CSE 373: Data Structures and Algorithms

Lecture 21: Graphs V

Dijkstra's algorithm

- **Dijkstra's algorithm**: finds shortest (minimum weight) path between a particular pair of vertices in a *weighted* directed graph with nonnegative edge weights
  - solves the "one vertex, shortest path" problem
  - basic algorithm concept: create a table of information about the currently known best way to reach each vertex (distance, previous vertex) and improve it until it reaches the best solution

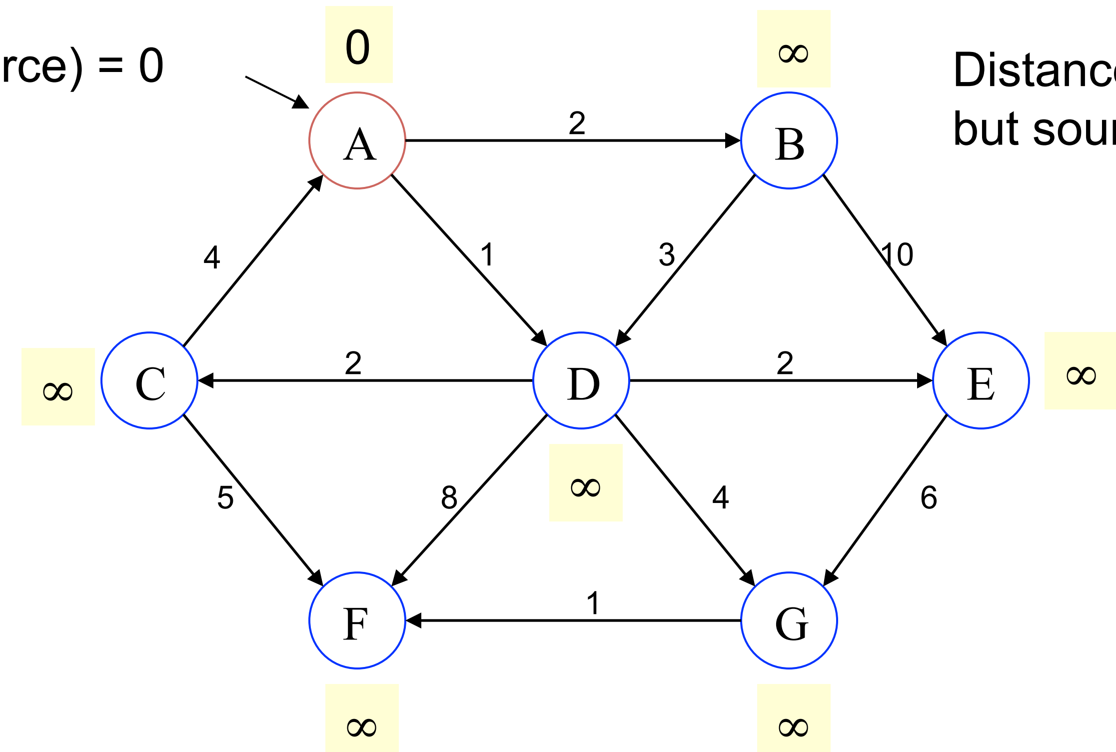
- in a graph where:
  - vertices represent cities,
  - edge weights represent driving distances between pairs of cities connected by a direct road,
    Dijkstra's algorithm can be used to find the shortest route between one city and any other

Dijkstra pseudocode

```
Dijkstra(v1, v2):
  for each vertex v:                    // Initialization
     v's distance := infinity.
     v's previous := none.
  v1's distance := 0.
  List := {all vertices}.

  while List is not empty:
     v := remove List vertex with minimum distance.
     mark v as known.
     for each unknown neighbor n of v:
        dist := v's distance + edge (v, n)'s weight.

        if dist is smaller than n's distance:
           n's distance := dist.
           n's previous := v.

  reconstruct path from v2 back to v1,
  following previous pointers.
```

