Sorting

(Chapter 7 in Weiss)

CSE 373
Data Structures & Algorithms
Ruth Anderson

Today's Outline

- Announcements
  - No class on Monday 5/31
  - Homework #6/7 due Thurs 6/3 at 11:45pm.
- **Graphs**
  - **Minimum Spanning Trees**
  - **Sorting**

Why Sort?

Sorting: *The Big Picture*

Given n comparable elements in an array, sort them in an increasing (or decreasing) order.

Simple algorithms: $O(n^2)$	Fancier algorithms: $O(n \log n)$	Comparison lower bound: $\Omega(n \log n)$	Specialized algorithms: $O(n)$	Handling huge data sets
Insertion sort Selection sort Bubble sort Shell sort …	Heap sort Merge sort Quick sort …		Bucket sort Radix sort	External sorting

Insertion Sort: Idea

- At the k^{th} step, put the k^{th} input element in the correct place among the first k elements
- **Result**: After the k^{th} step, the first k elements are sorted.

Runtime:

worst case :
best case :
average case :

Selection Sort: Idea

- Find **the** smallest element, put it 1st
- Find **the** next smallest element, put it 2nd
- Find **the** next smallest, put it 3rd
- And so on …

Student Activity

```
Mystery(int array a[]) {
  for (int p = 1; p < length; p++) {
     int tmp = a[p];
     for (int j = p; j > 0 && tmp < a[j-1]; j--)
          a[j] = a[j-1];
     a[j] = tmp;
  }
}
```

What sort is this?

What is its running time?
Best?
Avg?
Worst?

Selection Sort: Code

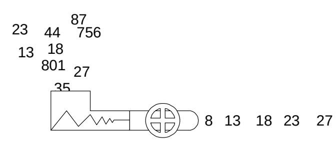
```
void SelectionSort (Array a[0..n-1]) {
     for (i=0, i<n; ++i) {
          j = Find index of smallest entry in a[i..n-1]
          Swap(a[i],a[j])
     }

}
```

Runtime:
worst case :
best case :
average case :

Student Activity

Sorts using other data structures:

HeapSort: Using Priority Queue ADT (heap)



Shove all elements into a priority queue, take them out smallest to largest.

Runtime:

AVL Sort

Runtime:

Would the simpler "Splay sort" take any longer than this?

Divide and conquer

- A common and important technique in algorithms
  - Divide problem into parts
  - Solve parts
  - Merge solutions

Divide and Conquer Sorting

- MergeSort:
  - Divide array into two halves
  - Recursively sort left and right halves
  - Merge halves
- QuickSort:
  - Partition array into small items and large items
  - Recursively sort the two smaller portions

Merge Sort?

Merge Sort

```
MergeSort (Array [1..n])
1. Split Array in half
2. Recursively sort each half
3. Merge two halves together
```

"The 2-pointer method"

```
Merge (a1[1..n],a2[1..n])
i1=1, i2=1
While (i1<n, i2<n) {
      if (a1[i1] < a2[i2]) {
            Next is a1[i1]
            i1++
      } else {
            Next is a2[i2]
            i2++
      }
}
Now throw in the dregs…
```


Perform mergesort

8	2	9	4	5	3	1	6

Divide:

Divide:

Divide:

Divide:

Merge:

Merge:

Merge:

Merge Sort: Complexity

Auxiliary array

- The merging requires an auxiliary array

2	4	8	9	1	3	5	6

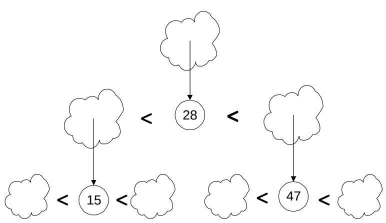
Properties of MergeSort

- Definition: In-place
  - Can be done without extra memory
- MergeSort: Not in-place
  - Requires Auxiliary array

Quicksort

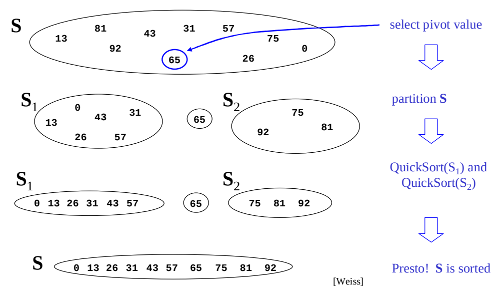
- Uses divide and conquer
- Doesn’t require O(N) extra space like MergeSort
- Partition into left and right
  - Left less than pivot
  - Right greater than pivot
- Recursively sort left and right
- Concatenate left and right

Quick Sort



1. Pick a “pivot”
2. Divide into less-than & greater-than pivot
3. Sort each side recursively

The steps of QuickSort



Selecting the pivot

- Ideas?

