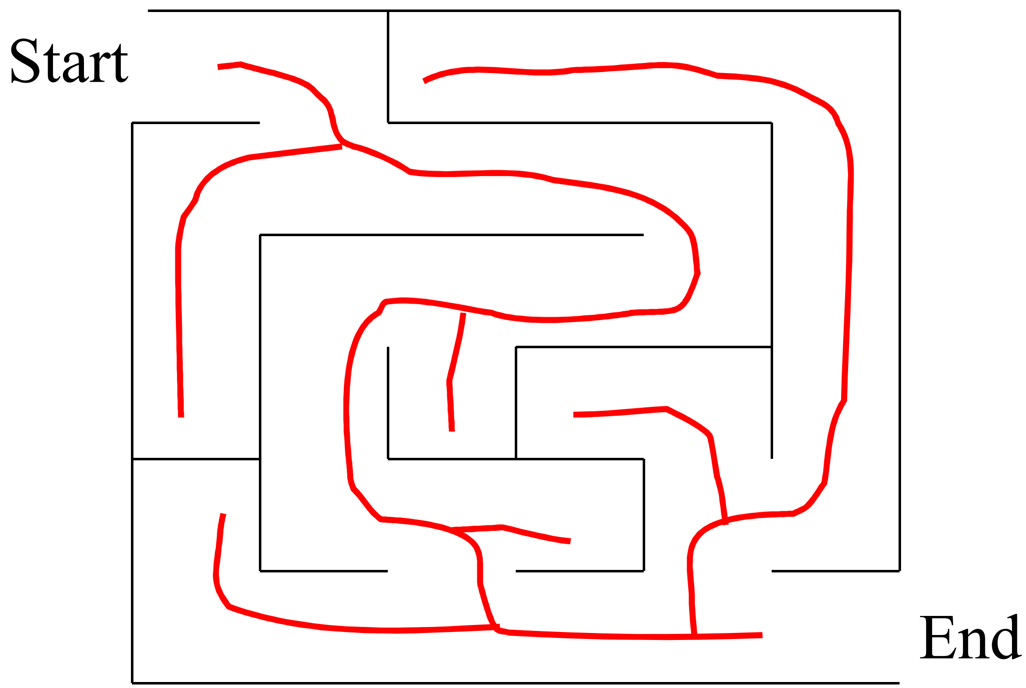
CSE 373
Data Structures & Algorithms

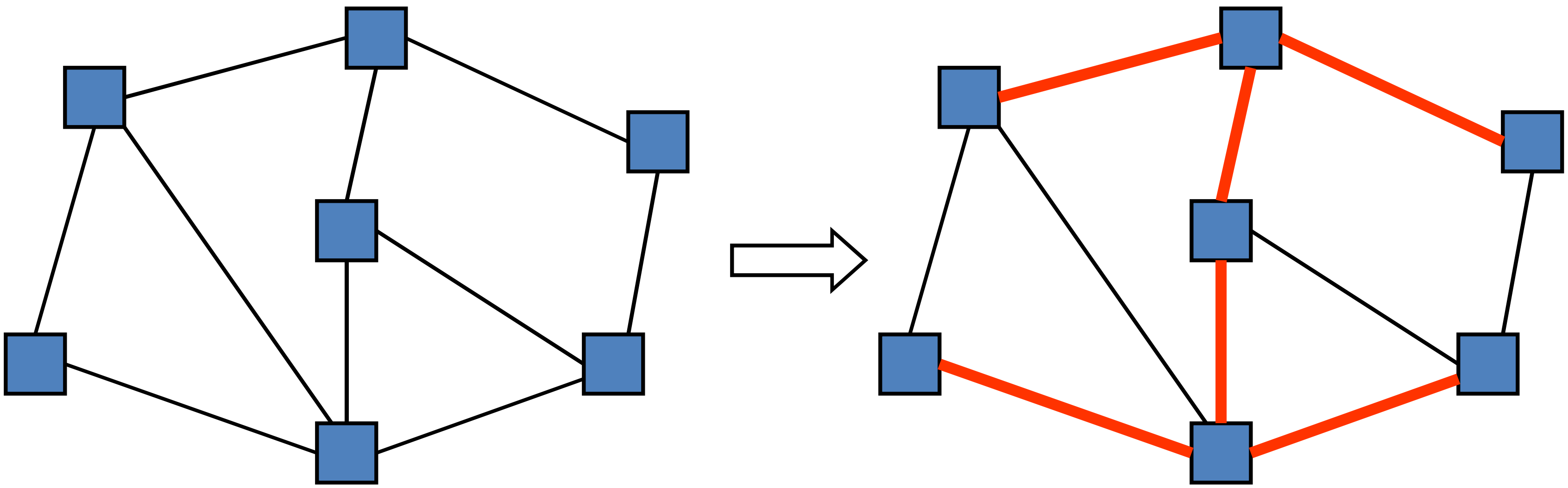
Lectures 21-22

Minimum Spanning Tree

A Hidden Tree



Spanning Tree in a Graph

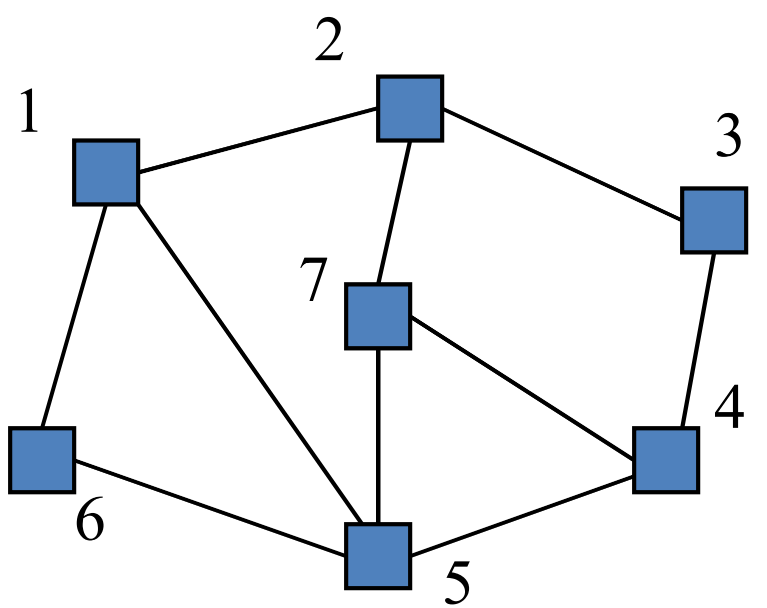


Vertex = router
Edge = link between routers

Spanning tree
- Connects all the vertices
- No cycles

Undirected Graph

- G = (V,E)
  - V is a set of vertices (or nodes)
  - E is a set of unordered pairs of vertices



V = {1,2,3,4,5,6,7}
E = {{1,2},{1,6},{1,5},{2,7},{2,3},
{3,4},{4,7},{4,5},{5,6}}

2 and 3 are adjacent
2 is incident to edge {2,3}

Spanning Tree Problem

- Input: An undirected graph G = (V,E). G is connected.
- Output: T contained in E such that
  - (V,T) is a connected graph
  - (V,T) has no cycles

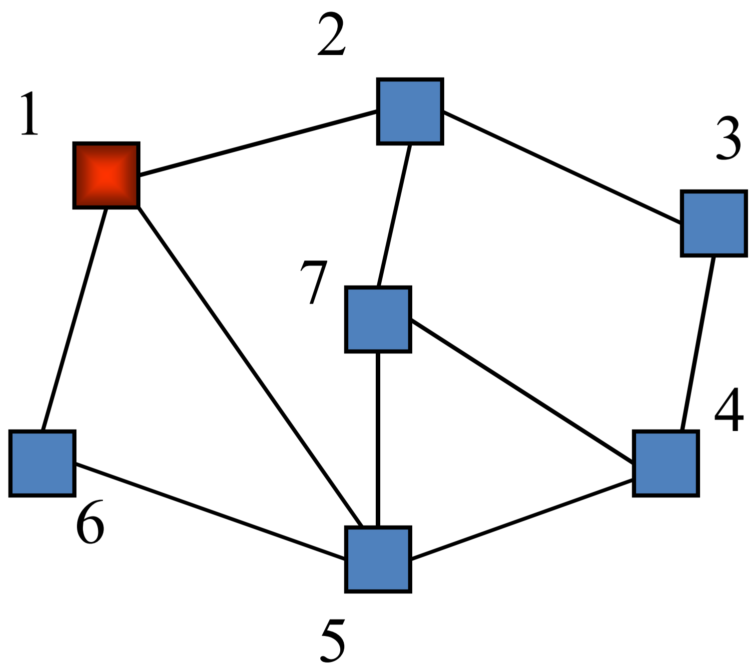
Spanning Tree Algorithm

```
ST(i: vertex)
    mark i;
    for each j adjacent to i do
        if j is unmarked then
            Add {i,j} to T;
            ST(j);
end{ST}
```