Example: Initialization

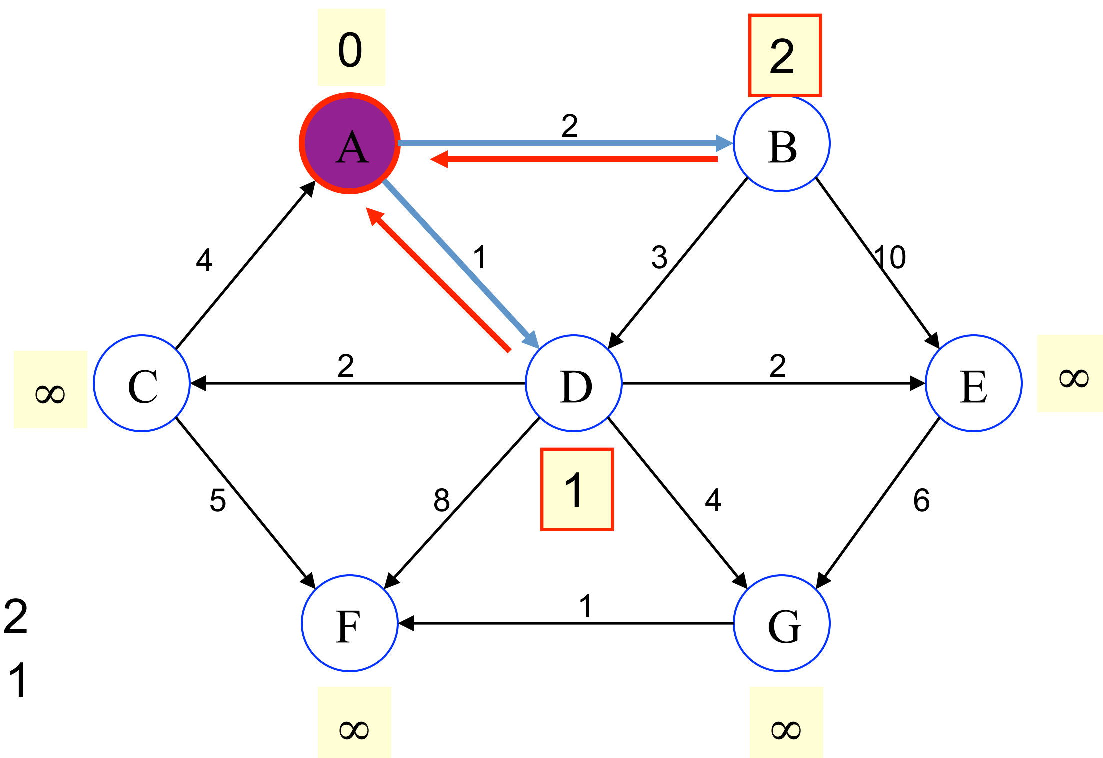
Distance(source) = 0



Distance (all vertices but source) = ∞

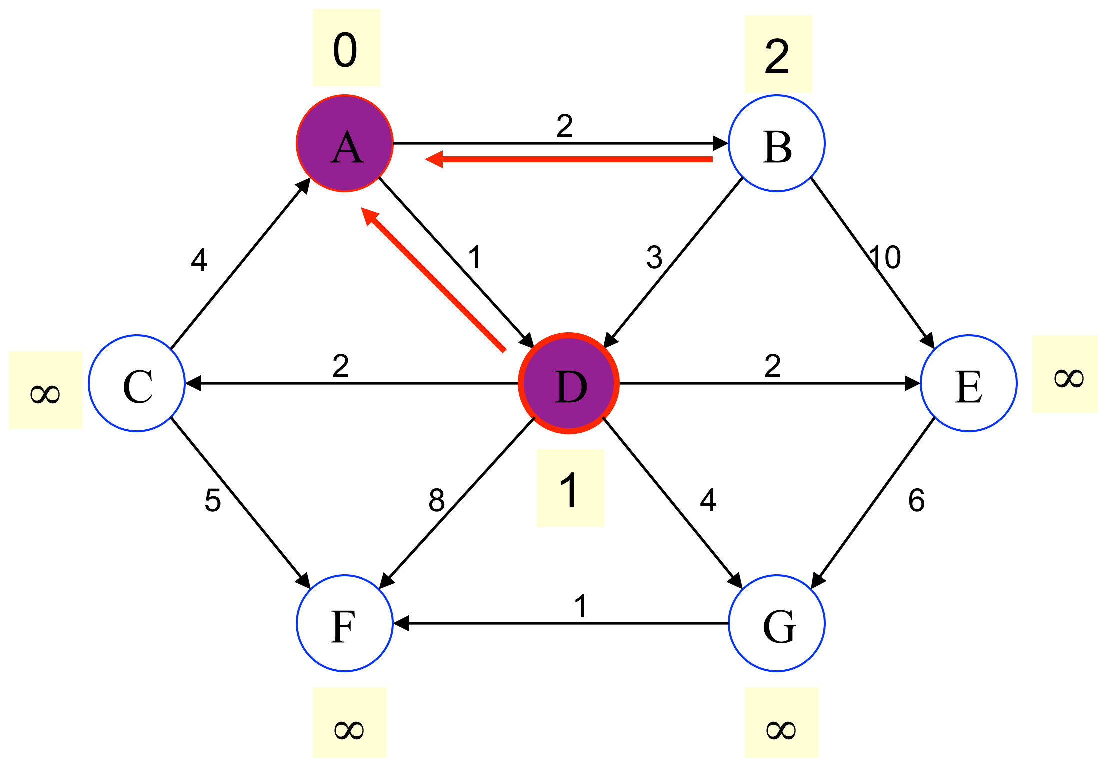
Pick vertex in List with minimum distance.

