Graphs: Traversals and Shortest Path Algorithms (Chapter 9)

CSE 373
Data Structures and Algorithms

Today’s Outline

• Announcements
  – Homework #6/7 coming soon.

• Graphs
  – Topological Sort
  – Shortest Paths Algorithms

Graph Traversals

• Breadth-first search
  – explore all adjacent nodes, then for each of those nodes explore all adjacent nodes

• Depth-first search
  – explore first child node, then its first child node, etc. until goal node is found or node has no children. Then backtrack, repeat with sibling.

• Both:
  – Work for arbitrary (directed or undirected) graphs
  – Must mark visited vertices so you do not go into an infinite loop!

• Either can be used to determine connectivity:
  – Is there a path between two given vertices?
  – Is the graph (weakly) connected?

• Which one:
  – Uses a queue?
  – Uses a stack?
  – Always finds the shortest path (for unweighted graphs)?

The Shortest Path Problem

Given a graph $G$, edge costs $c_{i,j}$, and vertices $s$ and $t$ in $G$, find the shortest path from $s$ to $t$.

For a path $p = v_0 \ v_1 \ v_2 \ ... \ v_k$

  – unweighted length of path $p = k$ (a.k.a. length)

  – weighted length of path $p = \sum_{i=0}^{k-1} c_{i,i+1}$ (a.k.a. cost)

Path length equals path cost when?

Single Source Shortest Paths (SSSP)

Given a graph $G$, edge costs $c_{i,j}$, and vertex $s$, find the shortest paths from $s$ to all vertices in $G$.

– Is this harder or easier than finding the shortest path from $s$ to $t$?

All Pairs Shortest Paths (APSP)

Given a graph $G$ and edge costs $c_{i,j}$, find the shortest paths between all pairs of vertices in $G$.

– Is this harder or easier than SSSP?

– Could we use SSSP as a subroutine to solve this?
Variations of SSSP

- Weighted vs. unweighted
- Directed vs undirected
- Cyclic vs. acyclic
- Positive weights only vs. negative weights allowed
- Shortest path vs. longest path
- ...

Applications

- Network routing
- Driving directions
- Cheap flight tickets
- Critical paths in project management (see textbook)
- ...

SSSP: Unweighted Version

Ideas?

void Graph::unweighted (Vertex s){
  Queue q(NUM_VERTICES);
  Vertex v, w;
  for all vertices v do {v.dist = INFINITY;}
  s.dist = 0;
  q.enqueue(s);
  while (!q.isEmpty()){
    v = q.dequeue();
    for each w adjacent to v
      if (w.dist == INFINITY)
        w.dist = v.dist + 1;
        w.path = v;
        q.enqueue(w);
  }
}

each edge examined at most once — if adjacency lists are used

each vertex enqueued at most once

SSSP: Unweighted Version

void Graph::unweighted (Vertex s){
  Queue q(NUM_VERTICES);
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  for all vertices v do {v.dist = INFINITY;}
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      if (w.dist == INFINITY)
        w.dist = v.dist + 1;
        w.path = v;
        q.enqueue(w);
  }
  
  total running time: O( )

Edsger Wybe Dijkstra

(1930-2002)

- Legendary figure in computer science; was a professor at University of Texas.
- Invented concepts of structured programming, synchronization, and "semaphores" for controlling computer processes.
- Supported teaching programming without computers (pencil and paper)
- 1972 Turing Award
- “In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind.”

Weighted SSSP: The Quest For Food

Can we calculate shortest distance to all nodes from MGH 241?
Dijkstra’s Algorithm for Single Source Shortest Path

• Similar to breadth-first search, but uses a heap instead of a queue:
  – Always select (expand) the vertex that has a lowest-cost path to the start vertex
• Correctly handles the case where the lowest-cost (shortest) path to a vertex is not the one with fewest edges

Dijkstra’s Algorithm: Idea

Adapt BFS to handle weighted graphs

Two kinds of vertices:
  – Finished or known vertices
    • Shortest distance has been computed
  – Unknown vertices
    • Have tentative distance

Dijkstra’s Algorithm: Idea

At each step:
1) Pick closest unknown vertex
2) Add it to known vertices
3) Update distances

Dijkstra’s Algorithm: Pseudocode

Initialize the cost of each node to \( \infty \)
Initialize the cost of the source to 0

While there are unknown nodes left in the graph
Select an unknown node \( b \) with the lowest cost
Mark \( b \) as known
For each node \( a \) adjacent to \( b \)
  if (!a.known)
    if (\( b \).dist + Cost ba < a.dist)
      decrease(a.dist to= \( b \).dist + Cost ba);
      a.path = b;

Important Features

• Once a vertex is made known, the cost of the shortest path to that node is known
• While a vertex is still not known, another shorter path to it might still be found
• The shortest path itself can be found by following the backward pointers stored in node.path
Dijkstra’s Alg: Implementation

Initialize the cost of each node to $\infty$

Initialize the cost of the source to 0

While there are unknown nodes left in the graph

Select the unknown node $b$ with the lowest cost

Mark $b$ as known

For each node $a$ adjacent to $b$

$a$’s cost = min($a$’s old cost, $b$’s cost + cost of $(b, a)$)

Running time?

Dijkstra’s Algorithm: a Greedy Algorithm

Greedy algorithms always make choices that currently seem the best

– Short-sighted – no consideration of long-term or global issues
– Locally optimal - does not always mean globally optimal!!

Dijkstra’s Algorithm: Summary

• Classic algorithm for solving SSSP in weighted graphs without negative weights
• A greedy algorithm (irrevocably makes decisions without considering future consequences)
• Intuition for correctness:
  – shortest path from source vertex to itself is 0
  – cost of going to adjacent nodes is at most edge weights
  – cheapest of these must be shortest path to that node
  – update paths for new node and continue picking cheapest path

Correctness: The Cloud Proof

How does Dijkstra’s decide which vertex to add to the Known set next?

• If path to $V$ is shortest, path to $W$ must be at least as long
  (or else we would have picked $W$ as the next vertex)

Correctness: Inside the Cloud

Prove by induction on # of nodes in the cloud:

Initial cloud is just the source with shortest path 0

Assume: Everything inside the cloud has the correct shortest path

Inductive step: Only when we prove the shortest path to some node $v$ (which is not in the cloud) is correct, we add it to the cloud

When does Dijkstra’s algorithm not work?
Dijkstra’s Algorithm vs BFS

At each step:
1) Pick closest unknown vertex
2) Add it to finished vertices
3) Update distances

Breadth-first Search

At each step:
1) Pick vertex from queue
2) Add it to visited vertices
3) Update queue with neighbors

The Trouble with Negative Weight Cycles

What’s the shortest path from A to E?

Problem?