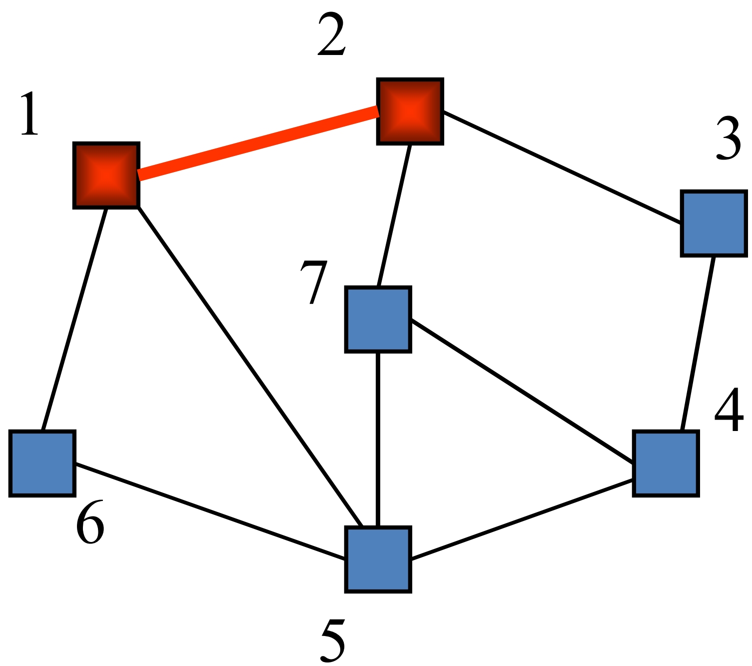
```
Main
T := empty set;
ST(1);
end{Main}
```

Example of Depth First Search

ST(1)



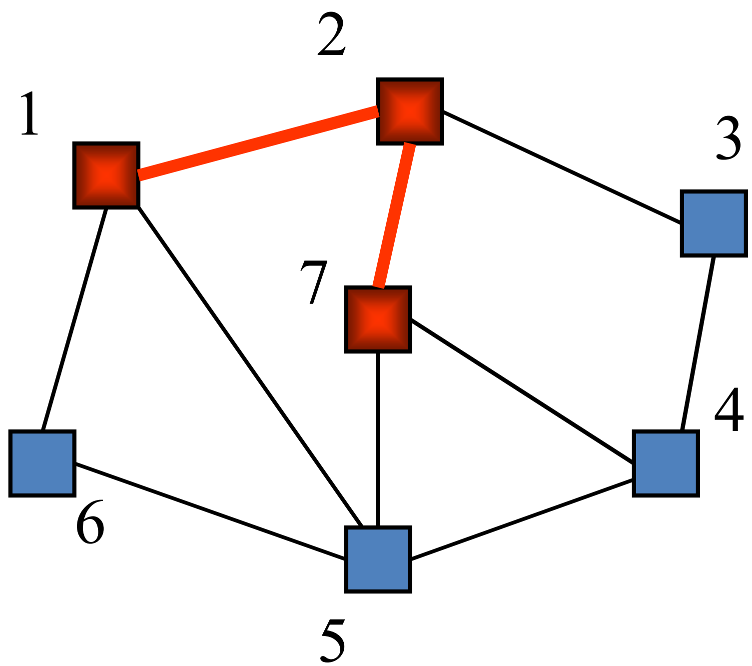
Example Step 2



ST(1)
ST(2)

{1,2}

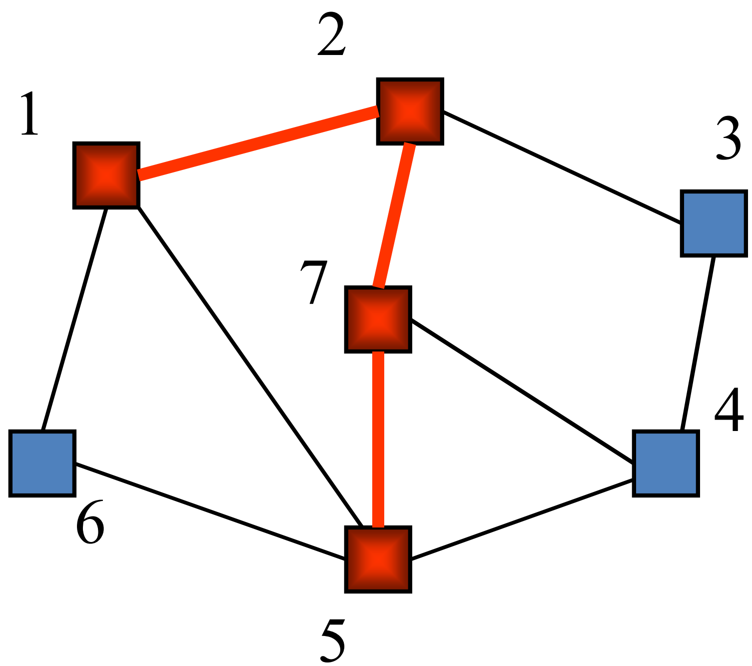
Example Step 3



ST(1)
ST(2)
ST(7)

{1,2} {2,7}

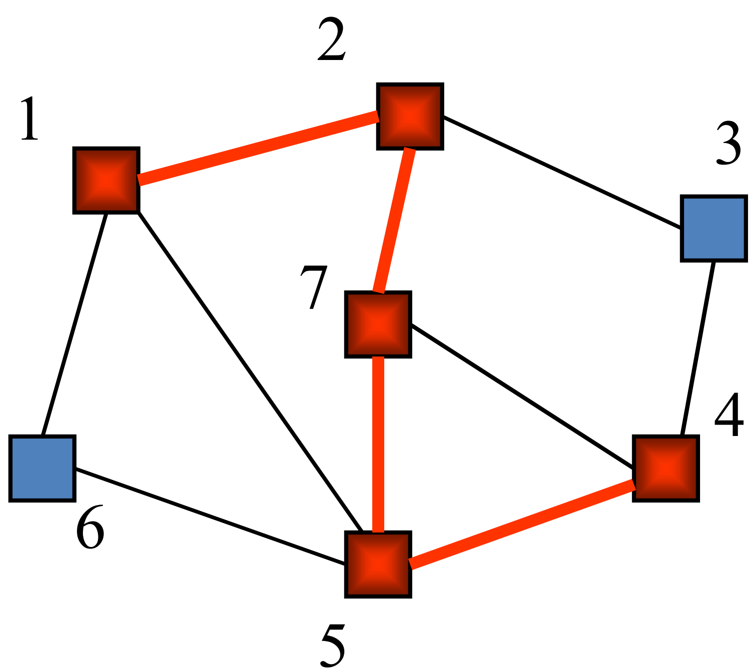
Example Step 4



ST(1)
ST(2)
ST(7)
ST(5)