Example: Update neighbors' distance



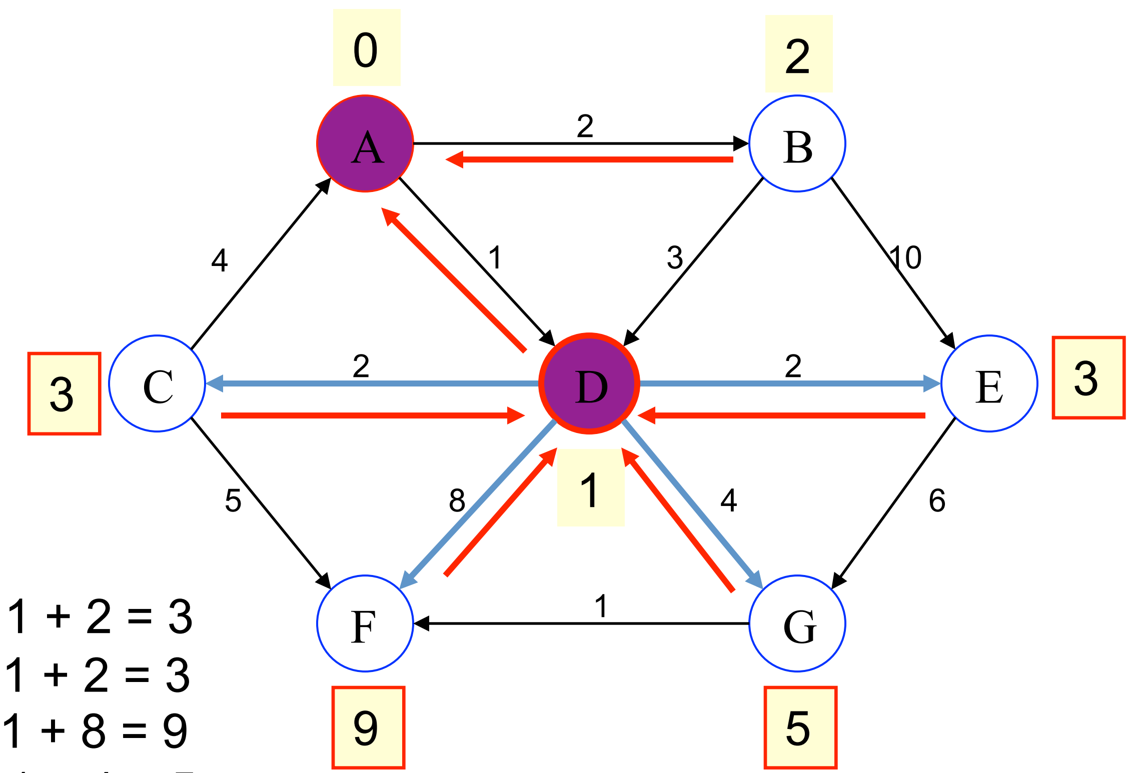
Distance(B) = 2
Distance(D) = 1

Example: Remove vertex with minimum distance



Pick vertex in List with minimum distance, i.e., D

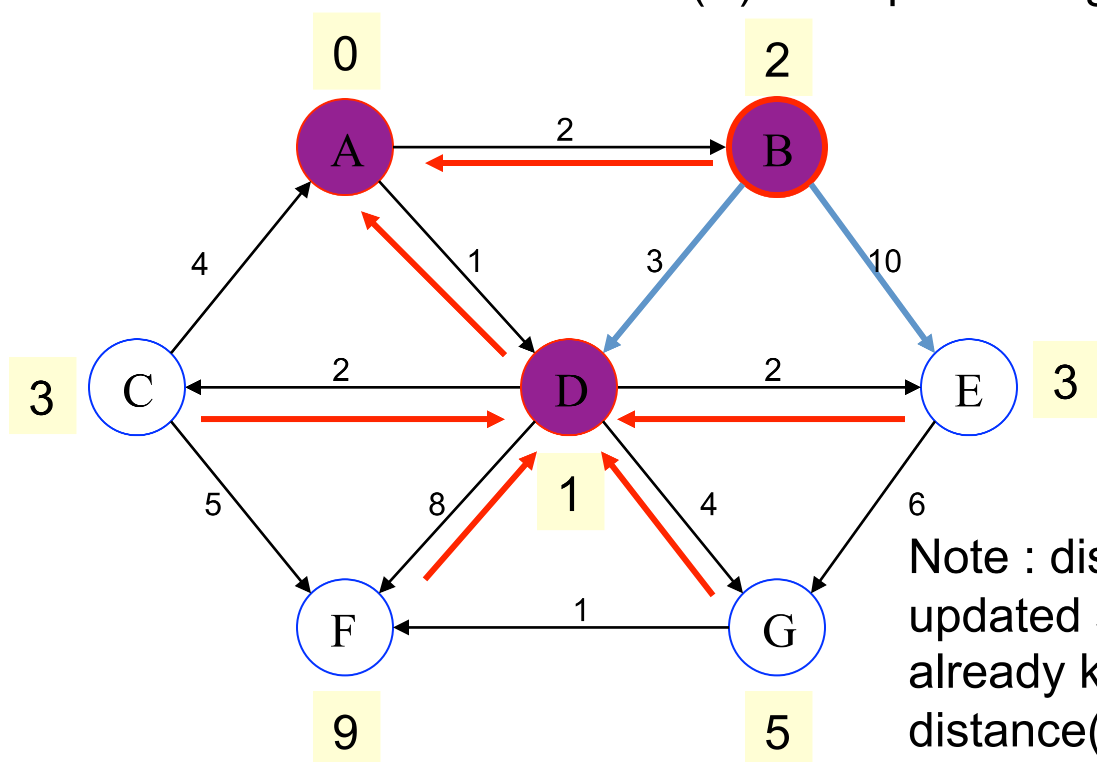
Example: Update neighbors



Distance(C) = 1 + 2 = 3
Distance(E) = 1 + 2 = 3
