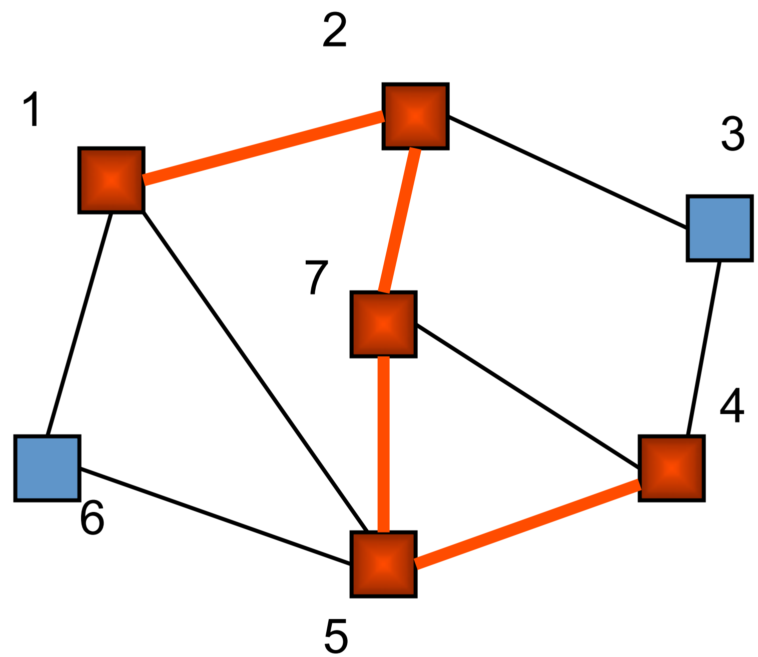
Example Step 4



ST(1)
ST(2)
ST(7)
ST(5)

{1,2} {2,7} {7,5}

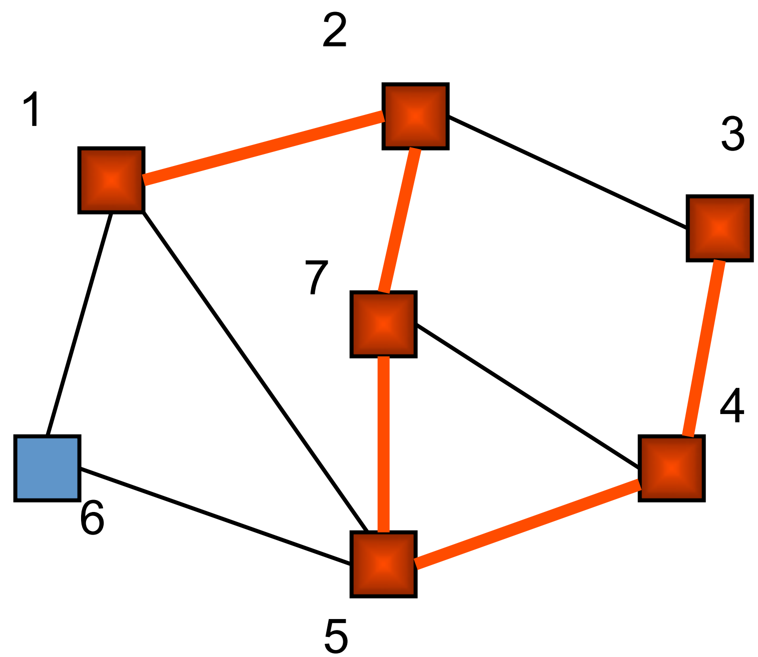
Example Step 5



ST(1)
ST(2)
ST(7)
ST(5)
ST(4)

{1,2} {2,7} {7,5} {5,4}

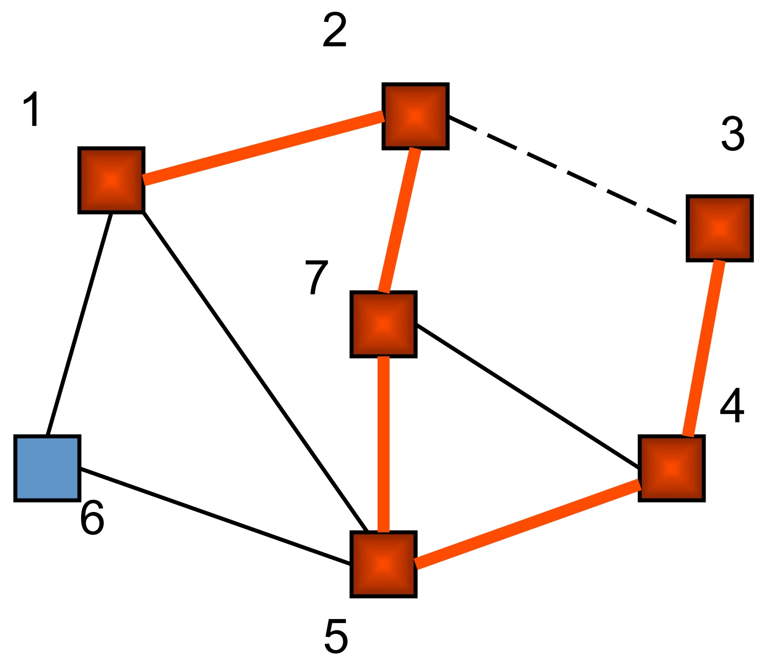
Example Step 6



ST(1)
ST(2)
ST(7)
ST(5)
ST(4)
ST(3)

{1,2} {2,7} {7,5} {5,4} {4,3}

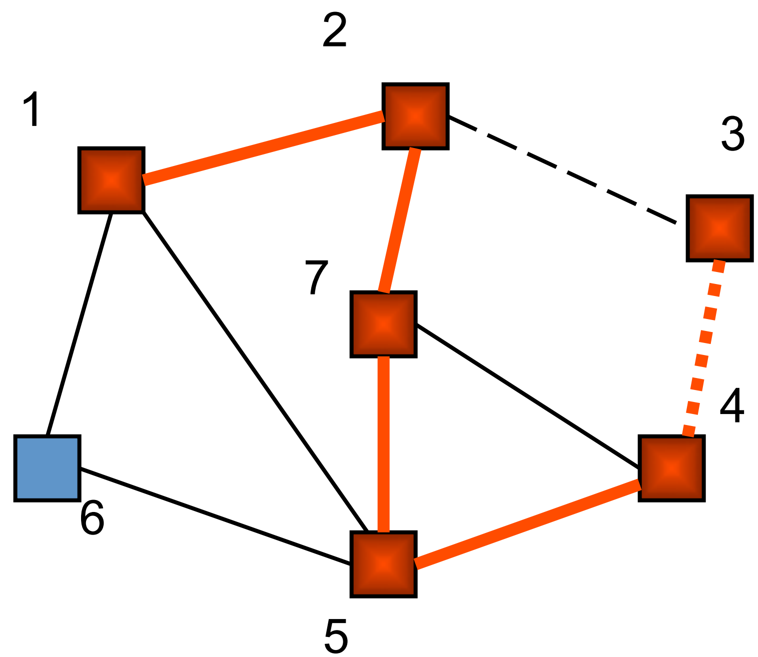
Example Step 7



ST(1)
ST(2)
ST(7)
ST(5)
ST(4)
ST(3)

{1,2} {2,7} {7,5} {5,4} {4,3}

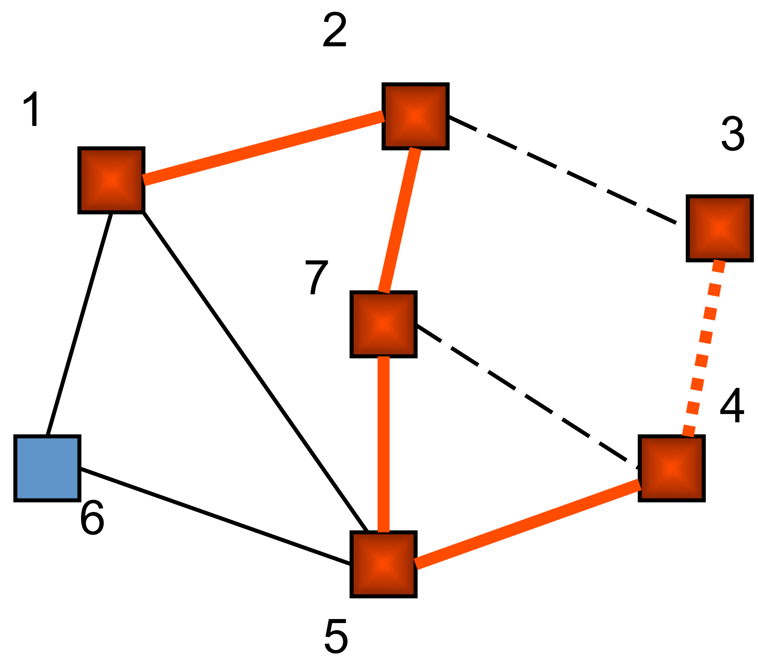
Example Step 8



ST(1)
ST(2)
ST(7)
ST(5)
ST(4)

