Disjoint Sets and Dynamic Equivalence Relations

CSE 373

Data Structures and Algorithms

Today's Outline

- **Announcements**
  - Assignment #4 due this Friday Feb 13th at the beginning of lecture.
- **Today's Topics:**
  - **Disjoint Sets & Dynamic Equivalence**

Desired Properties

- None of the boundary is deleted
- Every cell is reachable from every other cell.
- Only one path from any one cell to another (There are no cycles – no cell can reach itself by a path unless it retraces some part of the path.)

Number the Cells

We have disjoint sets P ={ {1}, {2}, {3}, {4},… {36} } each cell is unto itself.
We have all possible edges E ={ (1,2), (1,7), (2,8), (2,3), … } 60 edges total.

Start	1	2	3	4	5	6	
	7	8	9	10	11	12	
	13	14	15	16	17	18	
	19	20	21	22	23	24	
	25	26	27	28	29	30	
	31	32	33	34	35	36	End

Basic Algorithm

- P = set of sets of connected cells
- E = set of edges
- Maze = set of maze edges (initially empty)

```
While there is more than one set in P {
  pick a random edge (x,y) and remove from E
  u := Find(x);
  v := Find(y);
  if u ≠ v then   // removing edge (x,y) connects previously non-
                  // connected cells x and y - leave this edge removed!
     Union(u,v)
  else            // cells x and y were already connected, add this
                  // edge to set of edges that will make up final maze.
     add (x,y) to Maze
}
All remaining members of E together with Maze form the maze
```


Example Step

Pick (8,14)

Start	1	2	3	4	5	6	
	7	8	9	10	11	12	
	13	14	15	16	17	18	
	19	20	21	22	23	24	
	25	26	27	28	29	30	
	31	32	33	34	35	36	End

P
{1,2,7,8,9,13,19}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{14,20,26,27}
{15,16,21}
.
.
{22,23,24,29,30,32
33,34,35,36}

Example

P
{1,2,7,8,9,13,19}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{14,20,26,27}
{15,16,21}
.
.
{22,23,24,29,39,32
33,34,35,36}

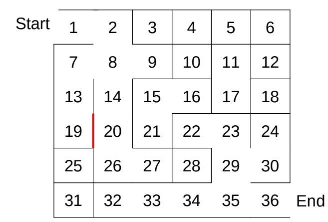
Find(8) = 7
Find(14) = 20

Union(7,20)

P
{1,2,7,8,9,13,19,14,20 26,27}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{15,16,21}
.
.
{22,23,24,29,39,32
33,34,35,36}

Example

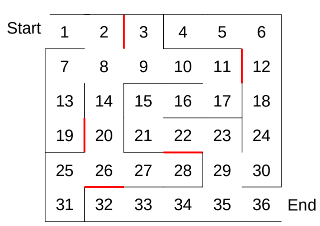
Pick (19,20)



P
{1,2,7,8,9,13,19
14,20,26,27}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{15,16,21}
.
.
{22,23,24,29,39,32
33,34,35,36}

Example at the End

P
{1,2,3,4,5,6,7,… 36}



Implementing the Disjoint Sets ADT

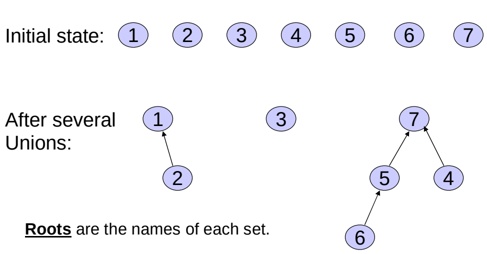
- *n* elements,
  Total Cost of: *m* finds, ≤ *n*-1 unions

can there be more unions?

- Target complexity: $O(m+n)$
  i.e. $O(1)$ amortized

- $O(1)$ worst-case for find as well as union would be great, but…
  Known result: both find and union *cannot* be done in worst-case $O(1)$ time

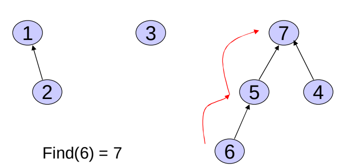
Up-Tree for Disjoint Union/Find



Roots are the names of each set.