Distance(F) = 1 + 8 = 9
Distance(G) = 1 + 4 = 5

Example: Continued...

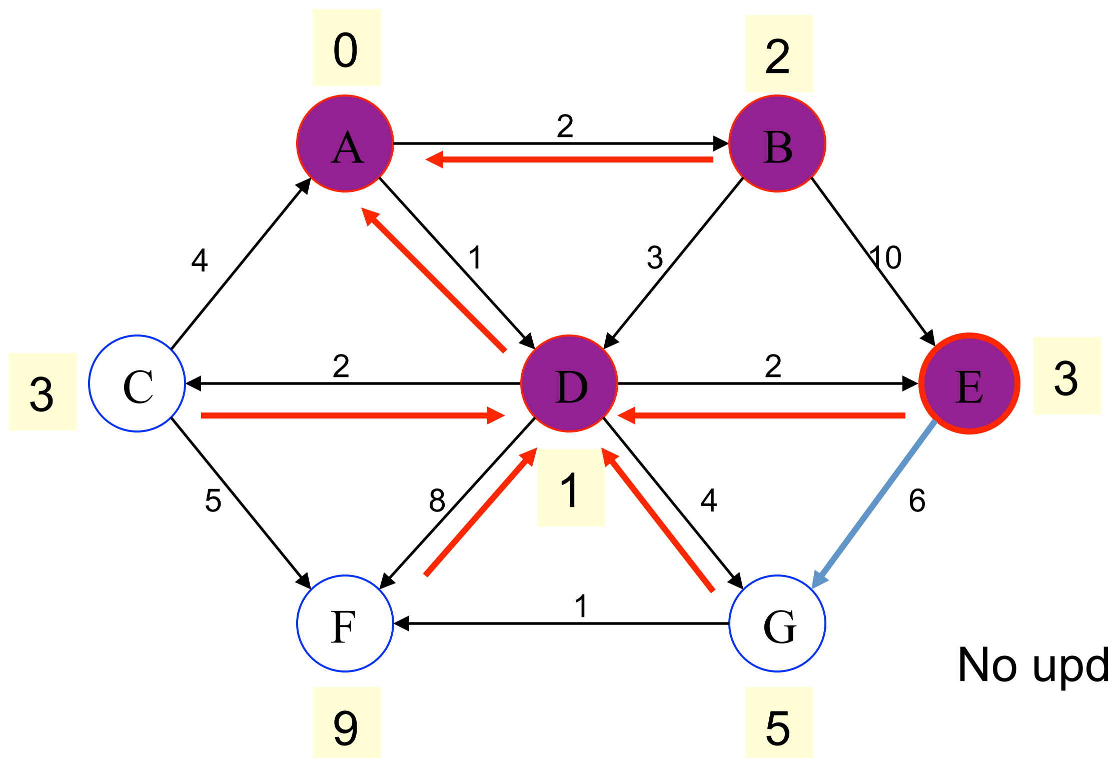
Pick vertex in List with minimum distance (B) and update neighbors



Note : distance(D) not updated since D is already known and distance(E) not updated since it is larger than previously computed

Example: Continued...