{1,2} {2,7} {7,5} {5,4} {4,3}

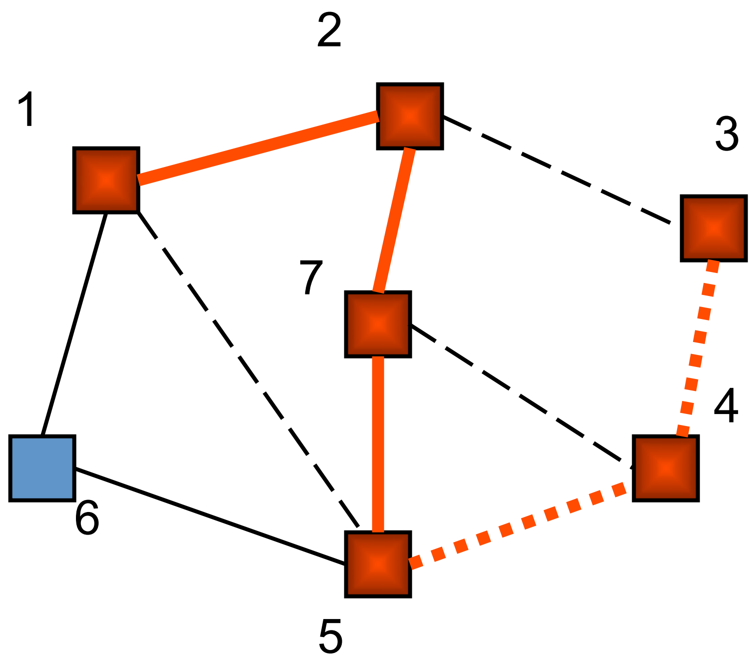
Example Step 9



ST(1)
ST(2)
ST(7)
ST(5)
ST(4)

{1,2} {2,7} {7,5} {5,4} {4,3}

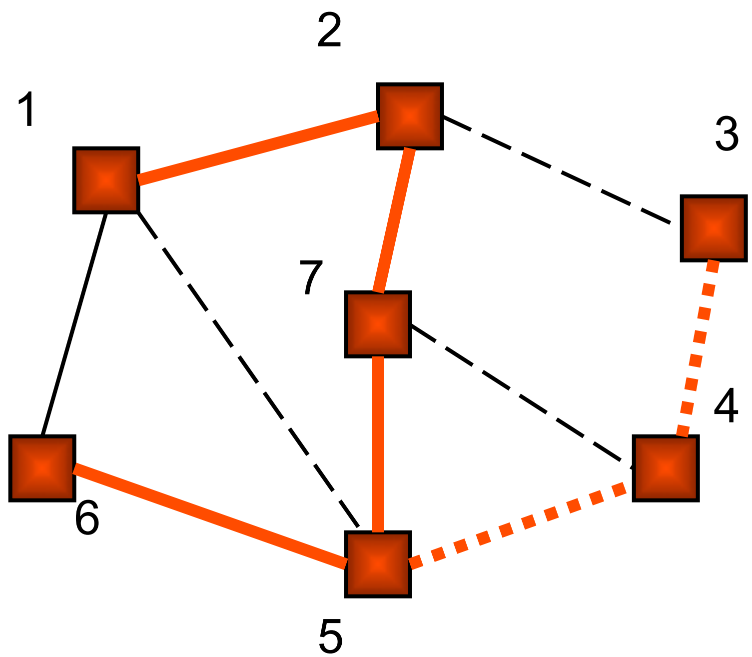
Example Step 10



ST(1)
ST(2)
ST(7)
ST(5)

{1,2} {2,7} {7,5} {5,4} {4,3}

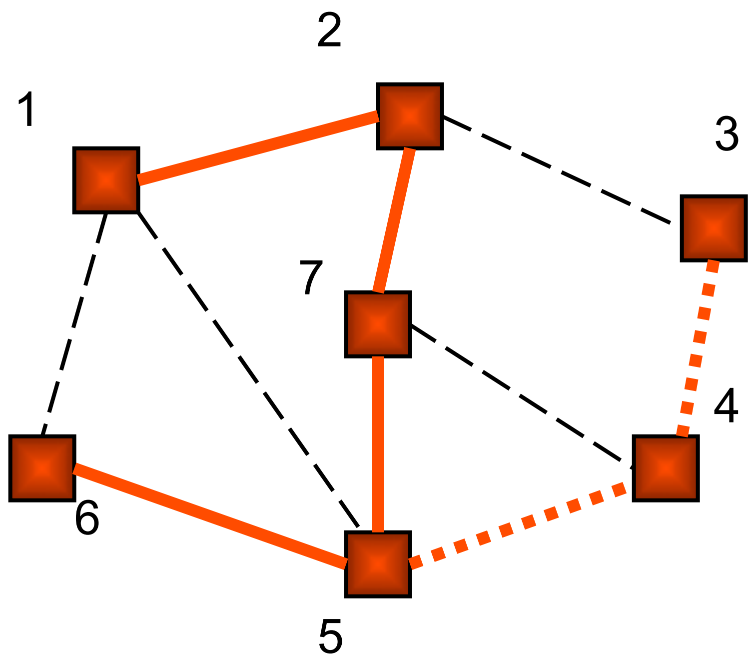
Example Step 11



ST(1)
ST(2)
ST(7)
ST(5)
ST(6)

{1,2} {2,7} {7,5} {5,4} {4,3} {5,6}

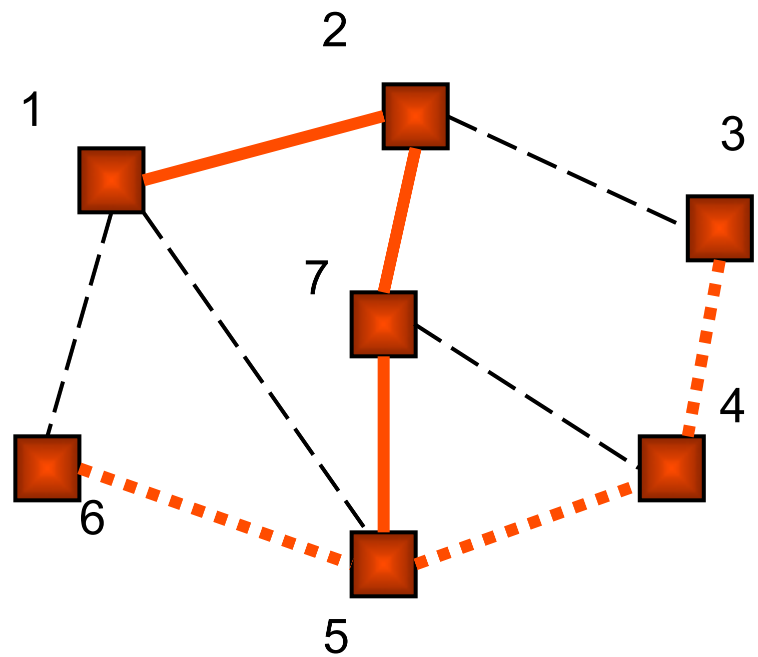
Example Step 12



ST(1)
ST(2)
ST(7)
ST(5)
ST(6)

{1,2} {2,7} {7,5} {5,4} {4,3} {5,6}

Example Step 13

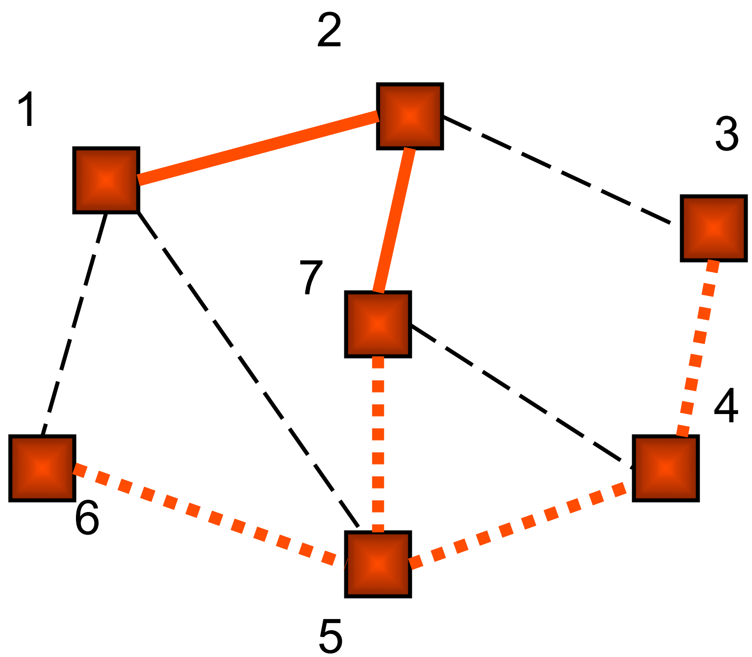


ST(1)
ST(2)
ST(7)
ST(5)

{1,2} {2,7} {7,5} {5,4} {4,3} {5,6}

Example Step 14

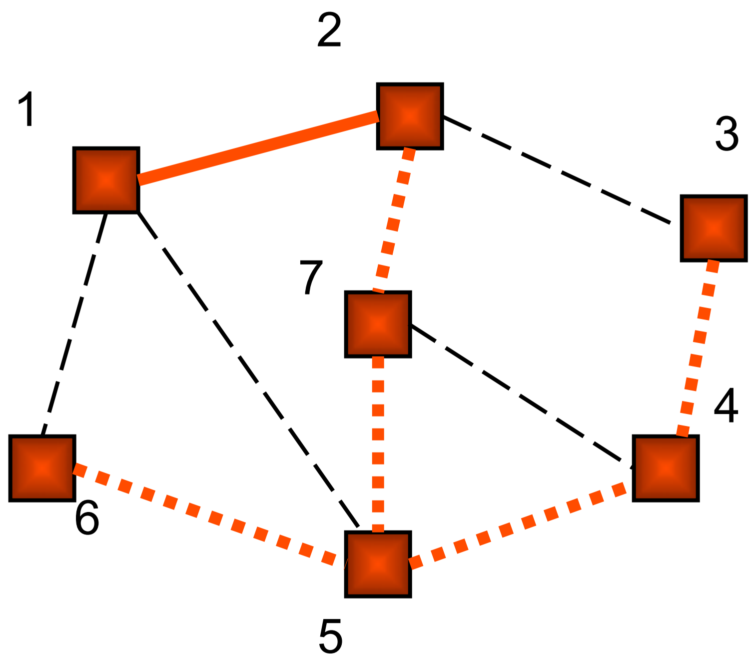
ST(1)
ST(2)
ST(7)