Perform quicksort

8	2	9	4	5	3	1	6

Divide:

Divide:

Divide:

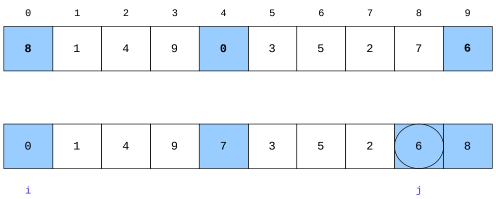
Divide:

Merge:

Merge:

Merge:

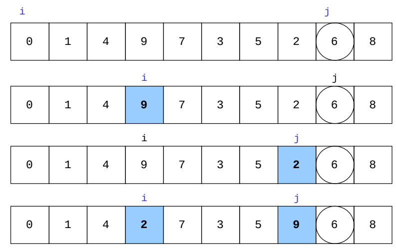
QuickSort Example



•Choose the pivot as the median of three.

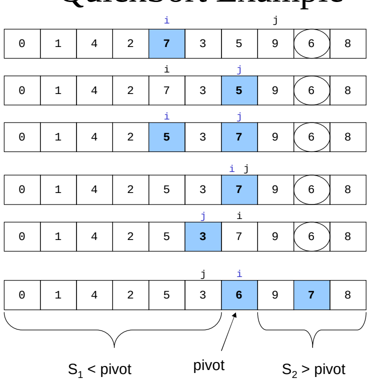
•Place the pivot and the largest at the right and the smallest at the left

QuickSort Example



•Move i to the right to be larger than pivot.
•Move j to the left to be smaller than pivot.
•Swap

QuickSort Example



Recursive Quicksort

```
Quicksort(A[]: integer array, left,right : integer): {
pivotindex : integer;
if left + CUTOFF ≤ right then
  pivot := median3(A,left,right);
  pivotindex := Partition(A,left,right-1,pivot);
  Quicksort(A, left, pivotindex – 1);
  Quicksort(A, pivotindex + 1, right);
else
  Insertionsort(A,left,right);
}
```

Don't use quicksort for small arrays.
CUTOFF = 10 is reasonable.

Cutoff for quicksort

- Quicksort performs poorly on small sets
  - In fact insertion sort does better
- Small sets occur often due to the recursion
- So below a certain set size, or cutoff, switch to insertion sort

Student Activity

Recurrence Relations

Write the recurrence relation for QuickSort:

- Best Case:

- Worst Case:

QuickSort: Best case complexity

QuickSort: Worst case complexity

QuickSort: Average case complexity

Turns out to be O(*n* log *n*)

See Section 7.7.5 for an idea of the proof.
Don't need to know proof details for this course.

Quicksort Complexity

- Worst case: $O(n^2)$
- Best case: O(n log n)
- Average Case: O(n log n)

Mergesort and massive data

- MergeSort is the basis of massive sorting
- Quicksort and Heapsort both jump all over the array, leading to expensive random disk accesses
- Mergesort scans linearly through arrays, leading to (relatively) efficient sequential disk access
- In-memory sorting of reasonable blocks can be combined with larger mergesorts
- Mergesort can leverage multiple disks

Features of Sorting Algorithms

- In-place
  - Sorted items occupy the same space as the original items. (No copying required, only O(1) extra space if any.)
- Stable
  - Items in input with the same value end up in the same order as when they began.

How fast can we sort?

- Heapsort, Mergesort, and Quicksort all run in O(N log N) best case running time
- Can we do any better?
- No, if the basic action is a comparison.

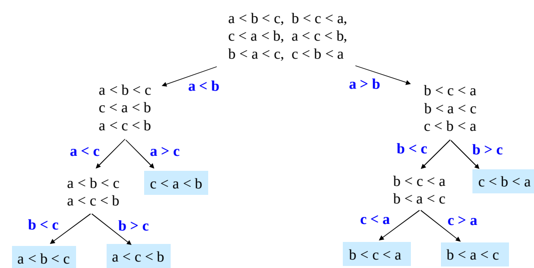
Sorting Model

- Recall our basic assumption: we can only compare two elements at a time
  - we can only reduce the possible solution space by half each time we make a comparison
- Suppose you are given N elements
  - Assume no duplicates
- How many possible orderings can you get?
  - Example: a, b, c (N = 3)

Permutations

- How many possible orderings can you get?
  - Example: a, b, c (N = 3)
  - (a b c), (a c b), (b a c), (b c a), (c a b), (c b a)
  - 6 orderings = 3·2·1 = 3! (ie, “3 factorial”)
  - All the possible permutations of a set of 3 elements
- For N elements
  - N choices for the first position, (N-1) choices for the second position, …, (2) choices, 1 choice
  - N(N-1)(N-2)⋯(2)(1)= N! possible orderings

Decision Tree



The leaves contain all the possible orderings of a, b, c

Student Activity

Lower bound on Height

- A binary tree of height h has **at most** *how many* leaves?