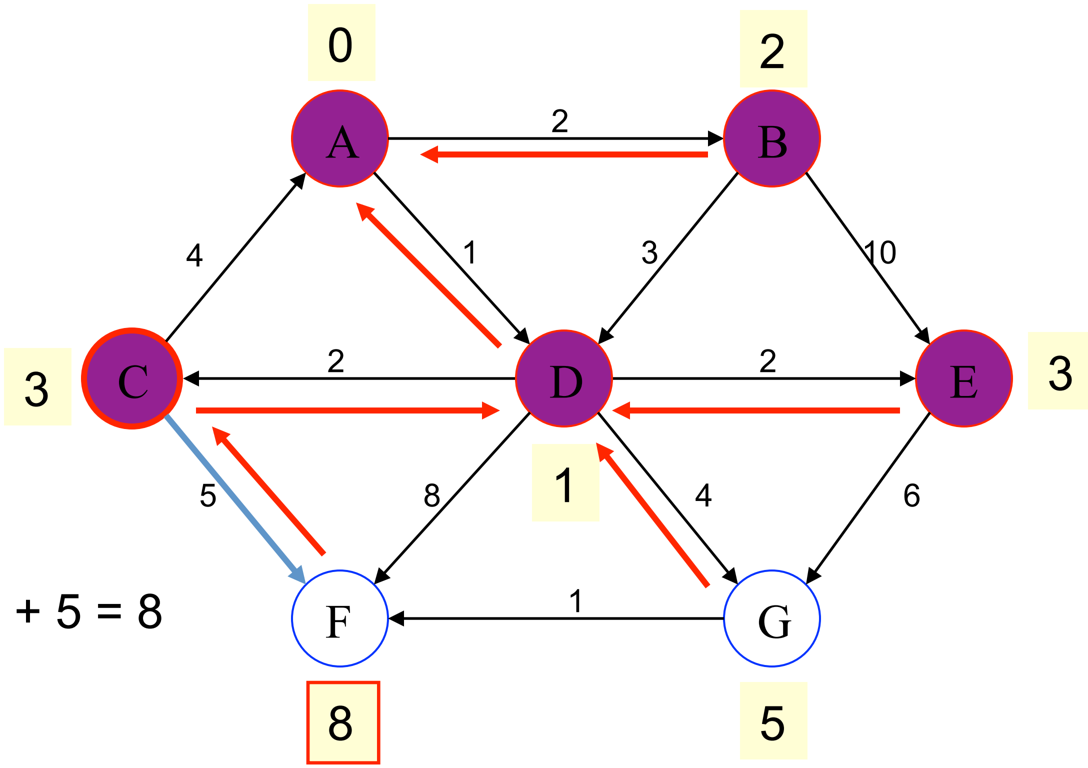
Pick vertex List with minimum distance (E) and update neighbors



No updating

Example: Continued...