{1,2} {2,7} {7,5} {5,4} {4,3} {5,6}

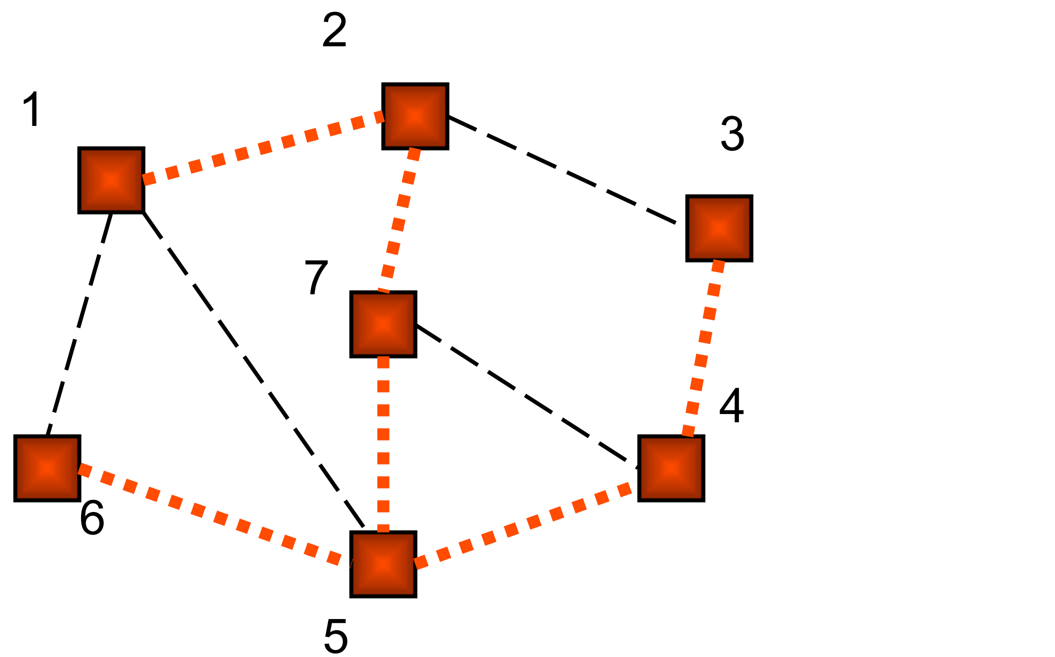
Example Step 15

ST(1)
ST(2)



{1,2} {2,7} {7,5} {5,4} {4,3} {5,6}

Example Step 16



{1,2} {2,7} {7,5} {5,4} {4,3} {5,6}

Minimum Spanning Tree Problem

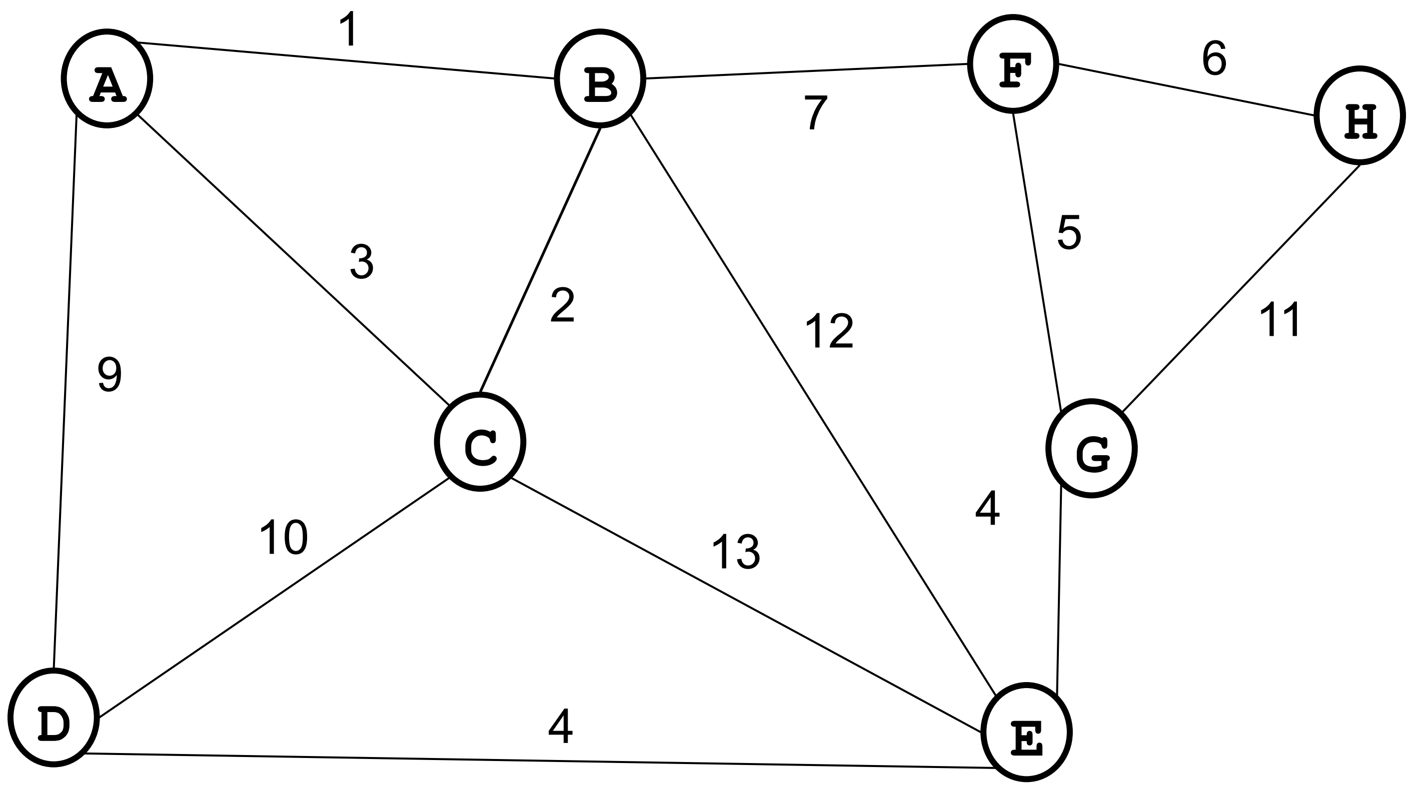
- Input: Undirected Graph $G = (V, E)$ and a cost function C from E to non-negative real numbers. $C(e)$ is the cost of edge e.
- Output: A spanning tree T with minimum total cost. That is: T that minimizes

$$C(T) = \sum_{e \in T} C(e)$$

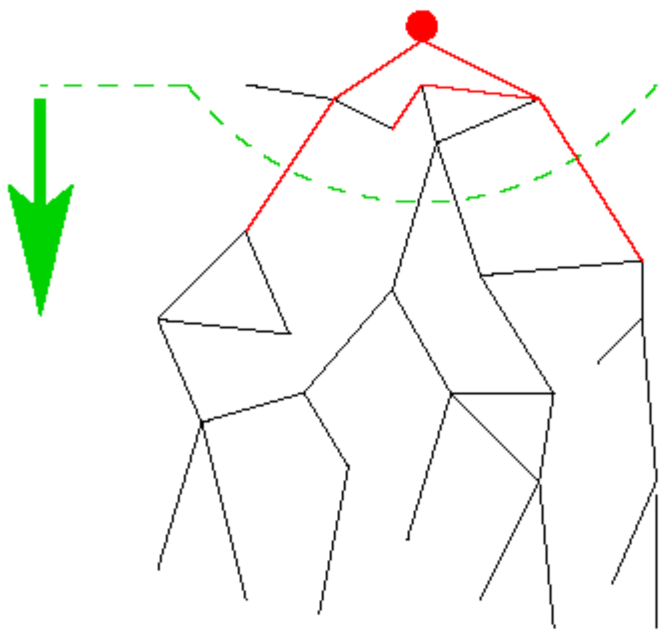
Observations about Spanning Trees

- For any spanning tree *T*, inserting an edge e_{new} not in *T* creates a cycle
- But
  - Removing any edge e_{old} from the cycle gives back a spanning tree
  - If e_{new} has a lower cost than e_{old} we have progressed!

Find the MST

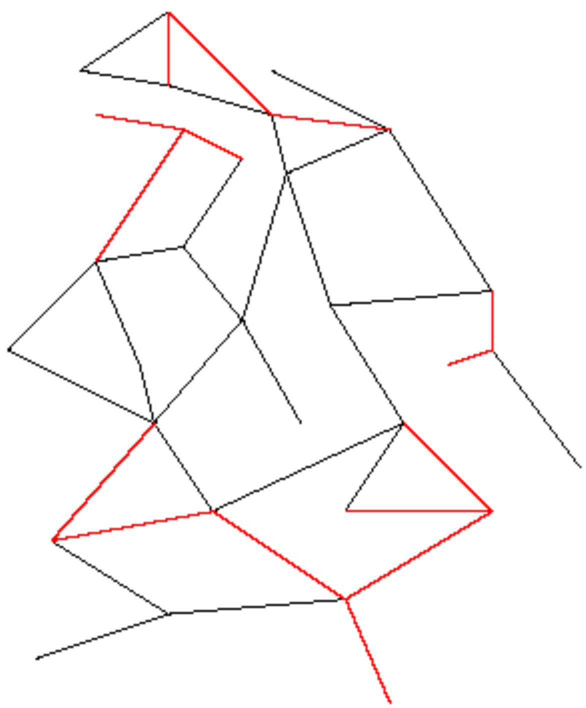


Two Different Approaches



Prim's Algorithm

Looks familiar!