  L

- A binary tree with L leaves has height **at least**:

  h

- The decision tree has how many leaves:
- So the decision tree has height:

  h

log(*N*!) is Ω(*N*log*N*)

$$
\begin{aligned}
\log(N!) &= \log\big(N\cdot(N-1)\cdot(N-2)\cdots(2)\cdot(1)\big)\\
&= \log N + \log(N-1) + \log(N-2) + \cdots + \log 2 + \log 1\\
&\ge \log N + \log(N-1) + \log(N-2) + \cdots + \log\frac{N}{2}\\
&\ge \frac{N}{2}\log\frac{N}{2}\\
&\ge \frac{N}{2}(\log N - \log 2) = \frac{N}{2}\log N - \frac{N}{2}\\
&= \Omega(N\log N)
\end{aligned}
$$

select just the first N/2 terms

each of the selected terms is ≥ logN/2

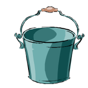
Ω(N log N)

- Run time of any comparison-based sorting algorithm is **Ω(N log N)**
- Can we do better if we don't use comparisons?

BucketSort (aka BinSort)

If all values to be sorted are *known* to be between 1 and *K*, create an array `count` of size *K*, **increment** counts while traversing the input, and finally output the result.

Example *K*=5. Input = (5,1,3,4,3,2,1,1,5,4,5)



`count` array	
1	
2	
3	
4	
5	

Running time to sort n items?

BucketSort Complexity: O(*n*+*K*)

- Case 1: *K* is a constant
  - BinSort is linear time
- Case 2: *K* is variable
  - Not simply linear time
- Case 3: *K* is constant but large (e.g. 2^{32})
  - ???

Fixing impracticality: RadixSort

- Radix = "The base of a number system"
  - We'll use 10 for convenience, but could be anything
- Idea: BucketSort on each **digit**, least significant to most significant (lsd to msd)

Radix Sort Example (1st pass)

Input data: 478, 537, 9, 721, 3, 38, 123, 67

Bucket sort by 1's digit

0	1	2	3	4	5	6	7	8	9
	721		3 123				537 67	478 38	9

After 1st pass: 721, 3, 123, 537, 67, 478, 38, 9

This example uses B=10 and base 10 digits for simplicity of demonstration. Larger bucket counts should be used in an actual implementation.

Radix Sort Example (2nd pass)

After 1st pass: 721, 3, 123, 537, 67, 478, 38, 9

Bucket sort by 10's digit

0	1	2	3	4	5	6	7	8	9
03 09		721 123	537 38			67	478		

After 2nd pass: 3, 9, 721, 123, 537, 38, 67, 478

Radix Sort Example (3rd pass)

After 2nd pass: 3, 9, 721, 123, 537, 38, 67, 478

Bucket sort by 100's digit

0	1	2	3	4	5	6	7	8	9
003 009 038 067	123			478	537		721		

After 3rd pass: 3, 9, 38, 67, 123, 478, 537, 721

Invariant: after k passes the low order k digits are sorted.

Student Activity

RadixSort

- Input:126, 328, 636, 341, 416, 131, 328

BucketSort on lsd:

BucketSort on next-higher digit:

BucketSort on msd:

Radixsort: Complexity

- How many passes?
- How much work per pass?
- Total time?
- Conclusion?
- In practice
  - RadixSort only good for large number of elements with relatively small values
  - Hard on the cache compared to MergeSort/QuickSort

Internal versus External Sorting

- Need sorting algorithms that minimize disk/tape access time
- **External sorting** – Basic Idea:
  - Load chunk of data into RAM, sort, store this "run" on disk/tape
  - Use the Merge routine from Mergesort to merge runs
  - Repeat until you have only one run (one sorted chunk)
  - Text gives some examples

Summary of sorting

- $O(n^2)$ average, worst case:
  - Selection Sort, Bubblesort, Insertion sort
- $O(n^{4/3})$ worst case:
  - Shell sort
- O(n log n) average case:
  - Heapsort: in-place, not stable
  - Mergesort: O(n) extra space, stable, massive data
  - Quicksort: Claimed fastest in practice, but $O(n^2)$ worst case. Recursion/stack requirement. Not stable.
- $\Omega(n \log n)$ worst and average case:
  - Any comparison-based sorting algorithm
- O(n)
  - Radix sort: Fast and stable. Not comparison based. Not in-place. Poor memory locality can undercut performance.