Asymptotic Analysis II

CSE 373
Data Structures & Algorithms
Ruth Anderson
Winter 2009

Today's Outline

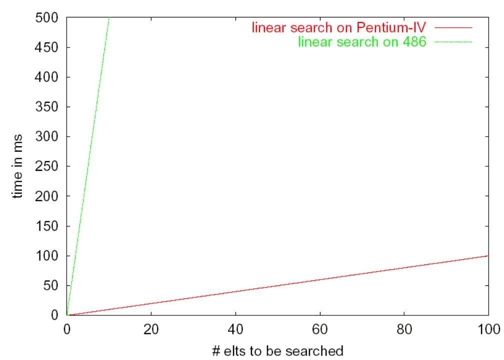
- **Announcements**
  - Assignment #1 due Thurs, Jan 15 at 11:45pm
  - Midterm Dates:
    - Midterm #1: Friday, Jan 30th
    - Midterm #2: Friday, February 27th
- **Asymptotic Analysis**

Linear Search vs Binary Search

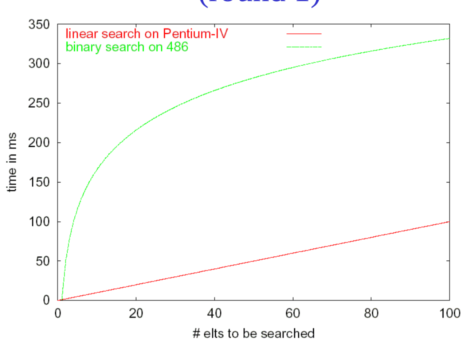
	Linear Search	Binary Search
Best Case		
Worst Case		

So … which algorithm is better?
What tradeoffs can you make?

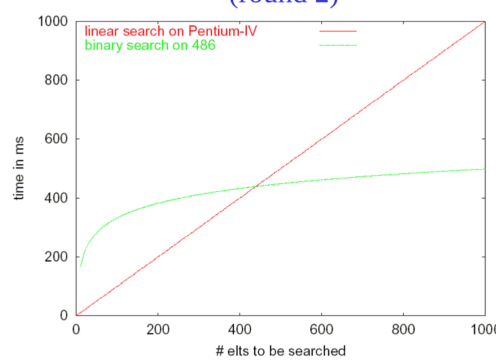
Fast Computer vs. Slow Computer



Fast Computer vs. Smart Programmer (round 1)



Fast Computer vs. Smart Programmer (round 2)



Asymptotic Analysis

- Asymptotic analysis looks at the *order* of the running time of the algorithm
  - A valuable tool when the input gets "large"
  - Ignores the *effects of different machines* or *different implementations* of the same algorithm
- Intuitively, to find the asymptotic runtime, throw away the constants and low-order terms
  - Linear search is $T(n) = 3n + 2 \in \Theta(\boldsymbol{n})$
  - Binary search is $T(n) = 4 \log_2 n + 4 \in \Theta(\mathbf{log}\ \boldsymbol{n})$

Remember: the fastest algorithm has the slowest growing function for its runtime

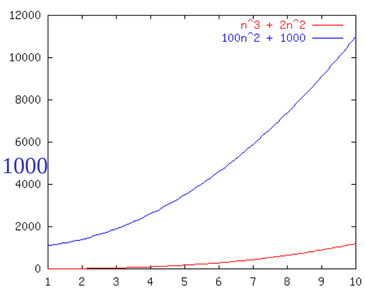
Asymptotic Analysis

- Eliminate low order terms
  - $4n + 5 \Rightarrow$
  - $0.5\ n \log n + 2n + 7 \Rightarrow$
  - $n^3 + 2^n + 3n \Rightarrow$
- Eliminate coefficients
  - $4n \Rightarrow$
  - $0.5\ n \log n \Rightarrow$
  - $n \log n^2$ =>

Order Notation: Intuition

$f(n) = n^3 + 2n^2$

$g(n) = 100n^2 + 1000$



Although not yet apparent, as n gets "sufficiently large", $f(n)$ will be "greater than or equal to" $g(n)$