{1,2} {2,7} {7,5}

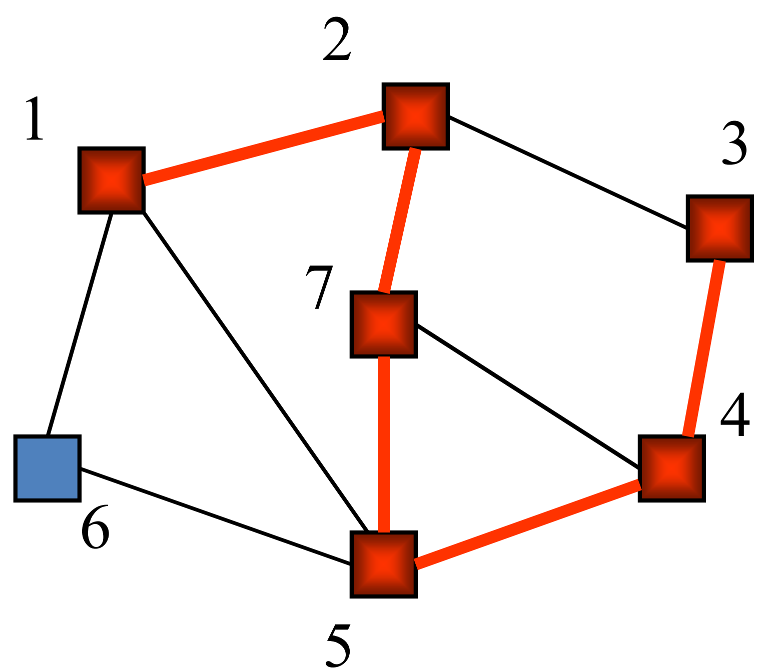
Example Step 5



ST(1)
ST(2)
ST(7)
ST(5)
ST(4)

{1,2} {2,7} {7,5} {5,4}

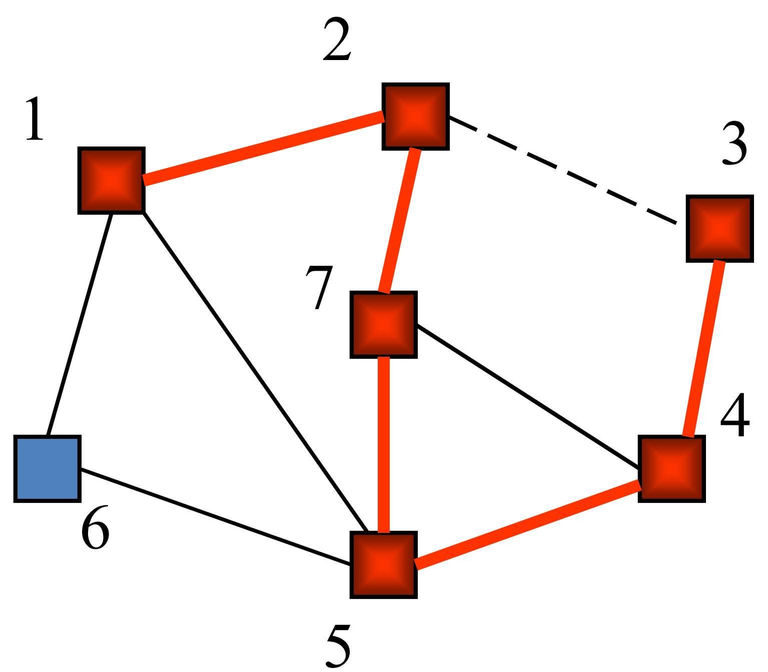
Example Step 6



ST(1)
ST(2)
ST(7)
ST(5)
ST(4)
ST(3)

{1,2} {2,7} {7,5} {5,4} {4,3}

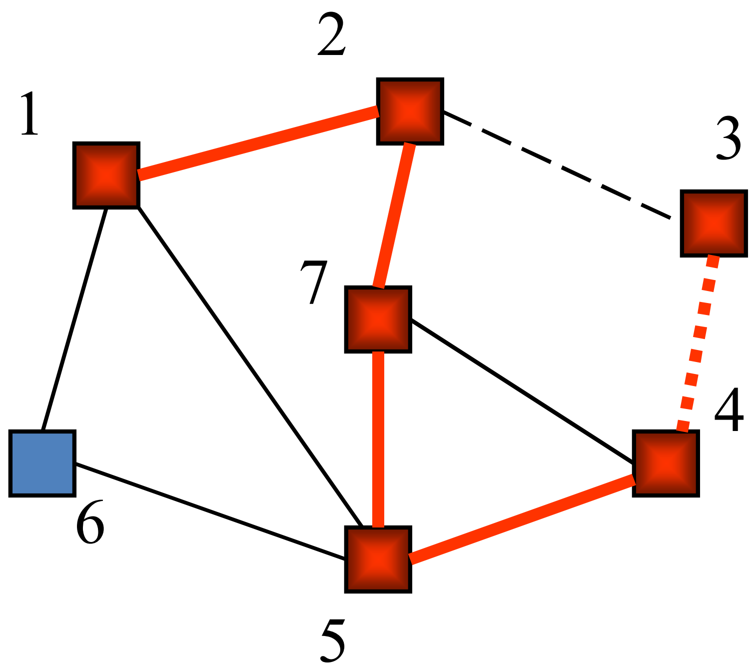
Example Step 7



ST(1)
ST(2)
ST(7)
ST(5)
ST(4)
ST(3)

{1,2} {2,7} {7,5} {5,4} {4,3}

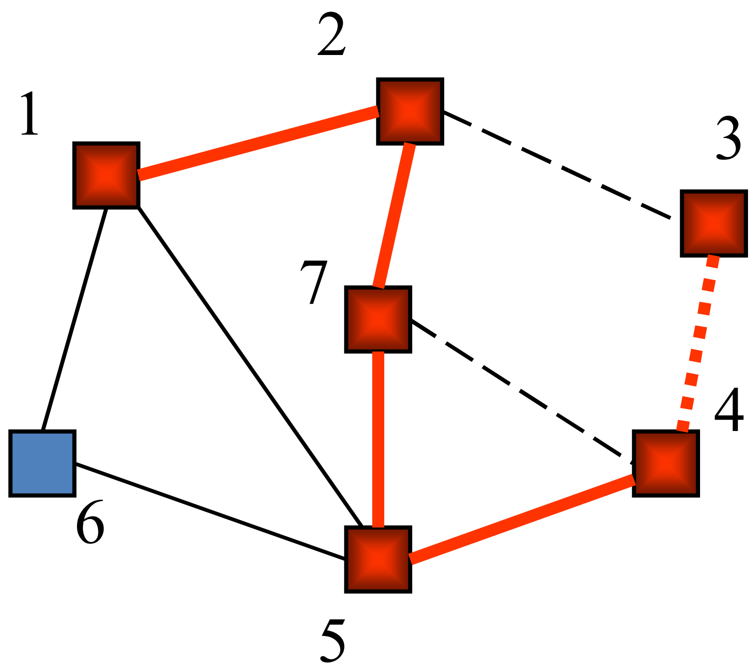
Example Step 8



ST(1)
ST(2)
ST(7)
ST(5)
ST(4)

{1,2} {2,7} {7,5} {5,4} {4,3}

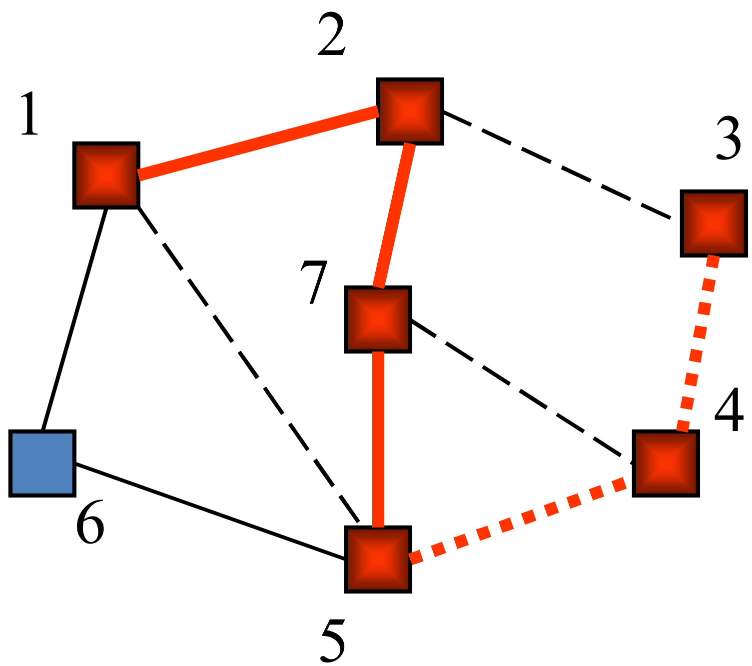
Example Step 9



ST(1)
ST(2)
ST(7)
ST(5)
ST(4)