Find Operation

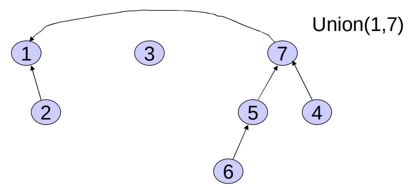
Find(x) - follow x to the root and return the root



Find(6) = 7

Union Operation

Union(x,y) - assuming x and y are roots, point y to x.

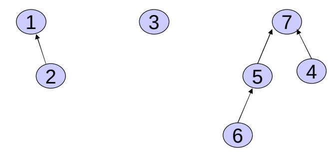


Simple Implementation

- Array of indices

	1	2	3	4	5	6	7
up	0	1	0	7	7	5	0

Up[x] = 0 means x is a root.



Implementation

```
int Find(int x) {

  while(up[x] != 0) {
    x = up[x];
  }

  return x;
}
```

```
void Union(int x, int y) {
   up[y] = x;
}
```

runtime for Union():

runtime for Find():

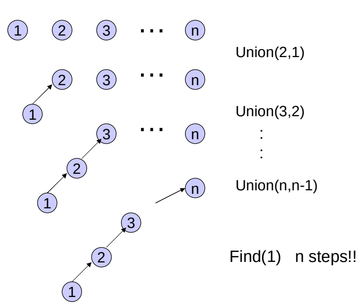
runtime for m Finds and n-1 Unions:

Now this doesn’t look good ☹

Can we do better? *Yes!*

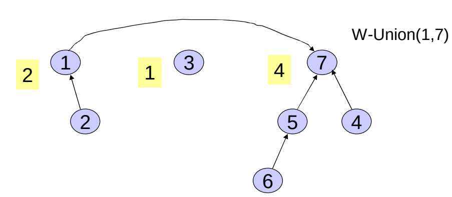
1. Improve union so that *find* only takes $\Theta(\log n)$
   - Union-by-size
   - Reduces complexity to $\Theta(m \log n + n)$
2. Improve find so that it becomes even better!
   - Path compression
   - Reduces complexity to almost $\Theta(m + n)$

A Bad Case

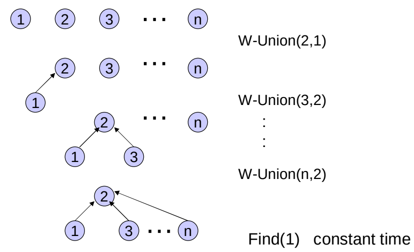


Weighted Union

- Weighted Union
  - Always point the *smaller* (total # of nodes) tree to the root of the larger tree



Example Again

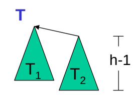


Analysis of Weighted Union

With weighted union an up-tree of height h has weight *at least* 2^h.

- Proof by induction
  - **Basis**: h = 0. The up-tree has one node, $2^0 = 1$
  - **Inductive step**: Assume true for all h’ < h.

Minimum weight up-tree of height h formed by weighted unions



$W(T_1) \geq W(T_2) \geq 2^{h-1}$

Weighted union   Induction hypothesis

$W(T) \geq 2^{h-1} + 2^{h-1} = 2^h$

Analysis of Weighted Union (cont)

Let T be an up-tree of weight n formed by weighted union. Let h be its height.

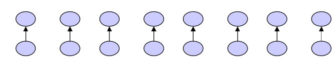
$$n \geq 2^h$$

$$\log_2 n \geq h$$

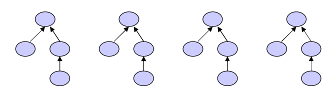
- Find(x) in tree T takes O(log n) time.
  - Can we do better?

Worst Case for Weighted Union

n/2 Weighted Unions

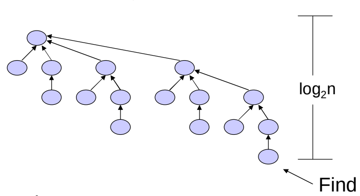


n/4 Weighted Unions



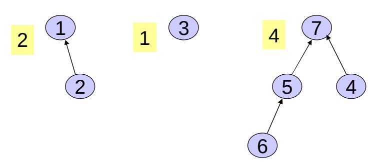
Example of Worst Case (cont’)

After n/2 + n/4 + …+ 1 Weighted Unions:



If there are $n = 2^k$ nodes then the longest path from leaf to root has length k.