Definition of Order Notation

- Upper bound: $T(n) = O(f(n))$ Big-O
  Exist constants c and n_0 such that
  $T(n) \leq c\, f(n)$ for all $n \geq n_0$
- Lower bound: $T(n) = \Omega(g(n))$ Omega
  Exist constants c and n_0 such that
  $T(n) \geq c\, g(n)$ for all $n \geq n_0$
- Tight bound: $T(n) = \Theta(f(n))$ Theta
  When both hold:
  $T(n) = O(f(n))$
  $T(n) = \Omega(f(n))$

Order Notation: Definition

O(f(*n*)) : <u>a set or class of functions</u>

$g(n) \in O(f(n))$ iff there exist constants c and n_0 such that:

$g(n) \leq c\, f(n)$ for all $n \geq n_0$

Example: $g(n) = 1000n$ vs. $f(n) = n^2$

Is $g(n) \in O(f(n))$?

Pick: $n_0 = 1000$, $c = 1$

Notation Notes

Note: Sometimes, you'll see the notation:

$g(n) = O(f(n))$.

This is equivalent to:

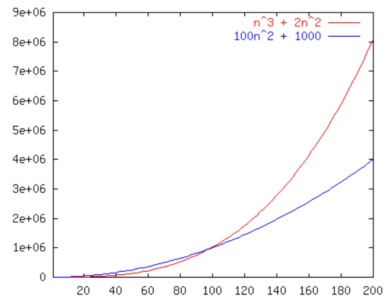
$g(n)$ is $O(f(n))$.

However: The notation

$O(f(n)) = g(n)$ is meaningless!

(in other words big-O "equality" is not symmetric)

Order Notation: Example



$100n^2 + 1000 \le 5\,(n^3 + 2n^2)$ for all $n \ge 19$

So $f(n)$ is O($g(n)$)

Big-O: Common Names

- constant: O(1)
- logarithmic: O(log n) ($\log_k n$, log n^2 is O(log n))
- log squared: O($\log^2$ n)
- linear: O(n)
- log-linear: O(n log n)
- quadratic: O(n^2)
- cubic: O(n^3)
- polynomial: O(n^k) (k is a constant)
- exponential: O(c^n) (c is a constant > 1)

Meet the Family

- O(f(n)) is the set of all functions asymptotically less than or equal to f(n)
  - o(f(n)) is the set of all functions asymptotically strictly less than f(n)
- Ω(f(n)) is the set of all functions asymptotically greater than or equal to f(n)
  - ω(f(n)) is the set of all functions asymptotically strictly greater than f(n)
- Θ(f(n)) is the set of all functions asymptotically equal to f(n)

Meet the Family, Formally

- $g(n) \in O(f(n))$ iff
  There exist c and n_0 such that $g(n) \le c\, f(n)$ for all $n \ge n_0$
  - $g(n) \in o(f(n))$ iff
    There exists a n_0 such that $g(n) < c\, f(n)$ for all c and $n \ge n_0$

  Equivalent to: $\lim_{n\to\infty} g(n)/f(n) = 0$
- $g(n) \in \Omega(f(n))$ iff
  There exist $c>0$ and n_0 such that $g(n) \ge c\, f(n)$ for all $n \ge n_0$
  - $g(n) \in \omega(f(n))$ iff
    There exists a n_0 such that $g(n) > c\, f(n)$ for all c and $n \ge n_0$

  Equivalent to: $\lim_{n\to\infty} g(n)/f(n) = \infty$
- $g(n) \in \Theta(f(n))$ iff
  $g(n) \in O(f(n))$ and $g(n) \in \Omega(f(n))$

Big-Omega et al. Intuitively

Asymptotic Notation	Mathematics Relation
O	≤
Ω	≥
Θ	=
o	<
ω	>

Pros and Cons of Asymptotic Analysis

Types of Analysis

Two orthogonal axes:

- bound flavor
  - upper bound (O, o)
  - lower bound (Ω, ω)
  - asymptotically tight (Θ)
- analysis case
  - worst case (adversary)
  - average case
  - best case
  - "amortized"

Algorithm Analysis Examples

- Consider the following program segment:

```
x:= 0;
for i = 1 to N do
  for j = 1 to i do
    x := x + 1;
```

- What is the value of x at the end?

