AVL Trees

(4.4 in Weiss)

CSE 373
Data Structures & Algorithms
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Spring 2010

Today's Outline

- **Announcements**
  - Assignment #2 due AT THE BEGINNING OF LECTURE, Fri, April 16, 2010.
- **Today's Topics:**
  - **Binary Search Trees (Weiss 4.1-4.3)**
  - **AVL Trees (Weiss 4.4)**

The AVL Balance Condition

Left and right subtrees of *every node*
have equal *heights* **differing by at most 1**

Define: **balance**(x) = height(x.left) – height(x.right)

AVL property: **$-1 \le$ balance(x) ≤ 1, for every node x**

- Ensures small depth
  - Will prove this by showing that an AVL tree of height h must have a lot of (i.e. $\Theta(2^h)$) nodes
- Easy to maintain
  - Using single and double rotations

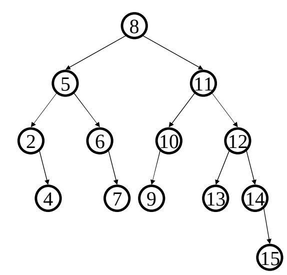
The AVL Tree Data Structure

Structural properties

1. Binary tree property (0,1, or 2 children)
2. Heights of left and right subtrees of *every node* **differ by at most 1**

Result:

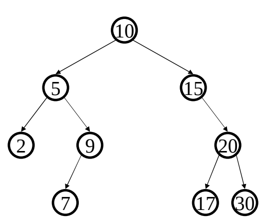
Worst case depth of any node is: O(log n)



Ordering property

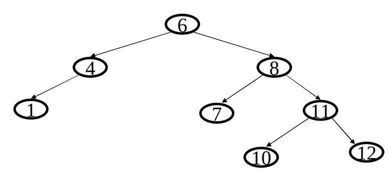
- Same as for BST

Is this an AVL Tree?



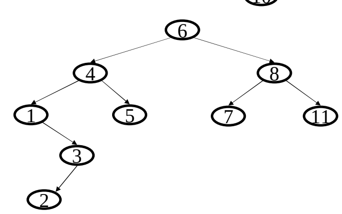
`NULL`s have height **`-1`**

Circle One:



AVL

Not AVL



AVL

Not AVL

Student Activity

If *not* AVL, put a **box** around nodes where AVL property is violated.

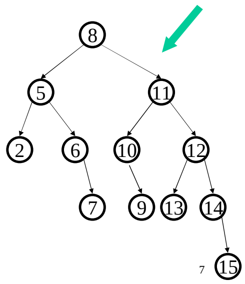
Proving Shallowness Bound

Let **S**(h) be the min # of nodes in an AVL tree of height h

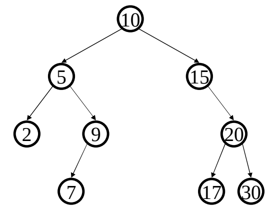
Claim: $\mathbf{S}(h) = \mathbf{S}(h-1) + \mathbf{S}(h-2) + 1$

Solution of recurrence: $\mathbf{S}(h) = \Theta(2^h)$ (like Fibonacci numbers)

AVL tree of height h=4 with the min # of nodes



Testing the Balance Property



We need to be able to:

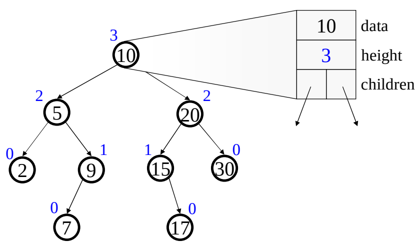
1.

2.

3.

`NULL`s have height **`-1`**

An AVL Tree



AVL trees: find, insert

- **AVL find**:
  - same as BST find.
- **AVL insert**:
  - same as BST insert, *except* may need to "fix" the AVL tree after inserting new value.

AVL tree insert

Let x be the node where an imbalance occurs.

Four cases to consider. The insertion is in the

1. left subtree of the left child of x.
2. right subtree of the left child of x.
3. left subtree of the right child of x.
4. right subtree of the right child of x.

