Graphs: Traversals and Shortest Path Algorithms

CSE 373
Data Structures and Algorithms

Today's Outline

- Announcements
  - Midterm #2 – Wed May 20
- **Graphs**
  - **Topological Sort**
  - **Shortest Paths Algorithms**

Graph Traversals

- **Breadth-first search**
  - explore *all* adjacent nodes, then for each of *those* nodes explore *all* adjacent nodes
- **Depth-first search**
  - explore first child node, then its first child node, etc. until goal node is found or node has no children. Then backtrack, repeat with sibling.
- Both:
  - Work for arbitrary (directed or undirected) graphs
  - Must mark visited vertices so you do not go into an infinite loop!
- Either can be used to determine connectivity:
  - Is there a **path** between two given vertices?
  - Is the graph (weakly) connected?
- Which one:
  - Uses a queue?
  - Uses a stack?
  - Always finds the shortest path (for unweighted graphs)?

The Shortest Path Problem

Given a graph *G*, edge costs $c_{i,j}$, and vertices *s* and *t* in *G*, find the shortest path from *s* to *t*.

For a path $p = v_0\ v_1\ v_2 \ldots v_k$

- *unweighted length* of path $p = k$ (a.k.a. *length*)
- *weighted length* of path $p = \sum_{i=0..k-1} c_{i,i+1}$ (a.k.a *cost*)

Path length equals path cost when ?

Single Source Shortest Paths (SSSP)

Given a graph *G*, edge costs $c_{i,j}$, and vertex *s*, find the shortest paths from *s* to all vertices in G.

All Pairs Shortest Paths (APSP)

Given a graph *G* and edge costs $c_{i,j}$, find the shortest paths between all pairs of vertices in G.

Variations of SSSP

- Weighted vs. unweighted
- Directed vs undirected
- Cyclic vs. acyclic
- Positive weights only vs. negative weights allowed
- Shortest path vs. longest path
- …

Applications

- Network routing
- Driving directions
- Cheap flight tickets
- Critical paths in project management (see textbook)
- …

SSSP: Unweighted Version

Ideas?

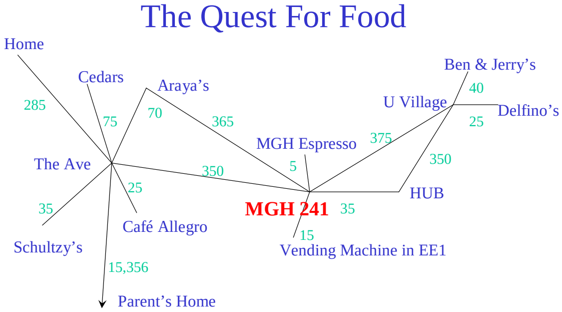

```
void Graph::unweighted (Vertex s){
  Queue q(NUM_VERTICES);
  Vertex v, w;
  q.enqueue(s);
  s.dist = 0;

  while (!q.isEmpty()){
    v = q.dequeue();
    for each w adjacent to v
      if (w.dist == INFINITY){
        w.dist = v.dist + 1;
        w.path = v;
        q.enqueue(w);
      }
    }
  }
```

each edge examined at most once – if adjacency lists are used

each vertex enqueued at most once

total running time: O(   )

Weighted SSSP: The Quest For Food



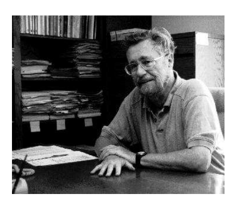
Can we calculate shortest distance to all nodes from MGH 241?

Dijkstra, Edsger Wybe

Legendary figure in computer science; was a professor at University of Texas.

Supported teaching introductory computer courses without computers (pencil and paper programming)

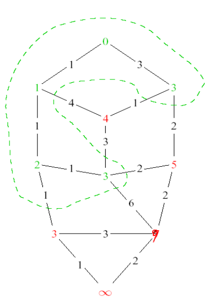
Supposedly wouldn't (until very late in life) read his e-mail; so, his staff had to print out messages and put them in his box.



E.W. Dijkstra (1930-2002)

1972 Turing Award Winner,
Programming Languages, semaphores, and …

Dijkstra’s Algorithm: Idea

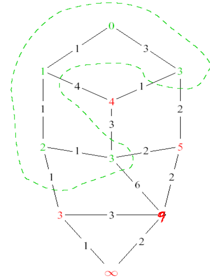


Adapt BFS to handle weighted graphs

Two kinds of vertices:

- Finished or known vertices
  - Shortest distance has been computed
- Unknown vertices
  - Have tentative distance

Dijkstra’s Algorithm: Idea



At each step:

1) Pick closest unknown vertex
2) Add it to known vertices
3) Update distances

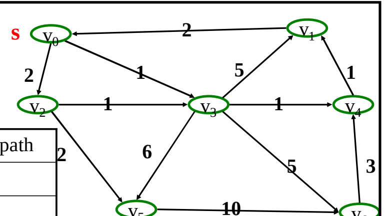
Dijkstra’s Algorithm: Pseudocode

Initialize the cost of each node to ∞

Initialize the cost of the source to 0

While there are unknown nodes left in the graph
  Select an unknown node *b* with the lowest cost
  Mark *b* as known
  For each node *a* adjacent to *b*
    a’s cost = min(*a*’s old cost, *b*’s cost + cost of (*b*, *a*))


```
void Graph::dijkstra(Vertex s){
  Vertex v,w;

  Initialize s.dist = 0 and set dist of all other
  vertices to infinity

  while (there exist unknown vertices, find the one b
  with the smallest distance)
    b.known = true;

    for each a adjacent to b
      if (!a.known)
        if (b.dist + Cost_ba < a.dist){
          decrease(a.dist to= b.dist + Cost_ba);
          a.path = b;
        }
  }
}
```

V	Known	Dist	path
v0			
v1			
v2			
v3			
v4			
v5			
v6			

Dijkstra’s Alg: Implementation

Initialize the cost of each node to ∞
Initialize the cost of the source to 0
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What data structures should we use?

Running time?

Dijkstra's Alg: Implementation

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Running time?

Dijkstra's Algorithm: a Greedy Algorithm

Greedy algorithms always make choices that *currently* seem the best

- Short-sighted – no consideration of long-term or global issues
- Locally optimal - does not always mean globally optimal!!

Dijkstra's Algorithm: Summary

- Classic algorithm for solving SSSP in weighted graphs *without negative weights*
- A *greedy* algorithm (irrevocably makes decisions without considering future consequences)
- Intuition for correctness:
  - shortest path from source vertex to itself is 0
  - cost of going to adjacent nodes is at most edge weights
  - cheapest of these must be shortest path to that node
  - update paths for new node and continue picking cheapest path