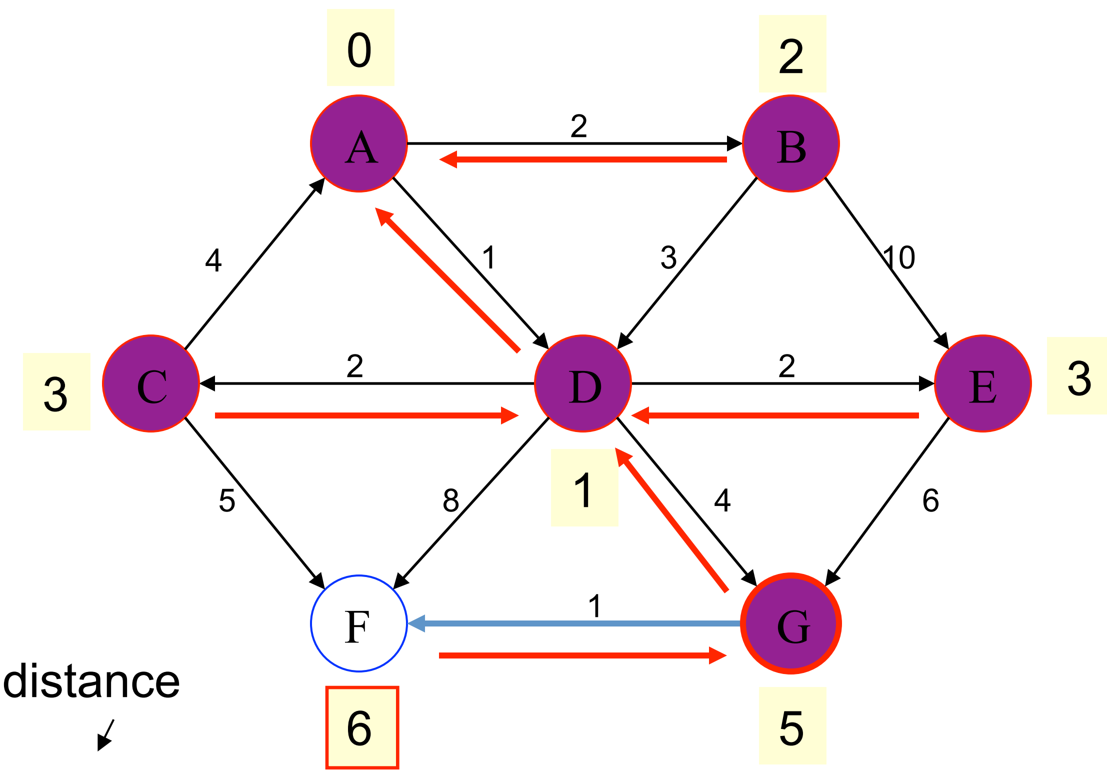
Pick vertex List with minimum distance (C) and update neighbors



Distance(F) = 3 + 5 = 8

Example: Continued...

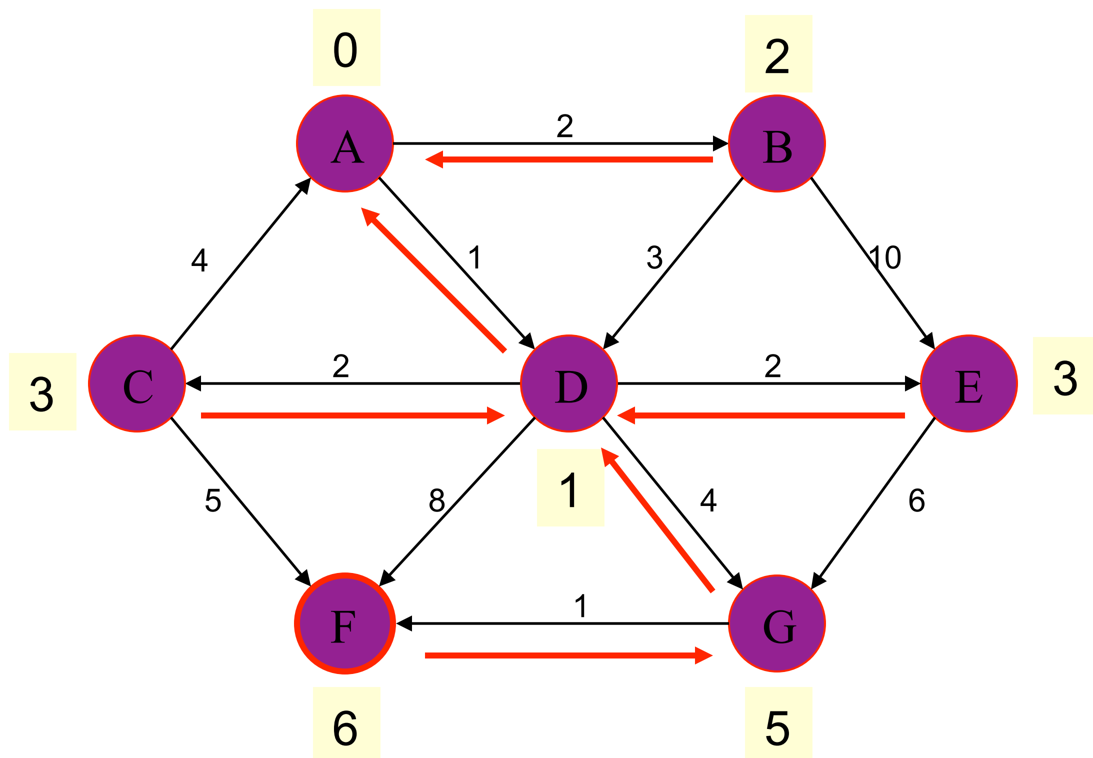
Pick vertex List with minimum distance (G) and update neighbors