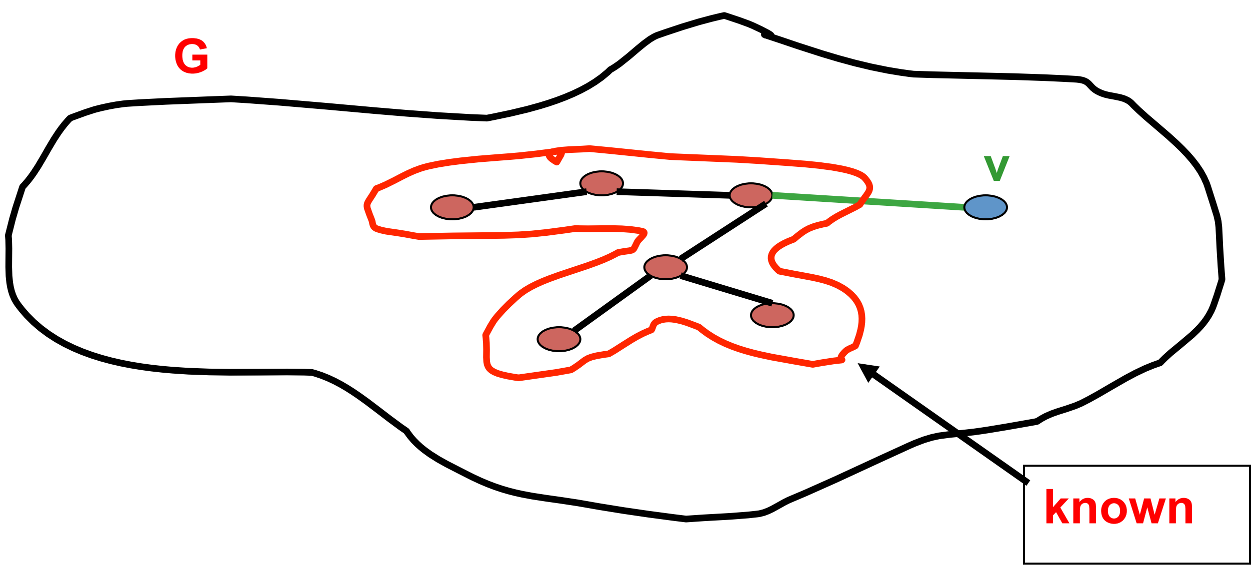


Kruskals's Algorithm

Completely different!

Prim's algorithm

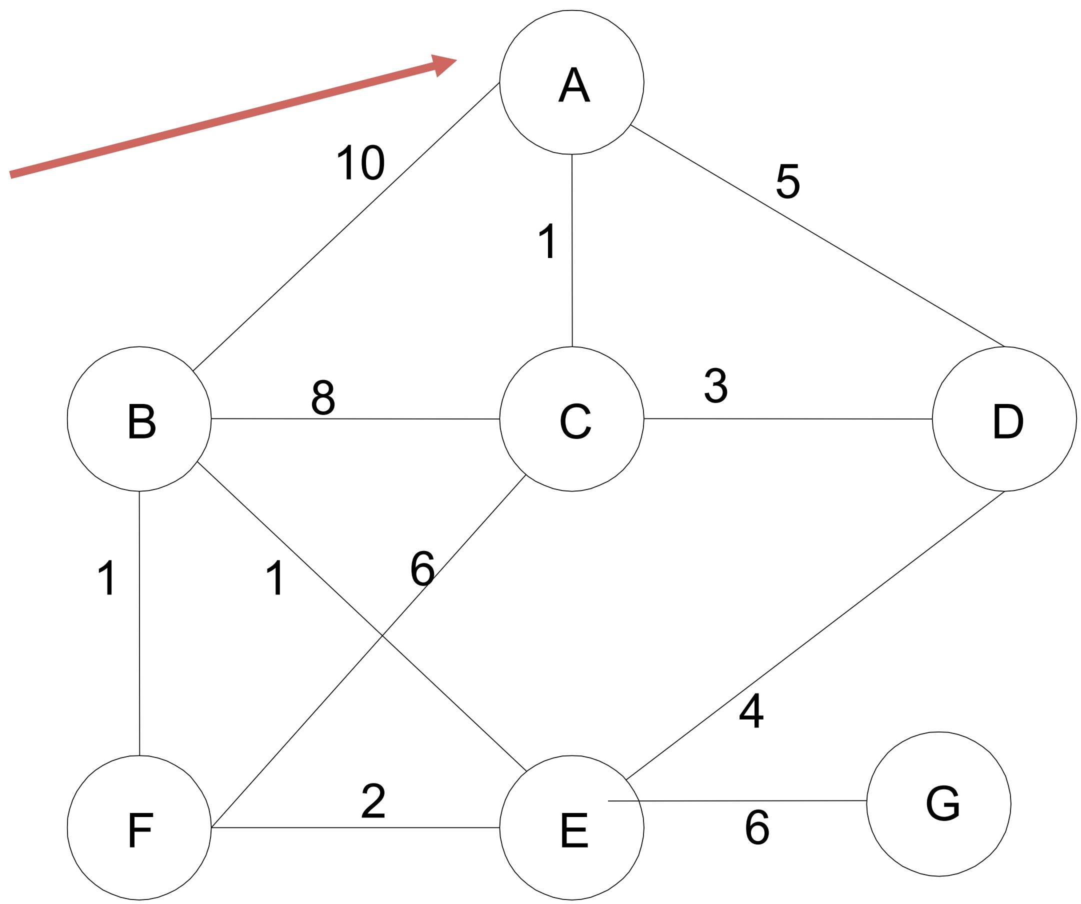
Idea: Grow a tree by adding an edge from the "known" vertices to the "unknown" vertices. Pick the edge with the smallest weight.



Prim's algorithm

Starting from empty *T*, choose a vertex at random and initialize

V = {A}, *T* ={}

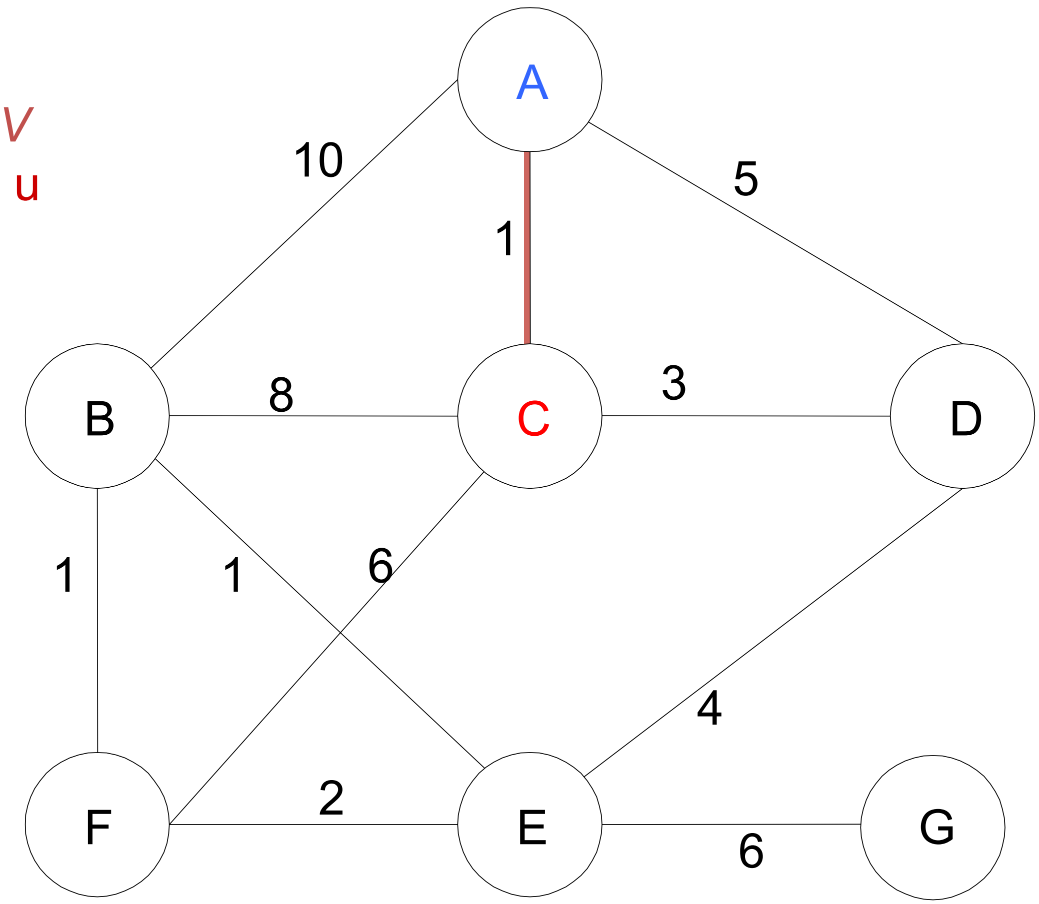


Prim's algorithm

Choose the vertex u not in *V* such that edge weight from u to a vertex in *V* is minimal (greedy!)