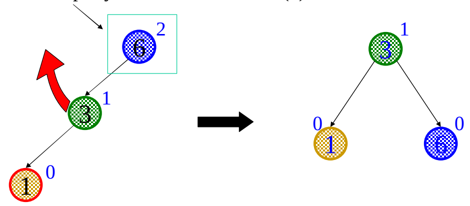
Idea: Cases 1 & 4 are solved by a single rotation.
Cases 2 & 3 are solved by a double rotation.

Bad Case #1

Insert(6)
Insert(3)
Insert(1)

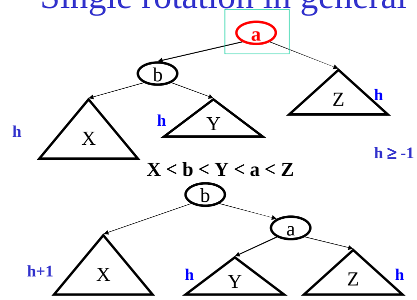
Fix: Apply Single Rotation

AVL Property violated at this node (x)



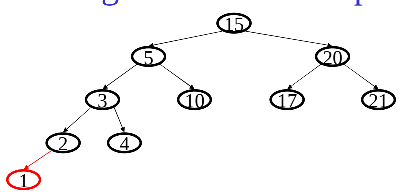
Single Rotation:
1. Rotate between x and child

Single rotation in general

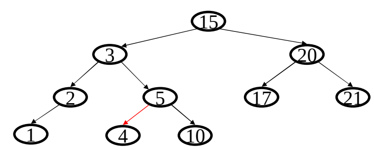


Height of tree before? Height of tree after? Effect on Ancestors?

Single rotation example



Soln:

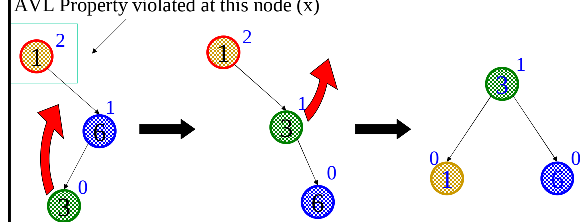


Bad Case #3

Insert(1)
Insert(6)
Insert(3)

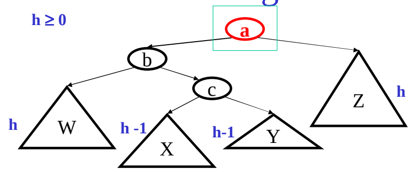
Fix: Apply Double Rotation

AVL Property violated at this node (x)

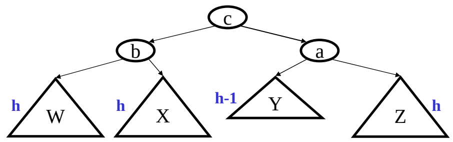


Double Rotation
1. Rotate between x’s child and grandchild
2. Rotate between x and x’s new child

Double rotation in general

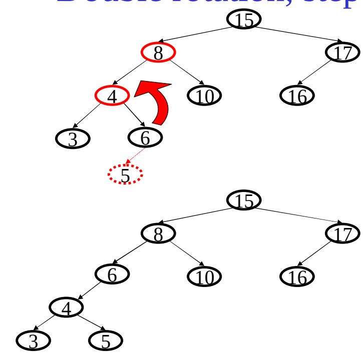


W < b <X < c < Y < a < Z

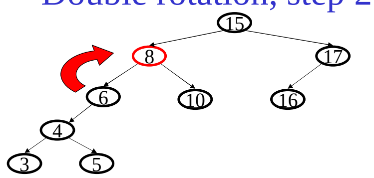


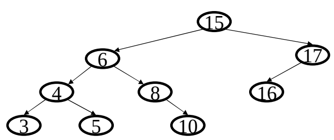
Height of tree before? Height of tree after? Effect on Ancestors?

Double rotation, step 1



Double rotation, step 2





Imbalance at node X

Single Rotation

1. Rotate between x and child

Double Rotation

1. Rotate between x's child and grandchild
2. Rotate between x and x's new child

Insert into an AVL tree: a b e c d

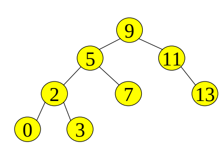
Student Activity

Circle your final answer

Single and Double Rotations:

Inserting what integer values would cause the tree to need a:

1. single rotation?