{1,2} {2,7} {7,5} {5,4} {4,3}

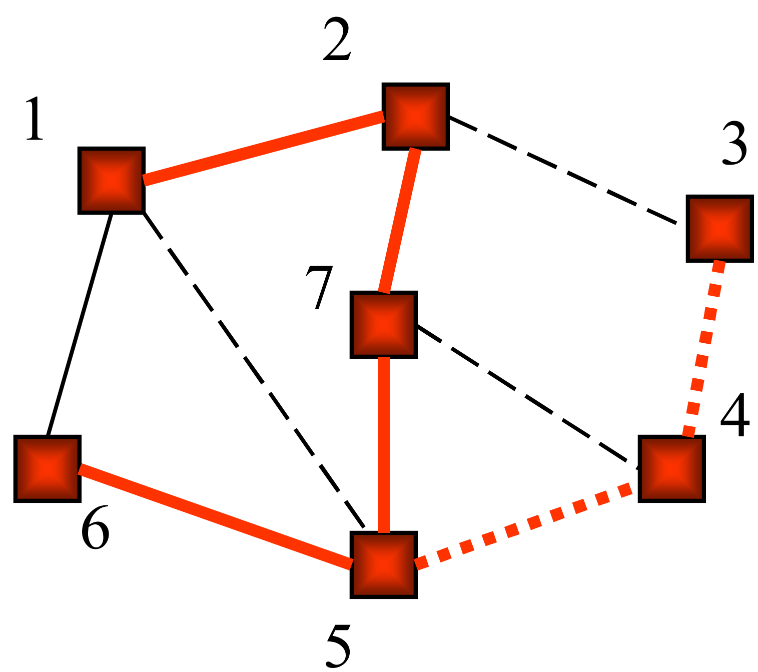
Example Step 10



ST(1)
ST(2)
ST(7)
ST(5)

{1,2} {2,7} {7,5} {5,4} {4,3}

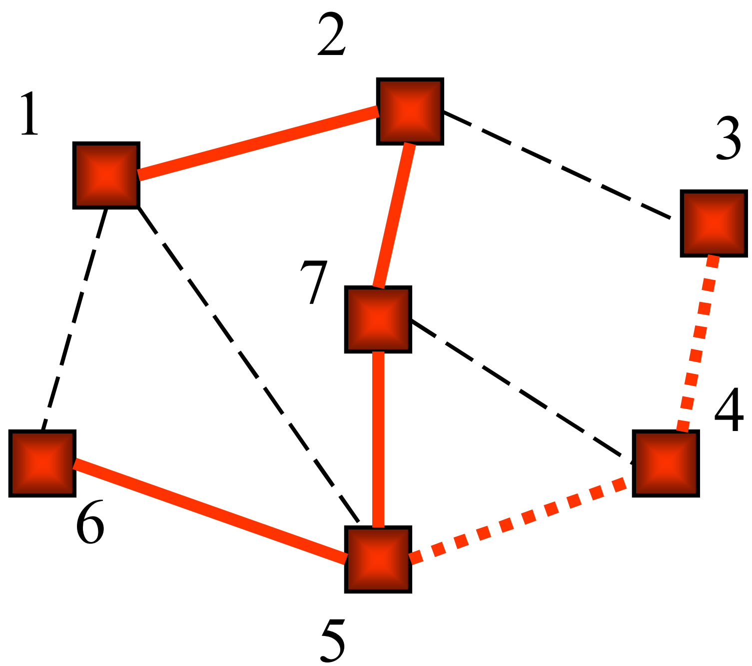
Example Step 11



ST(1)
ST(2)
ST(7)
ST(5)
ST(6)

{1,2} {2,7} {7,5} {5,4} {4,3} {5,6}

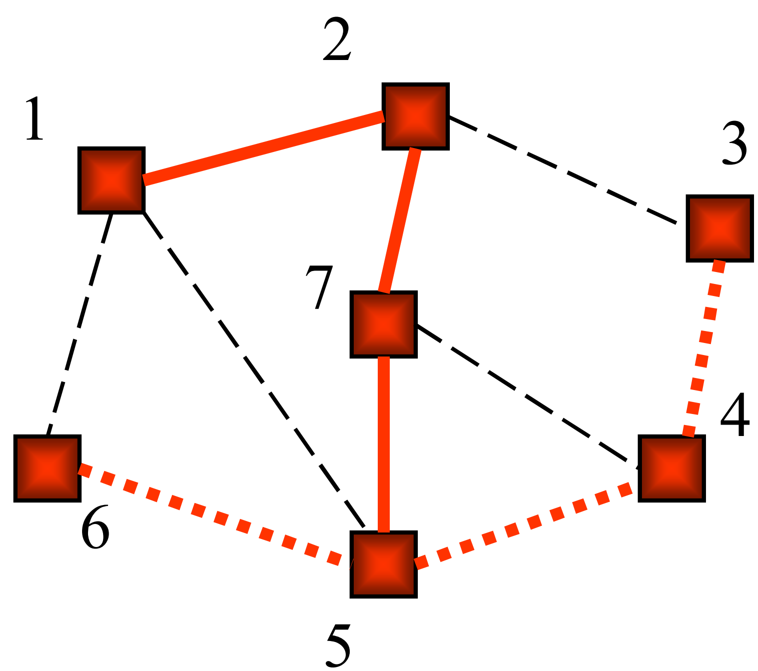
Example Step 12



ST(1)
ST(2)
ST(7)
ST(5)
ST(6)

{1,2} {2,7} {7,5} {5,4} {4,3} {5,6}

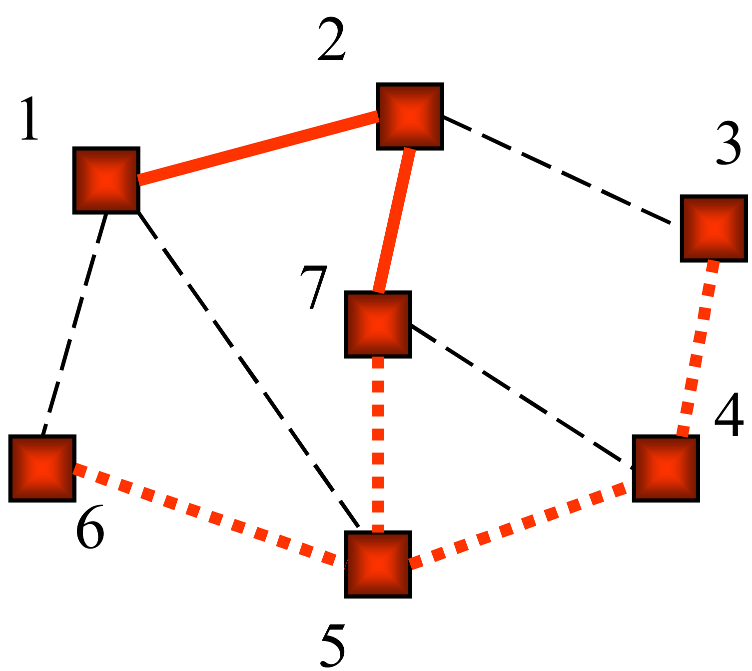
Example Step 13



ST(1)
ST(2)
ST(7)
ST(5)

{1,2} {2,7} {7,5} {5,4} {4,3} {5,6}

Example Step 14

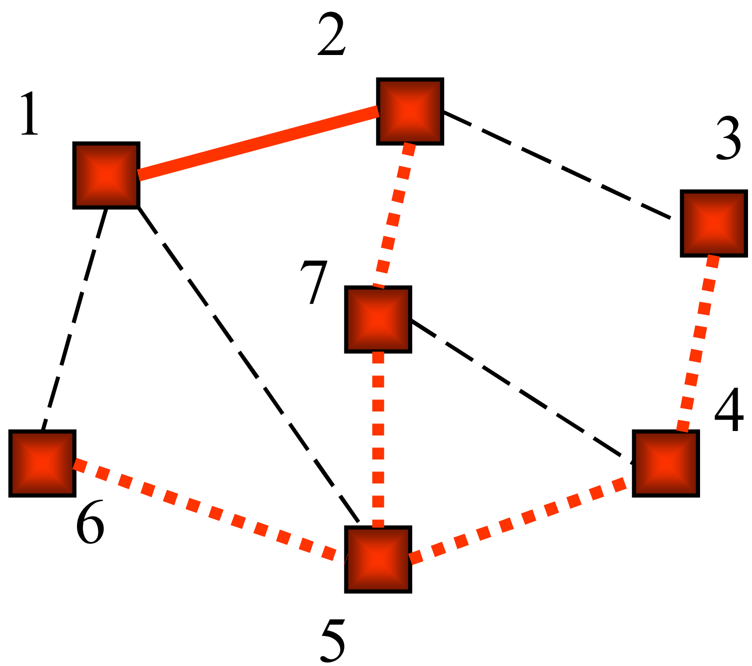


ST(1)
ST(2)
ST(7)