V = {A,C}

T = { (A,C) }

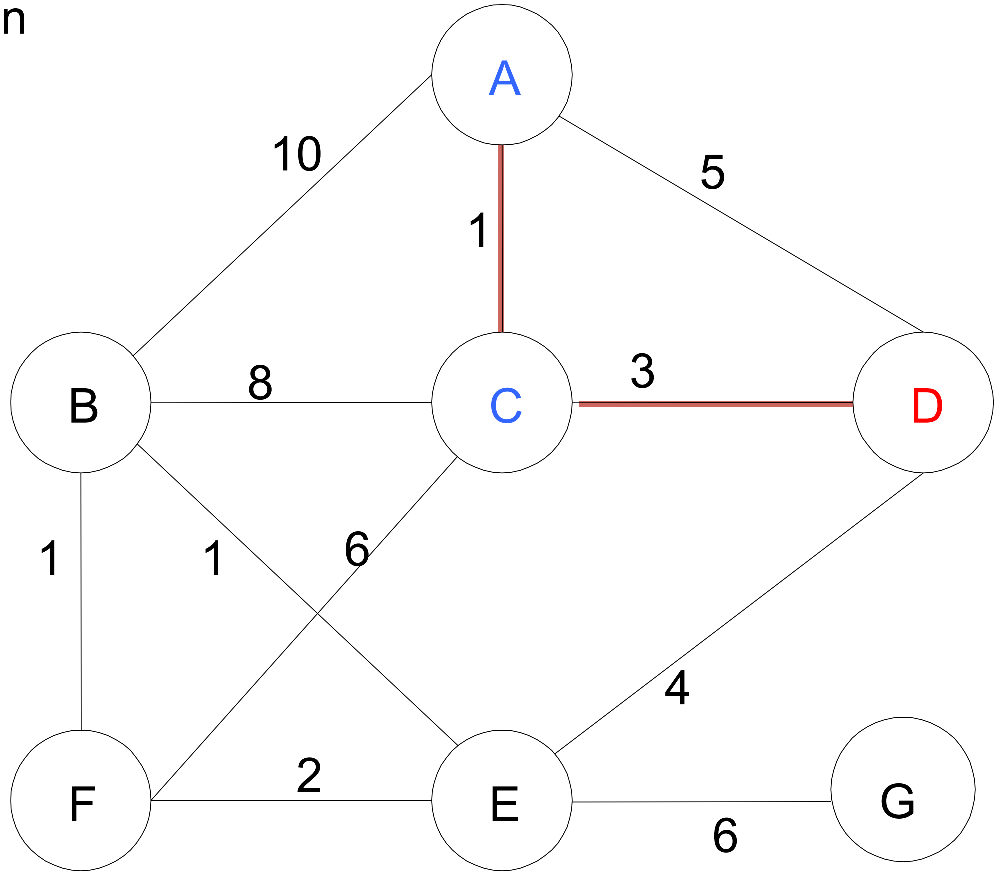


Prim's algorithm

Repeat until all vertices have been chosen

V = {A,C,D}

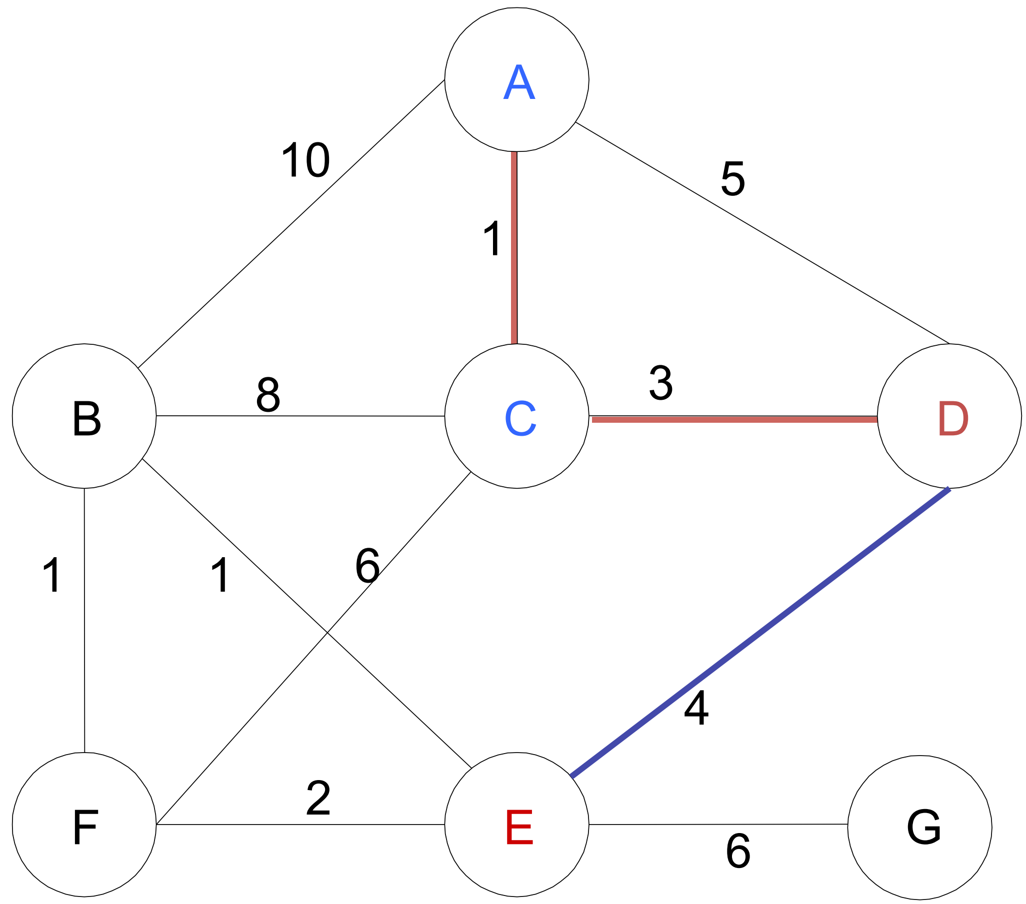
T= { (A,C), (C,D)}



Prim's algorithm

V = {A,C,D,E}

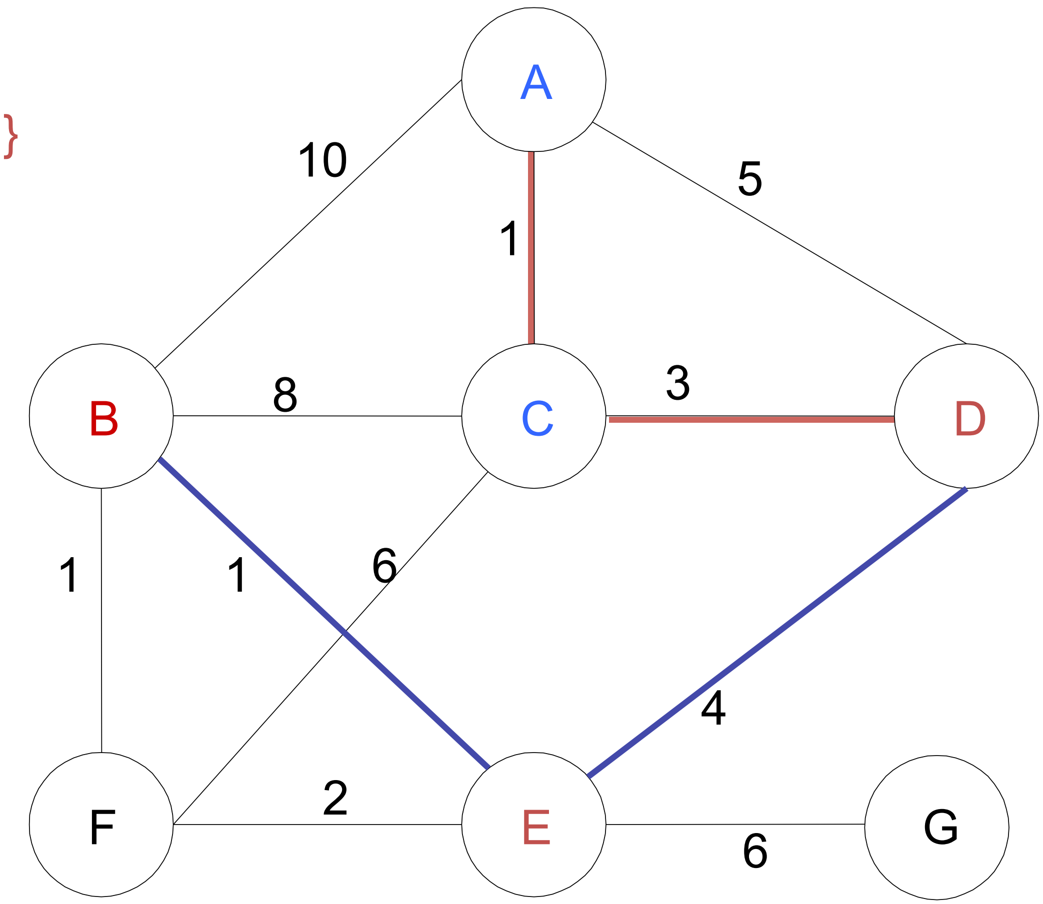
T = { (A,C), (C,D), (D,E) }



Prim's algorithm

V = {A,C,D,E,B}

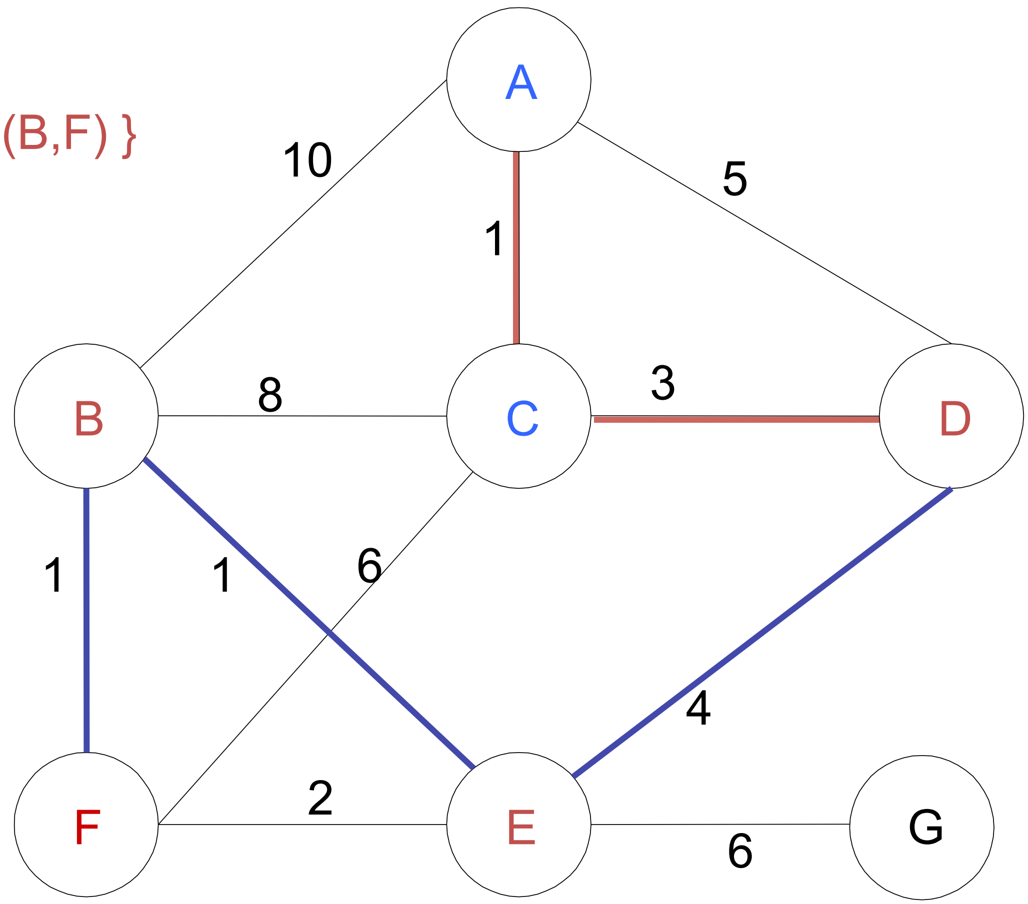
T = { (A,C), (C,D), (D,E), (E,B) }



Prim’s algorithm

V = {A,C,D,E,B,F}

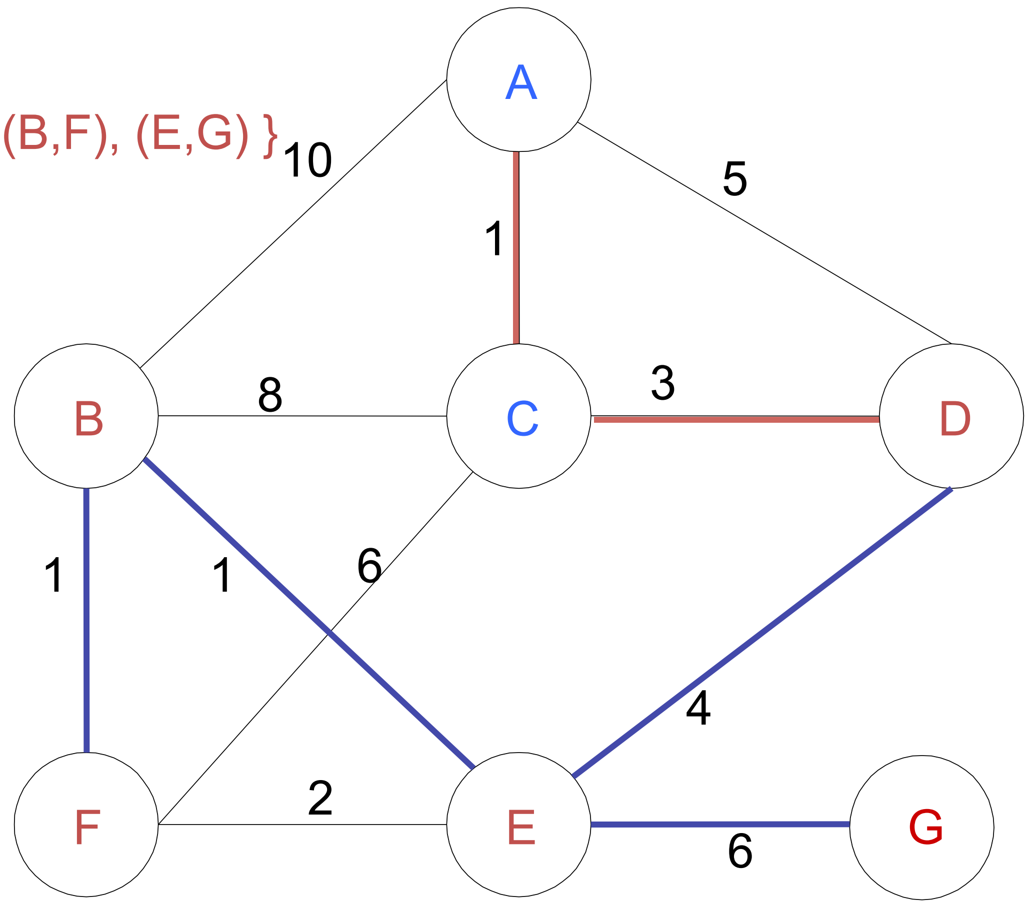
T = { (A,C), (C,D), (D,E), (E,B), (B,F) }



Prim's algorithm

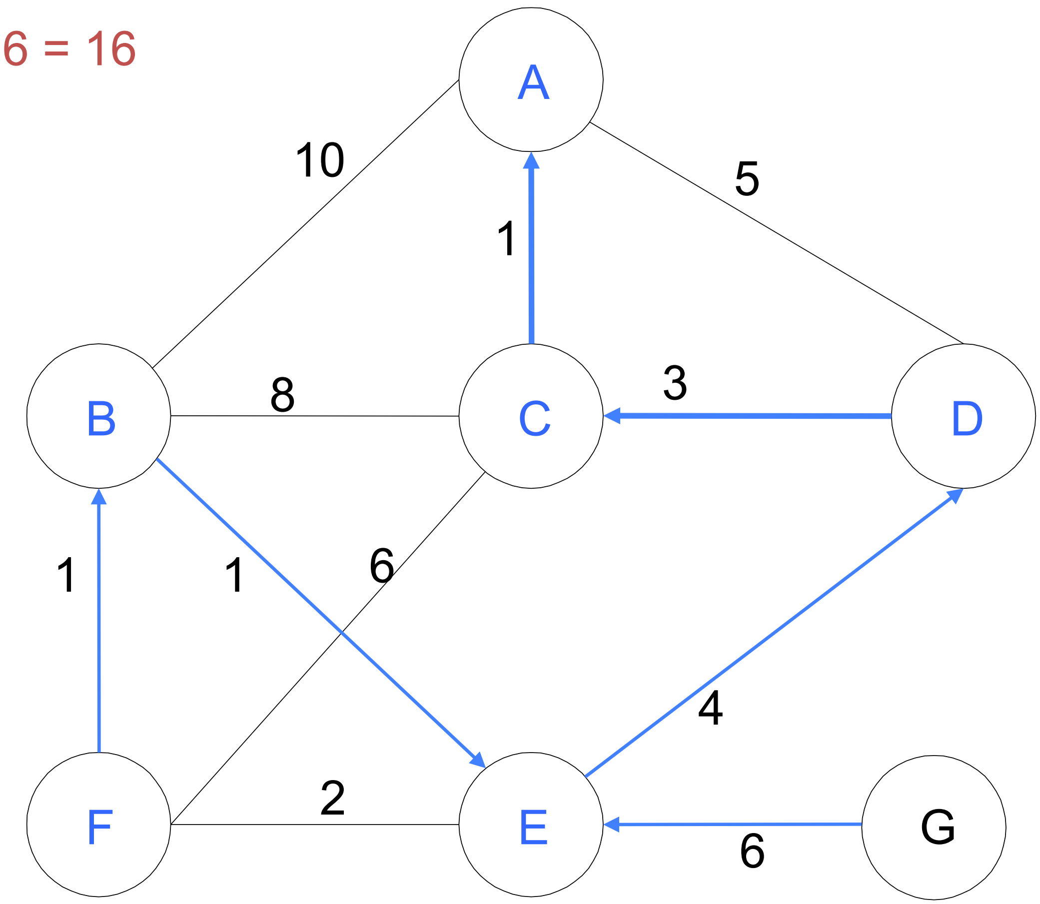
V = {A,C,D,E,B,F,G}

T = { (A,C), (C,D), (D,E), (E,B), (B,F), (E,G) }



Prim's algorithm

Final Cost: 1 + 3 + 4 + 1 + 1 + 6 = 16



Prim's Algorithm Implementation

```
Prim():
  for each vertex v:                    // Initialization
     v's distance := infinity.
     v's previous := none.
     mark v as unknown.
  choose random node v1.
  v1's distance := 0.
  List := {all vertices}.
  T := {}.

  while List is not empty:
     v := remove List vertex with minimum distance.
     add edge {v, v's previous} to T.
     mark v as known.
     for each unknown neighbor n of v:
        if distance(v, n) is smaller than n's distance:
           n's distance := distance(v, n).
           n's previous := v.

  return T.
```