Previous distance

Distance(F) = min (8, 5+1) = 6

Example (end)



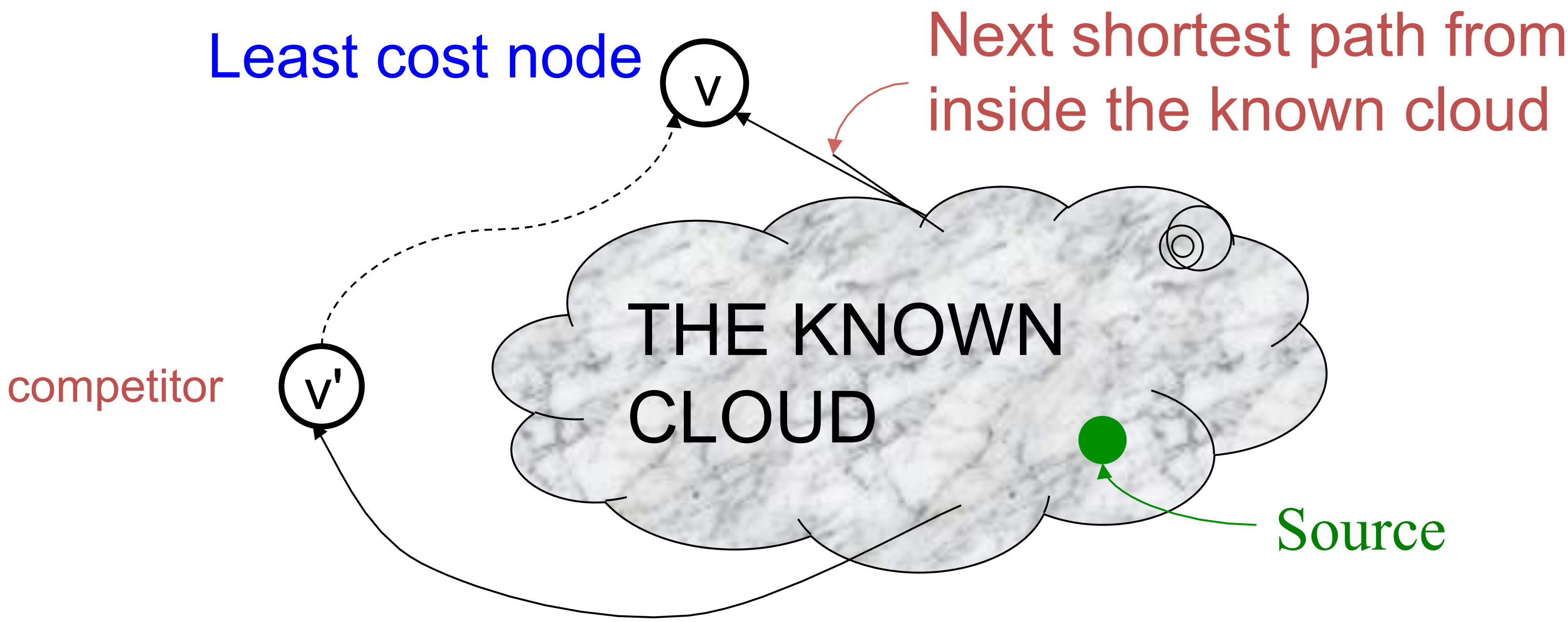
Pick vertex not in S with lowest cost (F) and update neighbors

Correctness

- Dijkstra's algorithm is a greedy algorithm
  - make choices that currently seem the best
  - locally optimal does not always mean globally optimal

- Correct because maintains following two properties:
  - for every known vertex, recorded distance is shortest distance to that vertex from source vertex
  - for every unknown vertex *v*, its recorded distance is shortest path distance to *v* from source vertex, considering only currently known vertices and *v*

"Cloudy" Proof: The Idea



- If the path to v is the next shortest path, the path to v' must be at least as long. Therefore, any path through v' to v cannot be shorter!