{1,2} {2,7} {7,5} {5,4} {4,3} {5,6}

Example Step 15

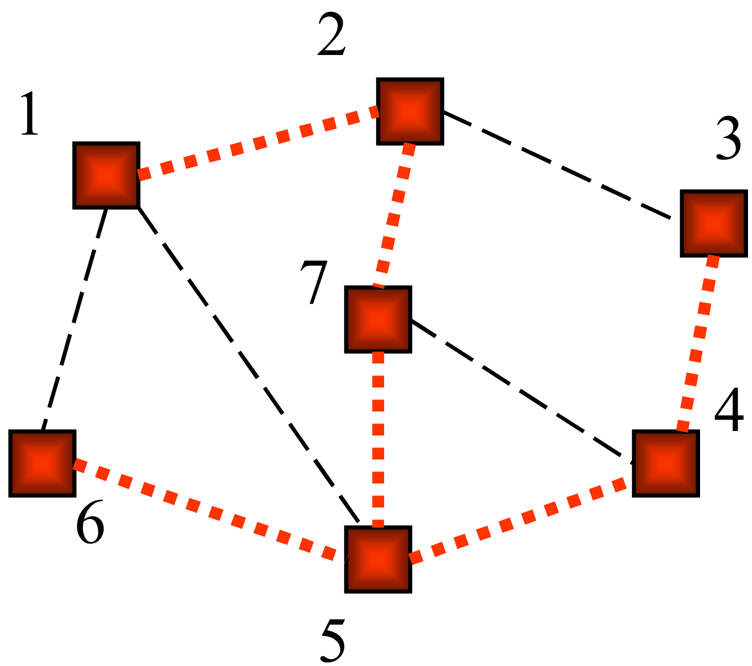
ST(1)
ST(2)



{1,2} {2,7} {7,5} {5,4} {4,3} {5,6}

Example Step 16

ST(1)



{1,2} {2,7} {7,5} {5,4} {4,3} {5,6}

Minimum Spanning Trees

Given an undirected graph **G**=(**V**,**E**), find a graph **G'**=(**V**, **E'**) such that:

- E' is a subset of E
- |E'| = |V| - 1
- G' is connected
- $\sum_{(u,v)\in E'} c_{uv}$ is minimal

G' is a **minimum spanning tree**.

Applications: wiring a house, power grids, Internet connections

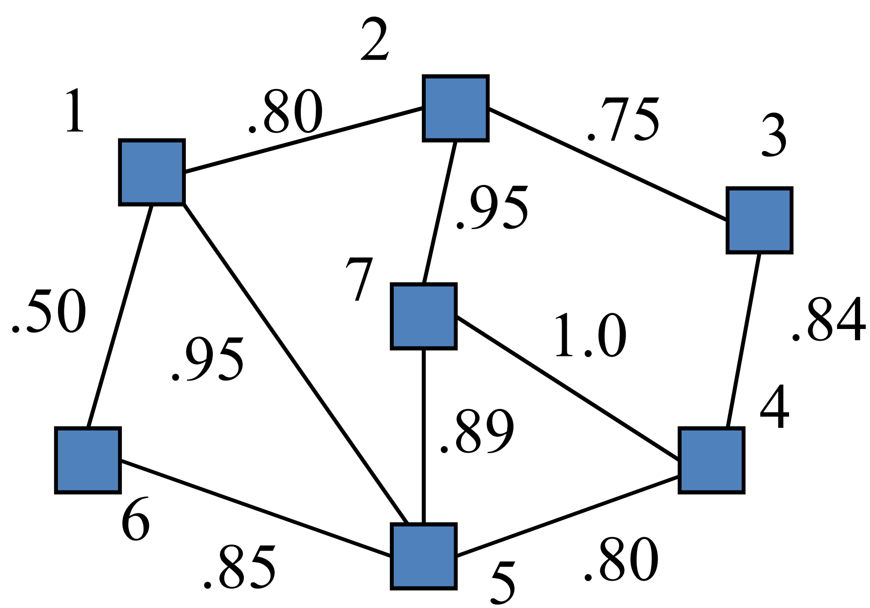
Minimum Spanning Tree Problem

- Input: Undirected Graph G = (V,E) and a cost function C from E to the reals. C(e) is the cost of edge e.
- Output: A spanning tree T with minimum total cost. That is: T that minimizes

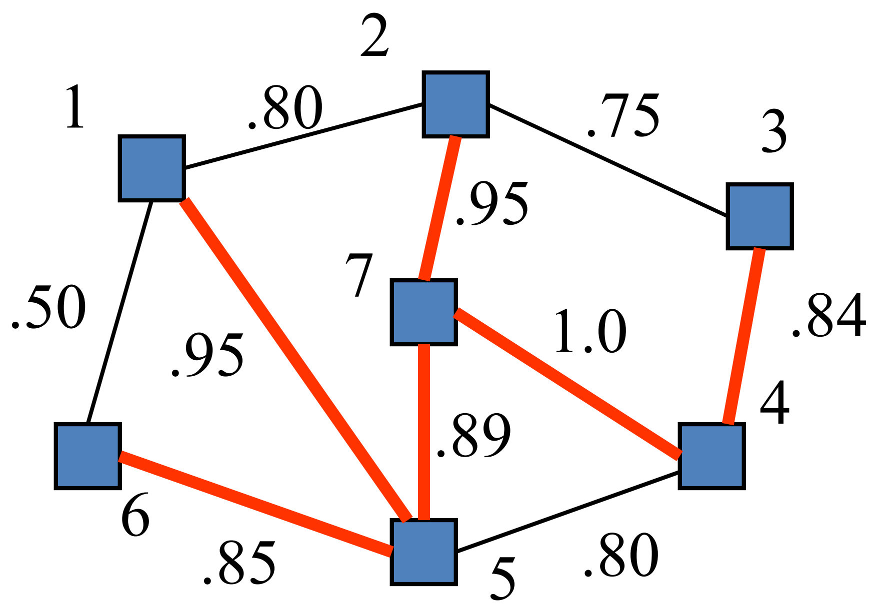
$$C(T) = \sum_{e \in T} C(e)$$

Best Spanning Tree

- Each edge has the probability that it won't fail
- Find the spanning tree that is least likely to fail



Example of a Spanning Tree



Probability of success = .85 x .95 x .89 x .95 x 1.0 x .84
= .5735

Minimum Spanning Tree Problem

- Input: Undirected Graph G = (V,E) and a cost function C from E to the reals. C(e) is the cost of edge e.
- Output: A spanning tree T with minimum total cost. That is: T that minimizes

$$C(T) = \sum_{e \in T} C(e)$$

Reducing Best to Minimum

Let P(e) be the probability that an edge doesn't fail.
Define:

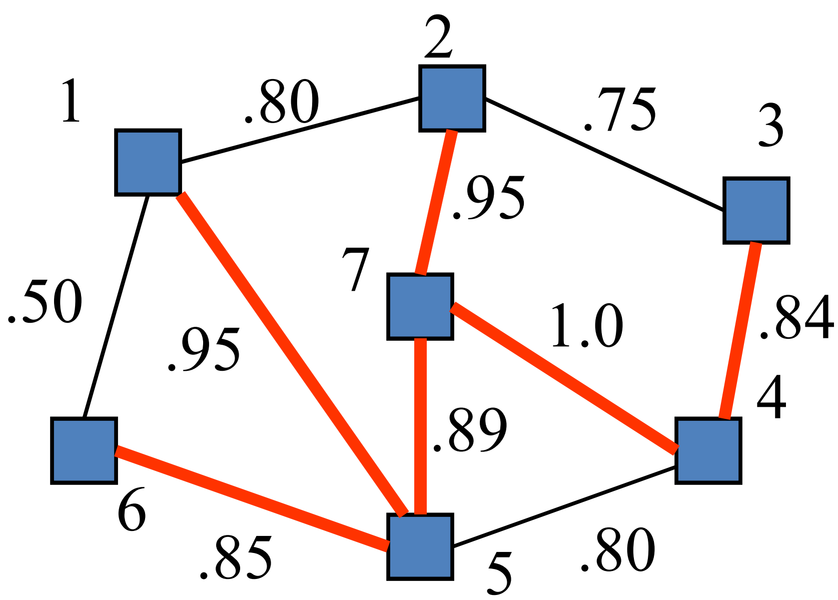
$$C(e) = -\log_{10}(P(e))$$

Minimizing $\sum_{e \in T} C(e)$

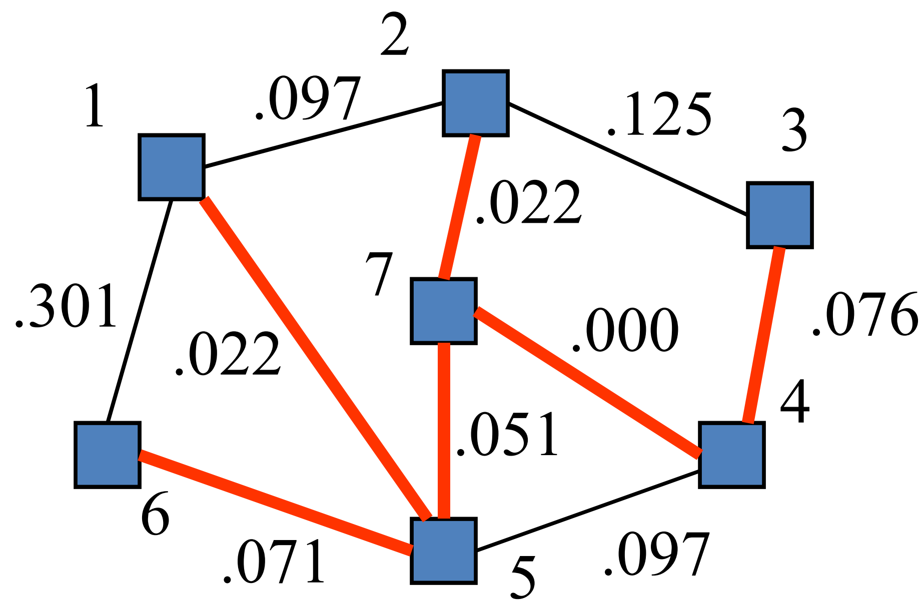
is equivalent to maximizing $\prod_{e \in T} P(e)$

because $\prod_{e \in T} P(e) = \prod_{e \in T} 10^{-C(e)} = 10^{-\sum_{e \in T} C(e)}$