Prim’s algorithm Analysis

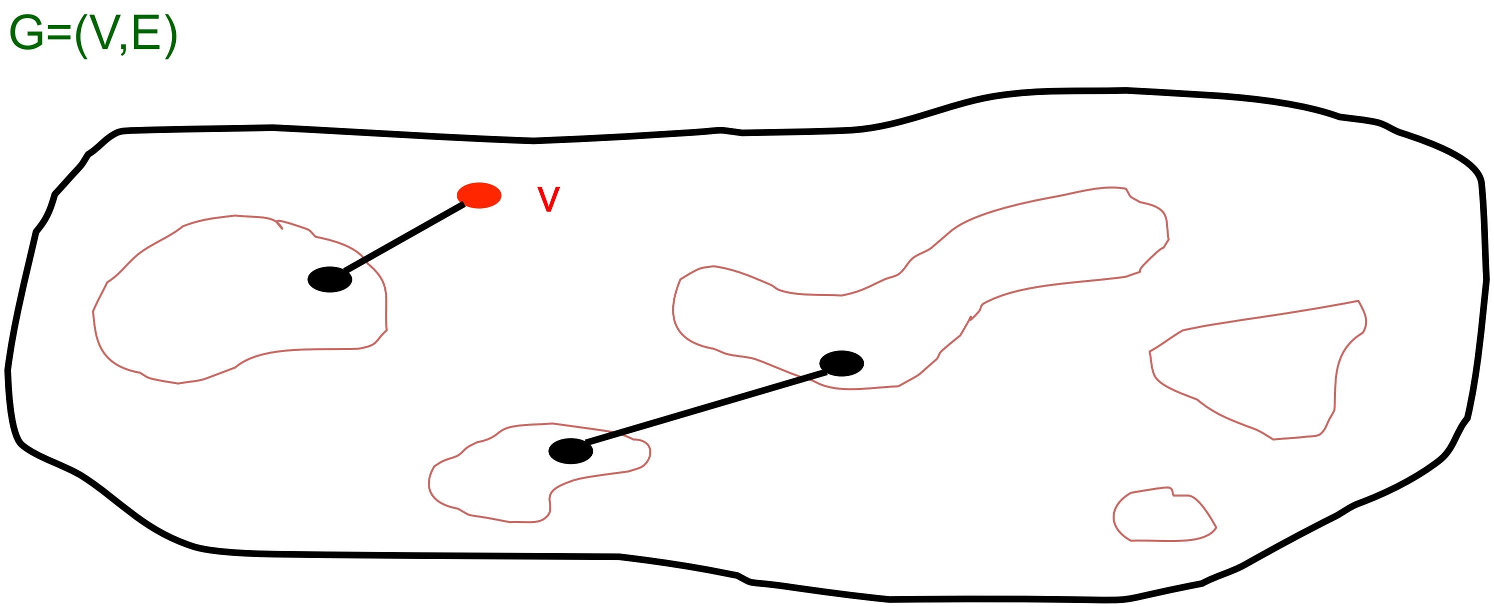
- How is it different from Djikstra's algorithm?

- If the step that removes unknown vertex with minimum distance is done with binary heap the running time is:

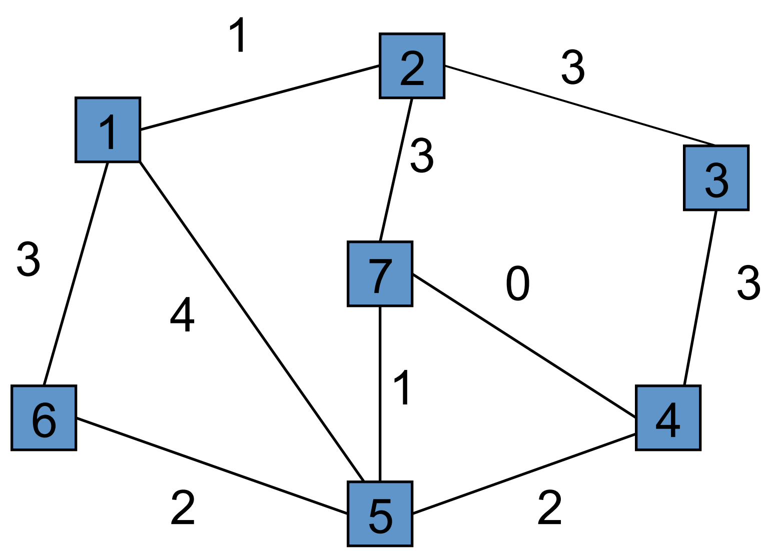
  $O(|E|\log|V|)$

Kruskal's MST Algorithm

Idea: Grow a forest out of edges that do not create a cycle. Pick an edge with the smallest weight.

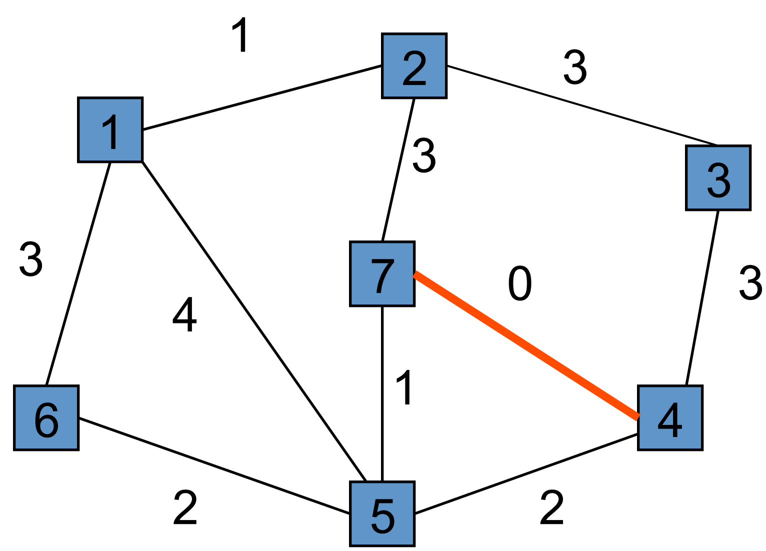


Example of Kruskal 1



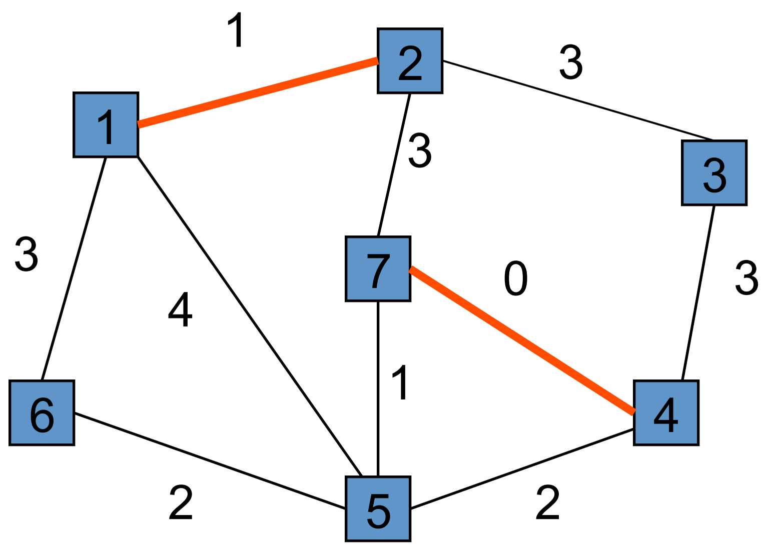
{7,4}	{2,1}	{7,5}	{5,6}	{5,4}	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
0	1	1	2	2	3	3	3	3	4

Example of Kruskal 2



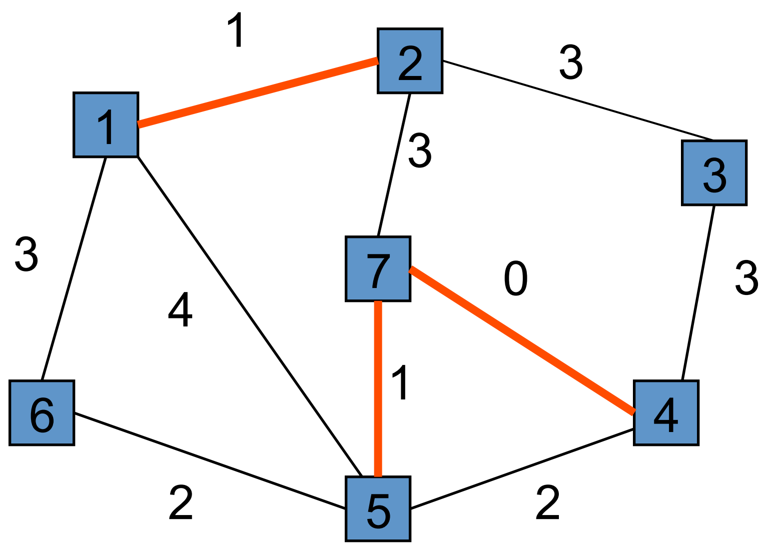
{7,4}	{2,1}	{7,5}	{5,6}	{5,4}	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
0	1	1	2	2	3	3	3	3	4