Array Implementation



	1	2	3	4	5	6	7
up	-1	1	-1	7	7	5	-1
weight	2		1				4

Weighted Union

```
W-Union(i,j : index){
  //i and j are roots
  wi := weight[i];
  wj := weight[j];
  if wi < wj then
    up[i] := j;
    weight[j] := wi + wj;
  else
    up[j] :=i;
    weight[i] := wi +wj;
}
```

new runtime for Union():

new runtime for Find():

runtime for m finds and n-1 unions =

Nifty Storage Trick

- Use the same array representation as before
- Instead of storing **-1** for the root, simply store **-size**

[Read section 8.4, page 299]

How about Union-by-height?

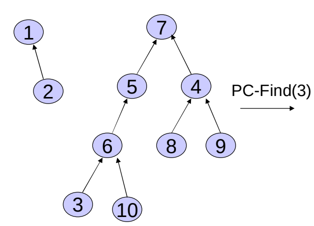
- Can still guarantee O(log *n*) worst case depth

  Left as an exercise!

- Problem: Union-by-height doesn't combine very well with the new find optimization technique we'll see next

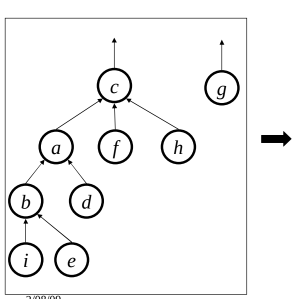
Path Compression

- On a Find operation point all the nodes on the search path directly to the root.

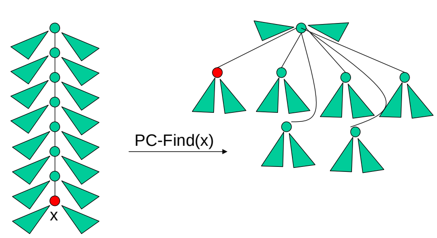


Student Activity

Draw the result of Find(e):



Self-Adjustment Works



Path Compression Find

```
PC-Find(i : index) {
  r := i;
  while up[r] ≠ -1 do //find root
    r := up[r];

  // Assert: r= the root, up[r] = -1
  if i ≠ r then   // if i was not a root

    temp := up[i];
    while temp ≠ r do // compress path
      up[i] := r;
      i := temp;
      temp := up[temp]

  return(r)
}
```

(New?) runtime for Find:

Interlude: A Really Slow Function

Ackermann's function is a really big function $A(x, y)$ with inverse $\alpha(x, y)$ which is really small

How fast does $\alpha(x, y)$ grow?

$\alpha(x, y) = 4$ for x **far** larger than the number of atoms in the universe (2^{300})

α shows up in:

- Computation Geometry (surface complexity)
- Combinatorics of sequences

A More Comprehensible Slow Function

log* x = number of times you need to compute log to bring value down to at most 1

E.g. log* 2 = 1
log* 4 = log* 2^2 = 2
log* 16 = log* 2^{2^2} = 3 (log log log 16 = 1)
log* 65536 = log* $2^{2^{2^2}}$ = 4 (log log log log 65536 = 1)
log* 2^{65536} = ………….. = 5

Take this: $\alpha(m,n)$ grows even slower than log* n !!

Complex Complexity of Union-by-Size + Path Compression

Tarjan proved that, with these optimizations, p union and find operations on a set of n elements have worst case complexity of $O(p \cdot \alpha(p, n))$

For *all practical purposes* this is amortized constant time: $O(p \cdot 4)$ for p operations!

- Very complex analysis

Disjoint Union / Find with Weighted Union and PC

- Worst case time complexity for a W-Union is O(1) and for a PC-Find is O(log n).
- Time complexity for m ≥ n operations on n elements is O(m log* n) where log* n is a very slow growing function.
  - Log * n < 7 for all reasonable n. Essentially constant time per operation!

Amortized Complexity

- For disjoint union / find with weighted union and path compression.
  - average time per operation is essentially a constant.
  - worst case time for a PC-Find is O(log n).
- An individual operation can be costly, but over time the average cost per operation is not.