Example of Reduction

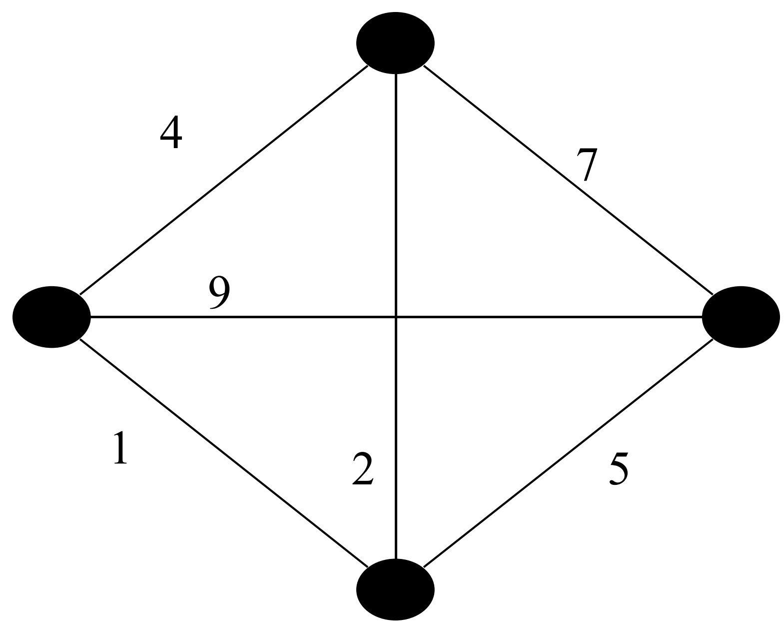


Best Spanning Tree Problem

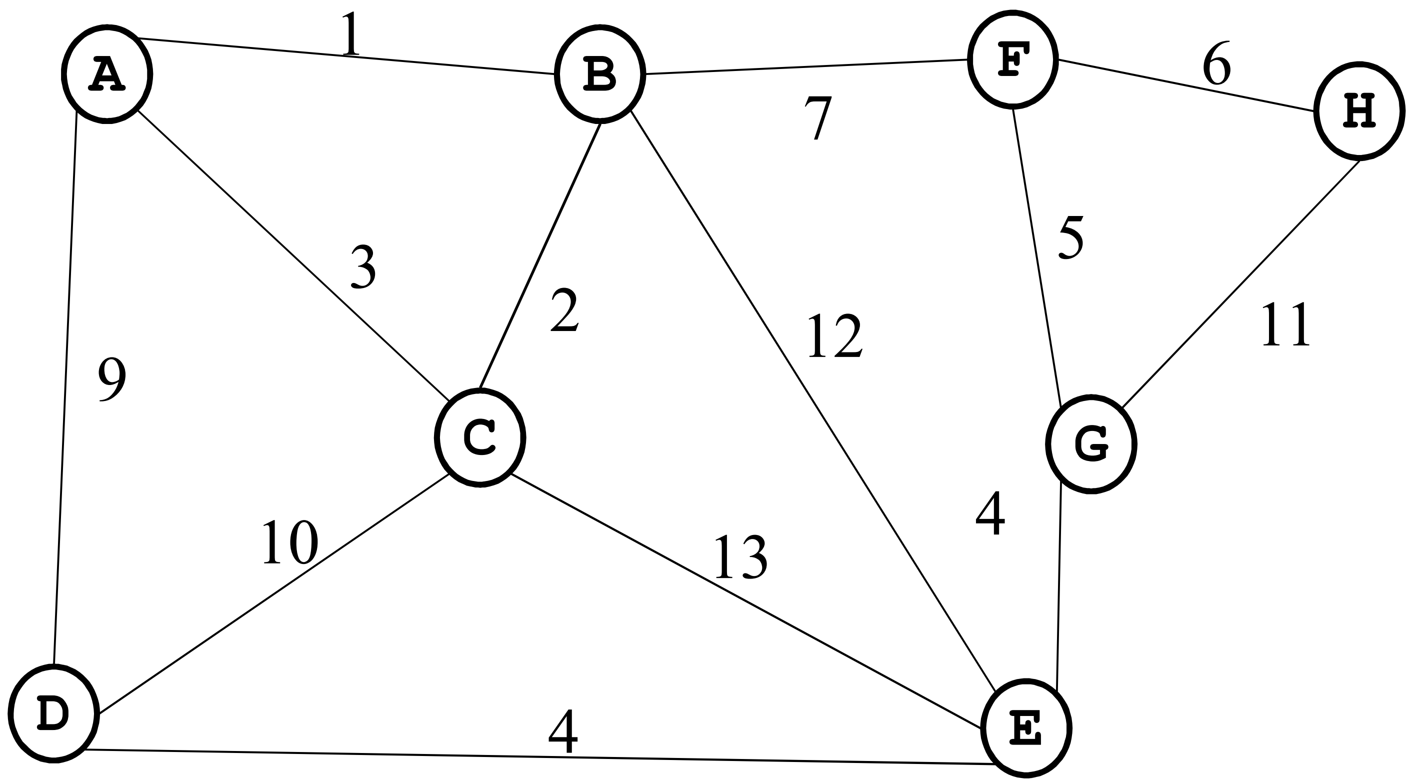


Minimum Spanning Tree Problem

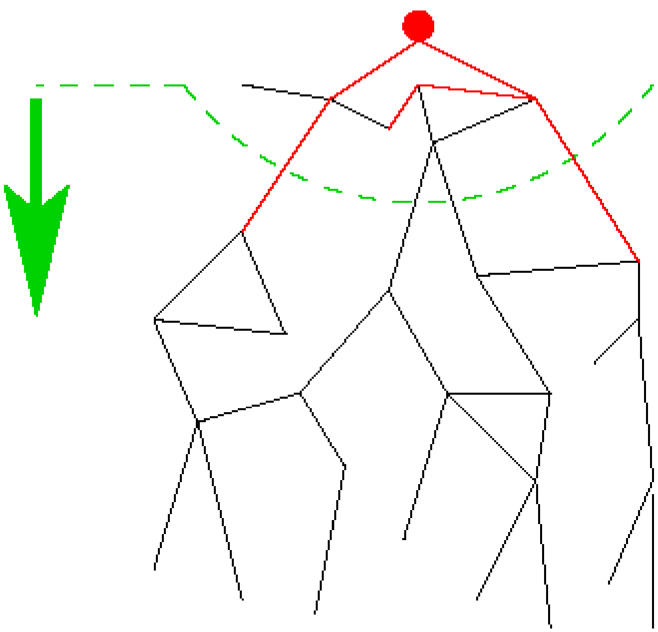
Find the MST



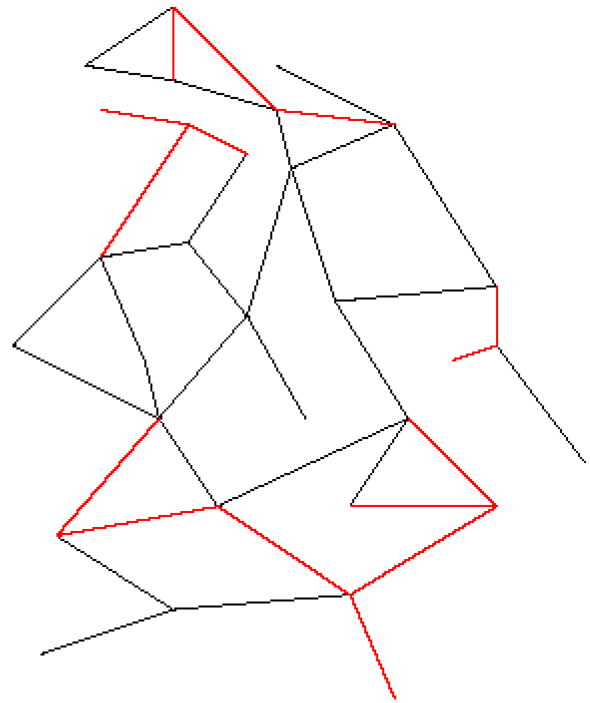
Find the MST



Two Different Approaches



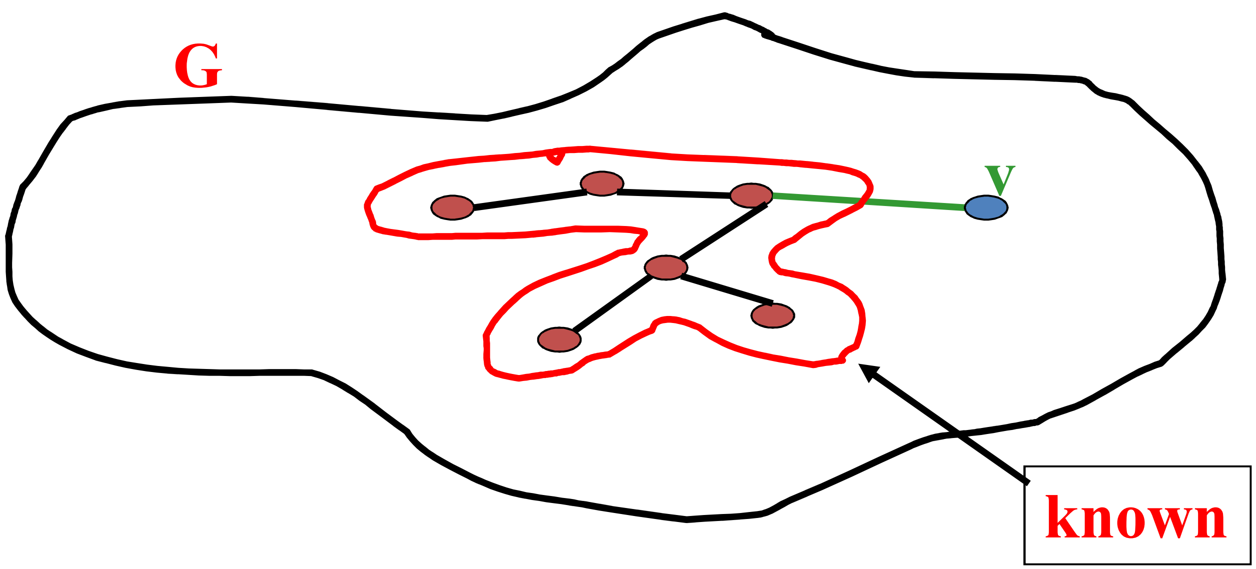
Prim's Algorithm
Looks familiar!



Kruskals's Algorithm
Completely different!

Prim’s algorithm

Idea: Grow a tree by adding an edge from the “known” vertices to the “unknown” vertices. Pick the edge with the smallest weight.



Prim's Algorithm for MST

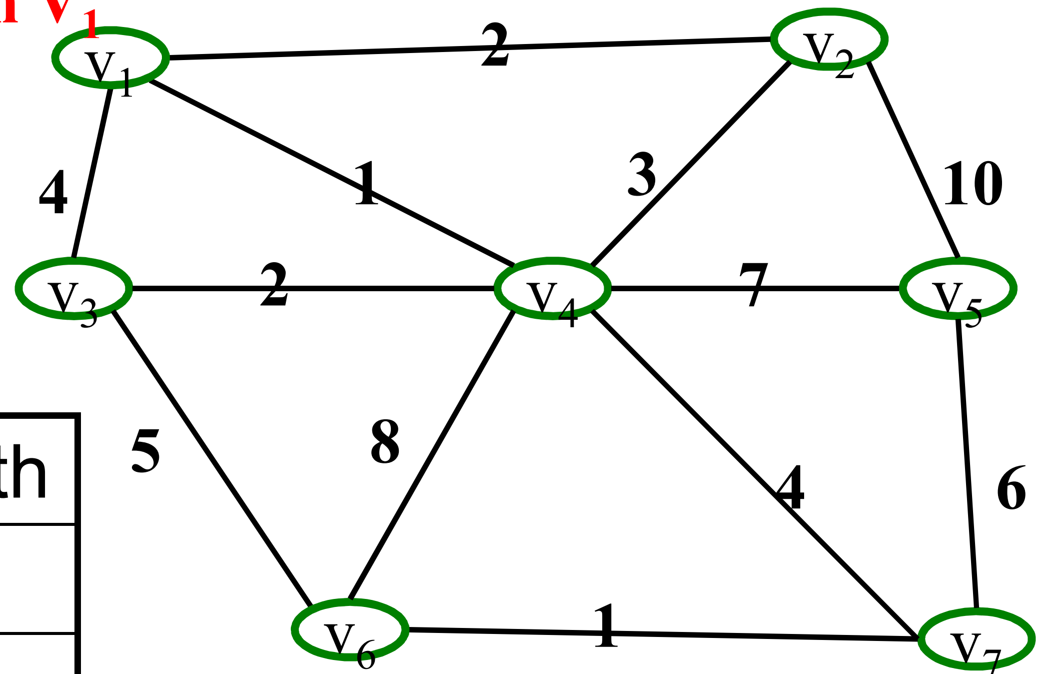
A *node-based* greedy algorithm

Builds MST by greedily adding nodes

1. Select a node to be the "root"
   - mark it as known
   - Update cost of all its neighbors
2. While there are unknown nodes left in the graph
   a. Select an unknown <u>node *b* with the smallest cost</u> from some *known* node *a*
   b. Mark *b* as known
   c. Add (*a*, *b*) to MST
   d. Update cost of all nodes adjacent to *b*

