The AVL Balance Condition

Left and right subtrees of every node have equal heights differing by at most 1

Define: balance(x) = height(x.left) – height(x.right)

AVL property: –1 ≤ balance(x) ≤ 1, for every node x

• Ensures small depth
  – Will prove this by showing that an AVL tree of height $h$ must have a lot of (i.e. $O(2^h)$) nodes
• Easy to maintain
  – Using single and double rotations

The AVL Tree Data Structure

Structural properties
1. Binary tree property (0,1, or 2 children)
2. Heights of left and right subtrees of every node differ by at most 1

Result:
Worst case depth of any node is: $O(\log n)$

Ordering property
– Same as for BST

Is this an AVL Tree?

NULLs have height –1

Circle One:

AVL
Not AVL

Student Activity:
If not AVL, put a box around nodes where AVL property is violated.
Let \( S(h) \) be the min # of nodes in an AVL tree of height \( h \)

Claim: \( S(h) = S(h-1) + S(h-2) + 1 \)

Solution of recurrence: \( S(h) = \Theta(2^h) \) (like Fibonacci numbers)

AVL tree of height 4 with the min # of nodes:

NULLs have height -1

We need to be able to:

1. 
2. 
3. 

We need to be able to:

1. 
2. 
3. 

AVL trees: find, insert

- **AVL find:**
  - same as BST find.

- **AVL insert:**
  - same as BST insert, except may need to “fix” the AVL tree after inserting new value.

Let \( x \) be the node where an imbalance occurs.

Four cases to consider. The insertion is in the

1. left subtree of the left child of \( x \).
2. right subtree of the left child of \( x \).
3. left subtree of the right child of \( x \).
4. right subtree of the right child of \( x \).

**Idea:** Cases 1 & 4 are solved by a single rotation.

Cases 2 & 3 are solved by a double rotation.
Fix: Apply Single Rotation

AVL Property violated at this node (x)

Single Rotation:
1. Rotate between x and child

Single rotation in general

Height of tree before?  Height of tree after?  Effect on Ancestors?

Single rotation example

Soln:

Bad Case #3

Insert(1)
Insert(6)
Insert(3)

Fix: Apply Double Rotation

AVL Property violated at this node (x)

Double Rotation
1. Rotate between x’s child and grandchild
2. Rotate between x and x’s new child
Double rotation in general

\[ h > 0 \]

W < b < c < Y < a < Z

Double rotation, step 1

Imbalance at node X

Single Rotation
1. Rotate between x and child

Double Rotation
1. Rotate between x’s child and grandchild
2. Rotate between x and x’s new child

Insert into an AVL tree: a b c d

Single and Double Rotations:
Inserting what integer values would cause the tree to need a:
1. single rotation?
2. double rotation?
3. no rotation?
Insertion into AVL tree

1. Find spot for new key
2. Hang new node there with this key
3. Search back up the path for imbalance
4. If there is an imbalance:
   - case #1: Perform single rotation and exit
   - case #2: Perform double rotation and exit
   Both rotations keep the subtree height unchanged. Hence only one rotation is sufficient!

Easy Insert

Hard Insert

Single Rotation

Hard Insert

Single Rotation (oops!)
AVL Trees Revisited

• Balance condition:
  - For every node $x$, $-1 \leq \text{balance}(x) \leq 1$
  - Strong enough: Worst case depth is $O(\log n)$
  - Easy to maintain: one single or double rotation

• Guaranteed $O(\log n)$ running time for
  - Find
  - Insert
  - Delete
  - buildTree

AVL Trees Revisited

• What extra info did we maintain in each node?

• Where were rotations performed?

• How did we locate this node?