2. double rotation?

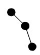
3. no rotation?

Student Activity

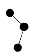
Insertion into AVL tree

1. Find spot for new key
2. Hang new node there with this key
3. Search back up the path for imbalance
4. If there is an imbalance:

   case #1: Perform single rotation and exit



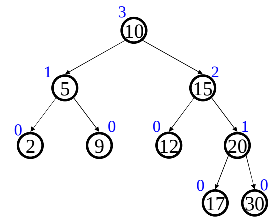
   case #2: Perform double rotation and exit



Both rotations keep the subtree height unchanged.
Hence only one rotation is sufficient!

Easy Insert

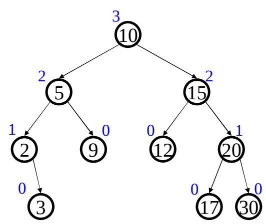
Insert(3)



Unbalanced?

Hard Insert

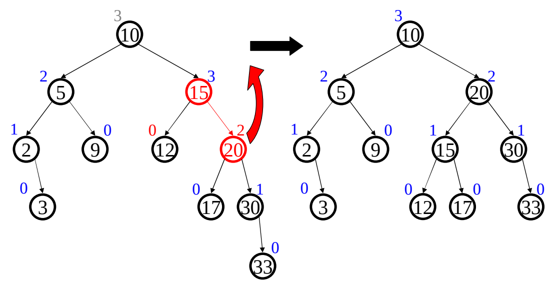
Insert(33)



Unbalanced?

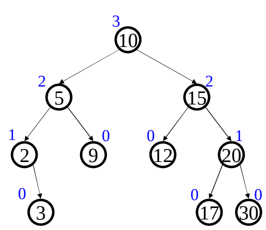
How to fix?

Single Rotation



Hard Insert

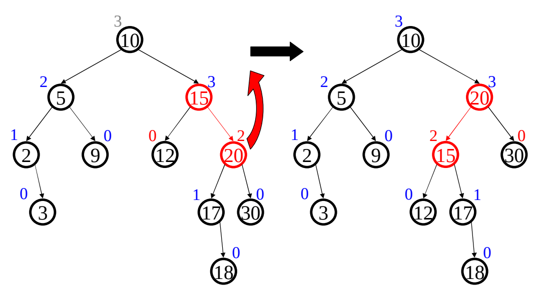
Insert(18)



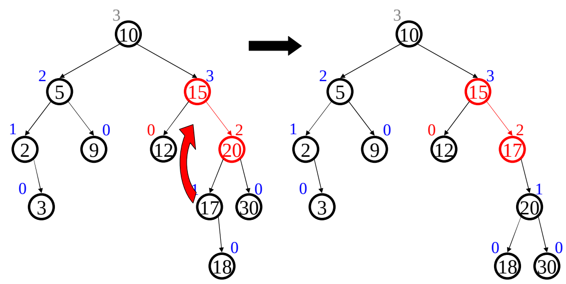
Unbalanced?

How to fix?

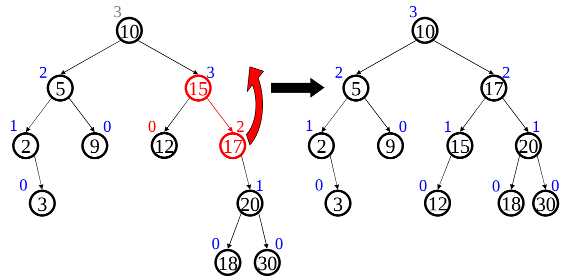
Single Rotation (oops!)



Double Rotation (Step #1)



Double Rotation (Step #2)



AVL Trees Revisited

- Balance condition:
  For every node *x*, $-1 \le \text{balance}(x) \le 1$
  - Strong enough : Worst case depth is O(log *n*)
  - Easy to maintain : *one* single or double rotation
- Guaranteed O(log *n*) running time for
  - Find ?
  - Insert ?
  - Delete ?
  - buildTree ?

AVL Trees Revisited

- What extra info did we maintain in each node?
- Where were rotations performed?
- How did we locate this node?