Example of Kruskal 2



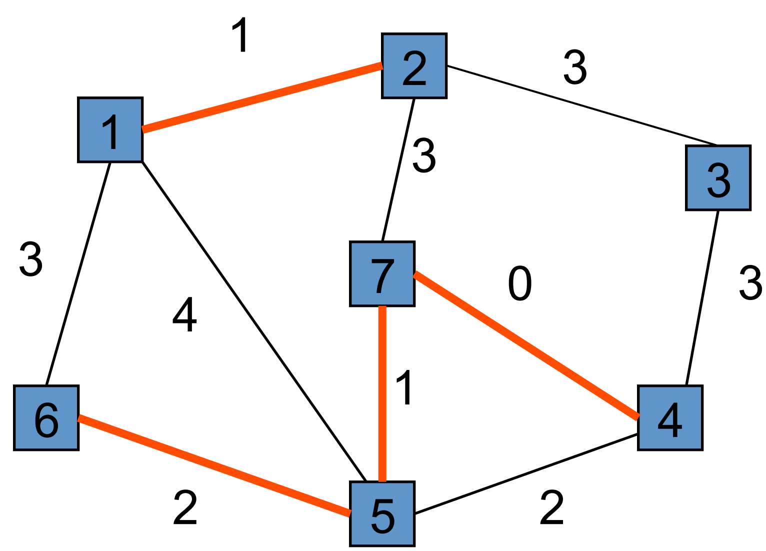
{7,4}	{2,1}	{7,5}	{5,6}	{5,4}	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
0	1	1	2	2	3	3	3	3	4

Example of Kruskal 3



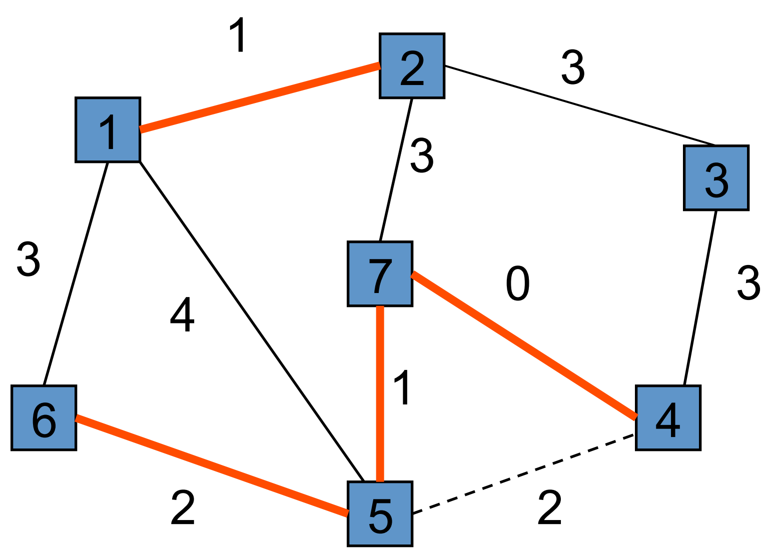
~~{7,4}~~	~~{2,1}~~	~~{7,5}~~	{5,6}	{5,4}	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
~~0~~	~~1~~	~~1~~	2	2	3	3	3	3	4

Example of Kruskal 4



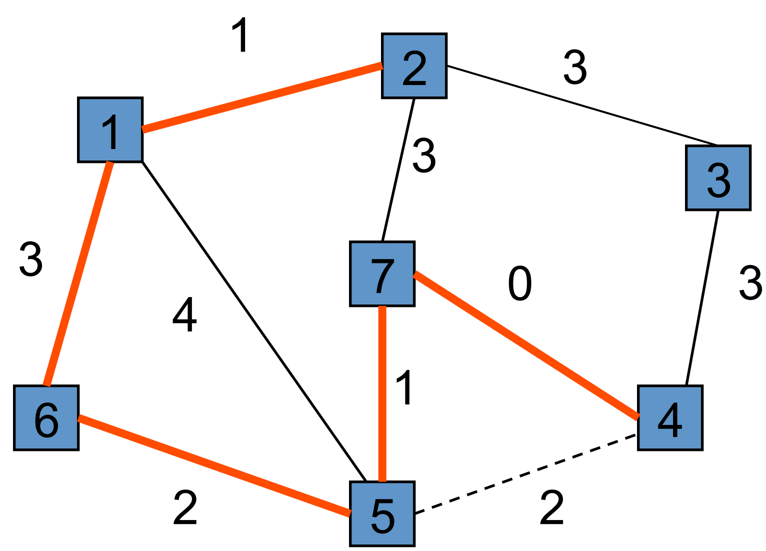
~~{7,4}~~	~~{2,1}~~	~~{7,5}~~	~~{5,6}~~	{5,4}	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
~~0~~	~~1~~	~~1~~	~~2~~	2	3	3	3	3	4

Example of Kruskal 5



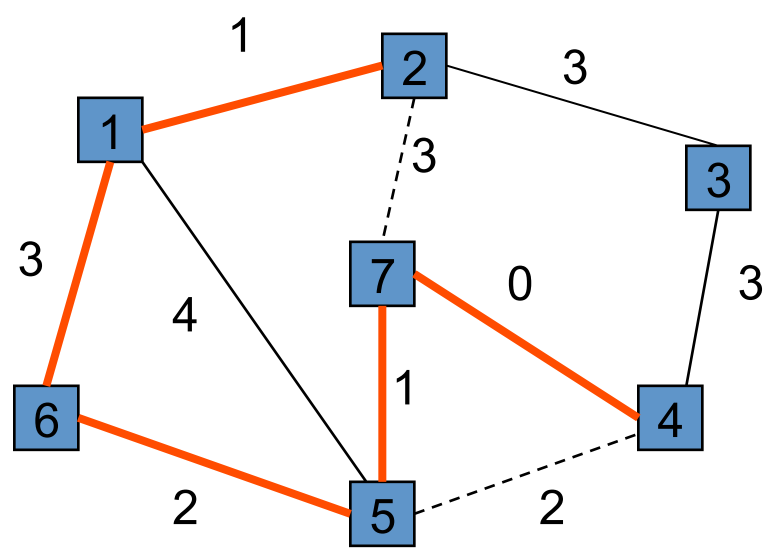
~~{7,4}~~	~~{2,1}~~	~~{7,5}~~	~~{5,6}~~	~~{5,4}~~	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
~~0~~	~~1~~	~~1~~	~~2~~	~~2~~	3	3	3	3	4

Example of Kruskal 6



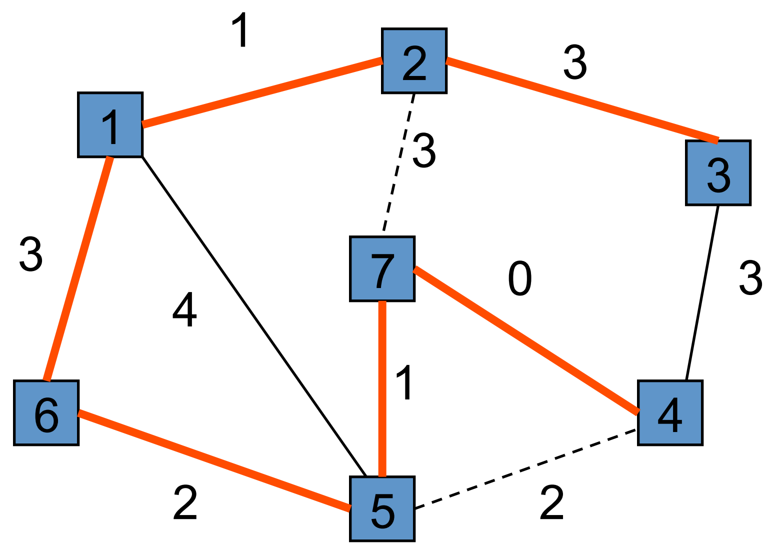
{7,4}	{2,1}	{7,5}	{5,6}	{5,4}	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
0	1	1	2	2	3	3	3	3	4

Example of Kruskal 7



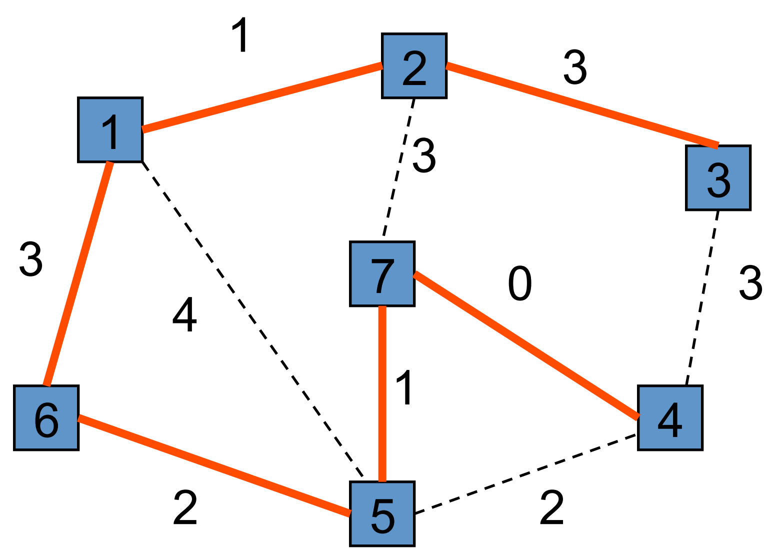
{7,4}	{2,1}	{7,5}	{5,6}	{5,4}	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
0	1	1	2	2	3	3	3	3	4

Example of Kruskal 7



{7,4}	{2,1}	{7,5}	{5,6}	{5,4}	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
0	1	1	2	2	3	3	3	3	4

Example of Kruskal 8,9



{7,4}	{2,1}	{7,5}	{5,6}	{5,4}	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
0	1	1	2	2	3	3	3	3	4

Kruskal's Algorithm Implementation

```
Kruskals():
   sort edges in increasing order of length (e_1, e_2, e_3, ..., e_m).

   T := {}.

   for i = 1 to m
      if e_i does not add a cycle:
         add e_i to T.

   return T.
```

- But how can we determine that adding e_i to *T* won't add a cycle?