Dijkstra pseudocode

```
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       v's distance := infinity.
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   List := {all vertices}.

   while List is not empty:
       v := remove List vertex with minimum distance.
       mark v as known.
       for each unknown neighbor n of v:
           dist := v's distance + edge (v, n)'s weight.

           if dist is smaller than n's distance:
               n's distance := dist.
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   reconstruct path from v2 back to v1,
   following previous pointers.
```

Time Complexity: Using List

The simplest implementation of the Dijkstra's algorithm stores vertices in an ordinary linked list or array

- Good for dense graphs (many edges)

- $|V|$ vertices and $|E|$ edges
- Initialization $O(|V|)$
- While loop $O(|V|)$
  - Find and remove min distance vertices $O(|V|)$
  - Potentially $|E|$ updates
    - Update costs $O(1)$
- Reconstruct path $O(|E|)$

Total time $O(|V^2| + |E|) = O(|V^2|)$

Time Complexity: Priority Queue

For sparse graphs, (i.e. graphs with much less than $|V^2|$ edges) Dijkstra's implemented more efficiently by *priority queue*

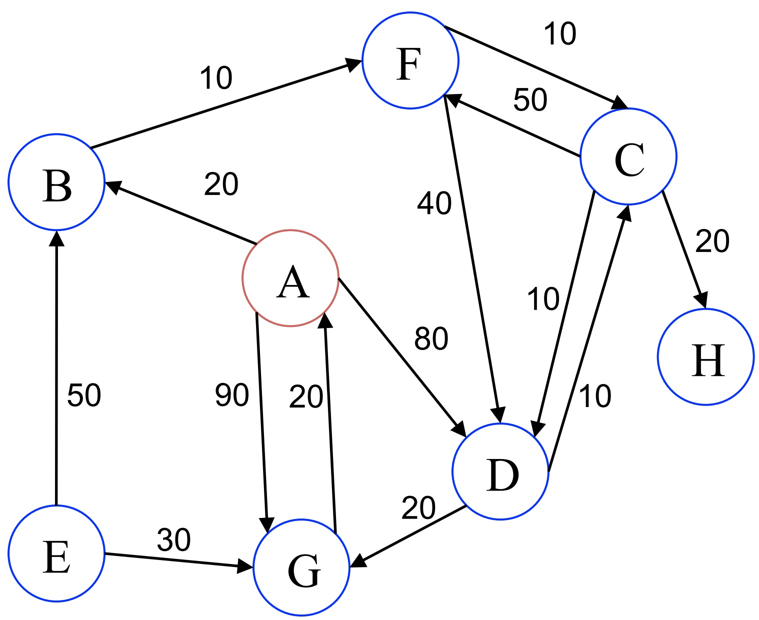
- Initialization $O(|V|)$ using $O(|V|)$ buildHeap
- While loop $O(|V|)$
  - Find and remove min distance vertices $O(\log |V|)$ using $O(\log |V|)$ deleteMin
  - Potentially $|E|$ updates
    - Update costs $O(\log |V|)$ using decreaseKey
- Reconstruct path $O(|E|)$

Total time $O(|V|\log|V| + |E|\log|V|) = O(|E|\log|V|)$

- $|V| = O(|E|)$ assuming a connected graph

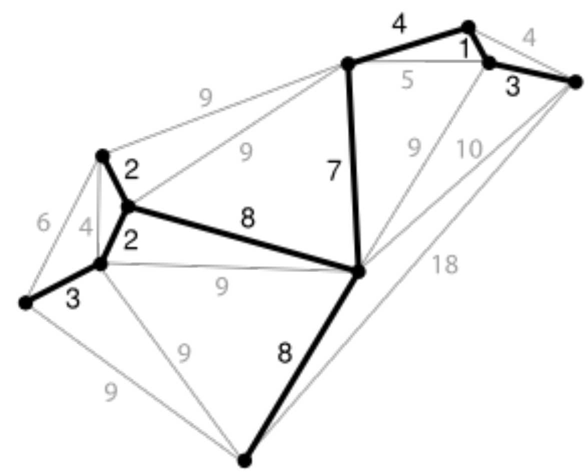
Dijkstra's Exercise

- Use Dijkstra's algorithm to determine the lowest cost path from vertex A to all of the other vertices in the graph. Keep track of previous vertices so that you can reconstruct the path later.



Minimum spanning tree

- **tree**: a connected, directed acyclic graph
- **spanning tree**: a subgraph of a graph, which meets the constraints to be a tree (connected, acyclic) and connects every vertex of the original graph
- **minimum spanning tree**: a spanning tree with weight less than or equal to any other spanning tree for the given graph



Min. span. tree applications

- Consider a cable TV company laying cable to a new neighborhood...
  - If it is constrained to bury the cable only along certain paths, then there would be a graph representing which points are connected by those paths.
  - Some of those paths might be more expensive, because they are longer, or require the cable to be buried deeper.
    - These paths would be represented by edges with larger weights.
  - A spanning tree for that graph would be a subset of those paths that has no cycles but still connects to every house.
    - There might be several spanning trees possible. A minimum spanning tree would be one with the lowest total cost.