Arithmetic Sequences

N = {0, 1, 2, ... } = natural numbers
[0, 1, 2, ...] is an infinite arithmetic sequence
[a, a+d, a+2d, a+3d, ...] is a general infinite arithmetic sequence.
There is a *constant difference* between terms.

$$1+2+3+\ldots+N=\sum_{i=1}^{N} i=\frac{N(N+1)}{2}$$

Analyzing the Loop

- Total number of times x is incremented is executed =

$$1+2+3+\ldots+N=\sum_{i=1}^{N} i=\frac{N(N+1)}{2}$$

- Congratulations - You've just analyzed your first program!
  - Running time of the program is proportional to N(N+1)/2 for all N
  - Big-O ??

Which Function Grows Faster?

$n^3 + 2n^2$ vs. $100n^2 + 1000$

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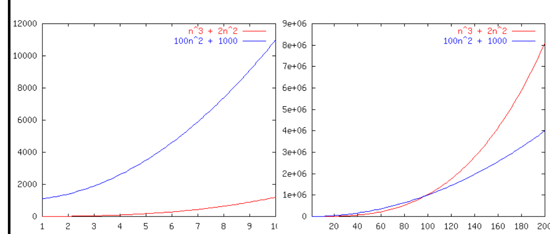

Which Function Grows Faster?

$n^{0.1}$ vs. `log n`

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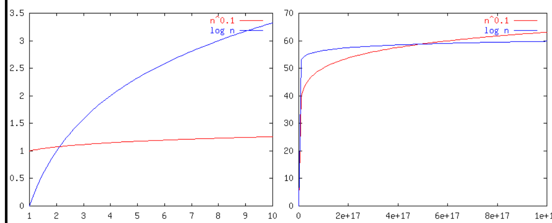

Which Function Grows Faster?

$5n^5$ vs. `n!`

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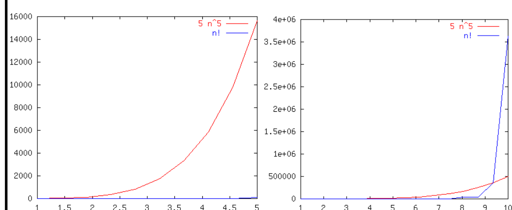
$5n^5$ vs. `n!`



Nested Loops

```
for i = 1 to n do
  for j = 1 to n do
    sum = sum + 1

for i = 1 to n do
  for j = 1 to n do
    sum = sum + 1
```


Nested Loops

```
for i = 1 to n do
  for j = 1 to n do
    if (cond) {
        do_stuff(sum)
    } else {
        for k  = 1 to n*n
            sum += 1
```

$16n^3 \log_8(10n^2) + 100n^2 = O(n^3 \log(n))$

- Eliminate low order terms
- Eliminate constant coefficients

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- Eliminate low order terms
- Eliminate constant coefficients

$$
\begin{aligned}
& 16n^3 \log_8(10n^2) + 100n^2 \\
& \Rightarrow 16n^3 \log_8(10n^2) \\
& \Rightarrow n^3 \log_8(10n^2) \\
& \Rightarrow n^3 \left[\log_8(10) + \log_8(n^2) \right] \\
& \Rightarrow n^3 \log_8(10) + n^3 \log_8(n^2) \\
& \Rightarrow n^3 \log_8(n^2) \\
& \Rightarrow n^3 2 \log_8(n) \\
& \Rightarrow n^3 \log_8(n) \\
& \Rightarrow n^3 \log_8(2) \log(n) \\
& \Rightarrow n^3 \log(n)
\end{aligned}
$$