Your Turn

Start with V_1



Find MST using Prim's

V	Kwn	Distance	path
v1			
v2			
v3			
v4			
v5			
v6			
v7			

Order Declared Known:

V_1

Prim's Algorithm Analysis

Running time:

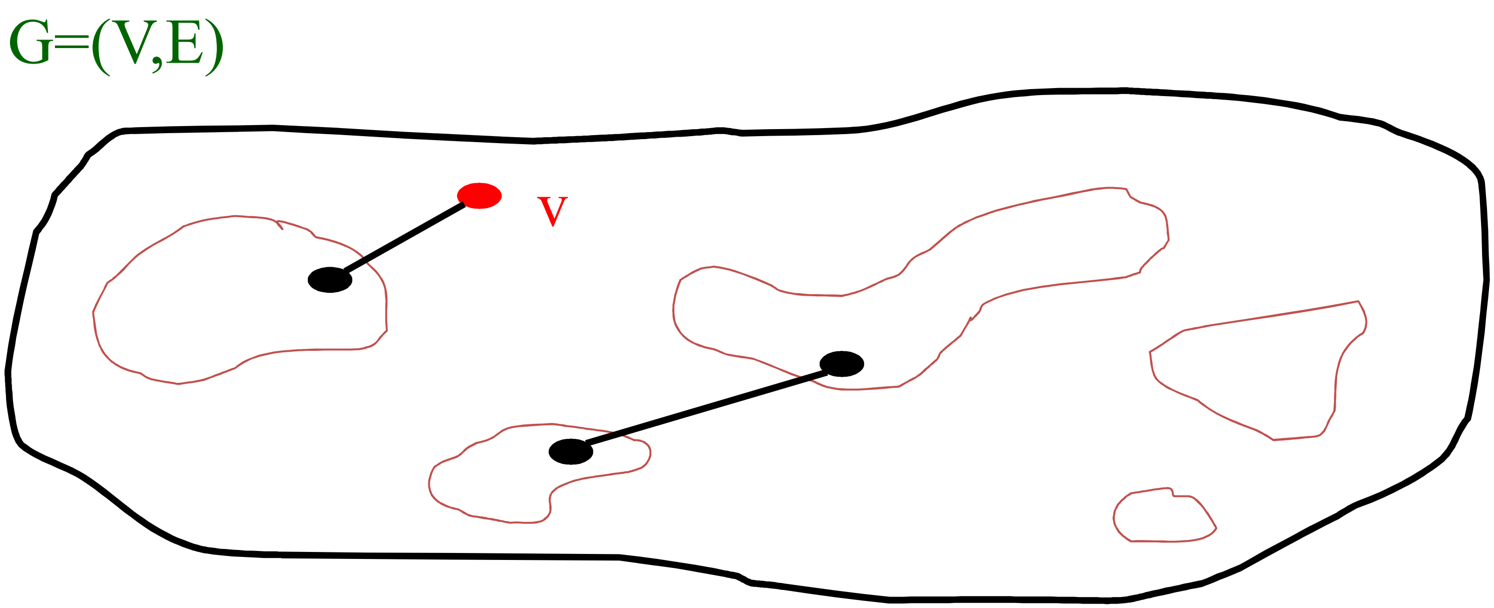
Same as Dijkstra's: $O(|E| \log |V|)$

Correctness:

Proof is similar to Dijkstra's

Kruskal's MST Algorithm

Idea: Grow a forest out of edges that do not create a cycle. Pick an edge with the smallest weight.



Kruskal's Algorithm for MST

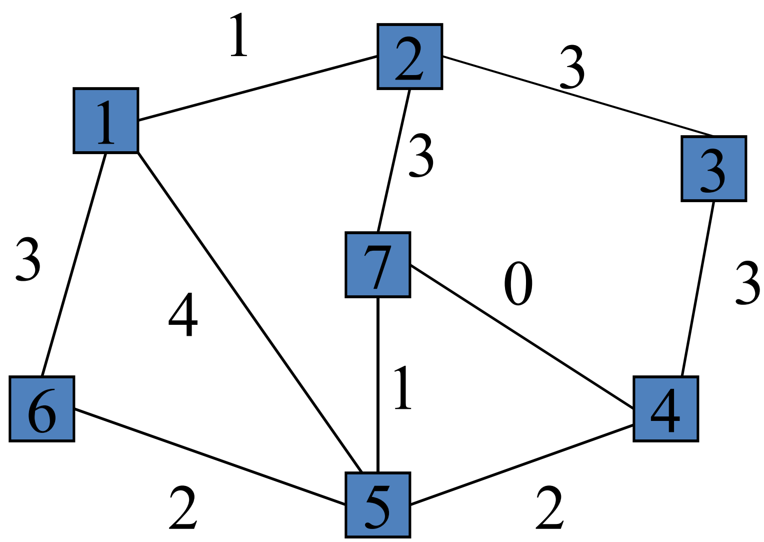
An *edge-based* greedy algorithm

Builds MST by greedily adding edges

1. Initialize with
   - empty MST
   - all vertices marked unconnected
   - all edges unmarked
2. While there are still unmarked edges
   a. Pick the lowest cost edge `(u,v)` and mark it
   b. If `u` and `v` are not already connected, add `(u,v)` to the MST and mark `u` and `v` as connected to each other

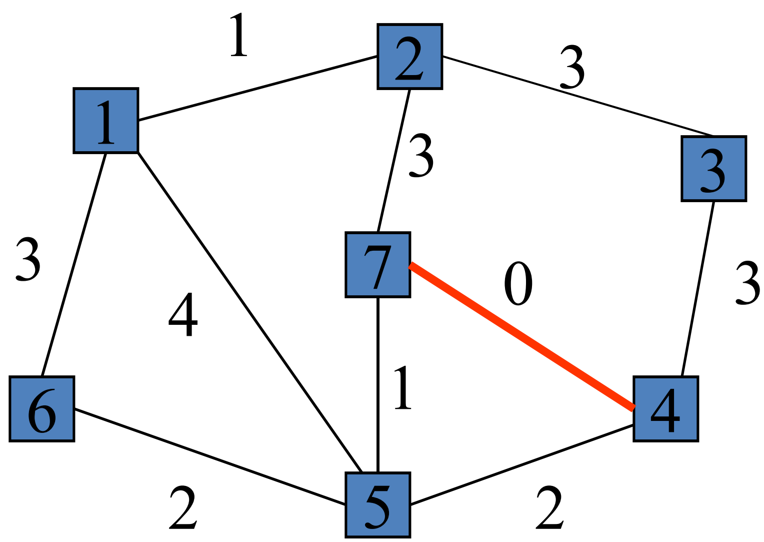
Doesn't it sound familiar?

Example of Kruskal 1



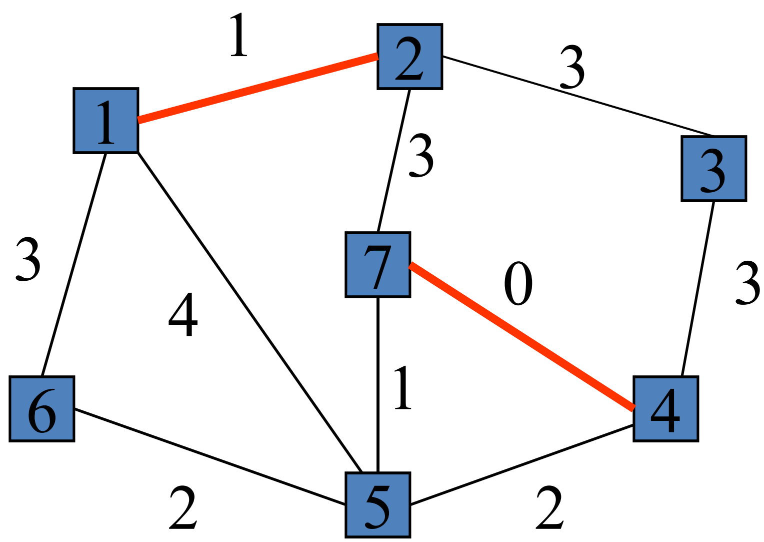
{7,4}	{2,1}	{7,5}	{5,6}	{5,4}	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
0	1	1	2	2	3	3	3	3	4

Example of Kruskal 2



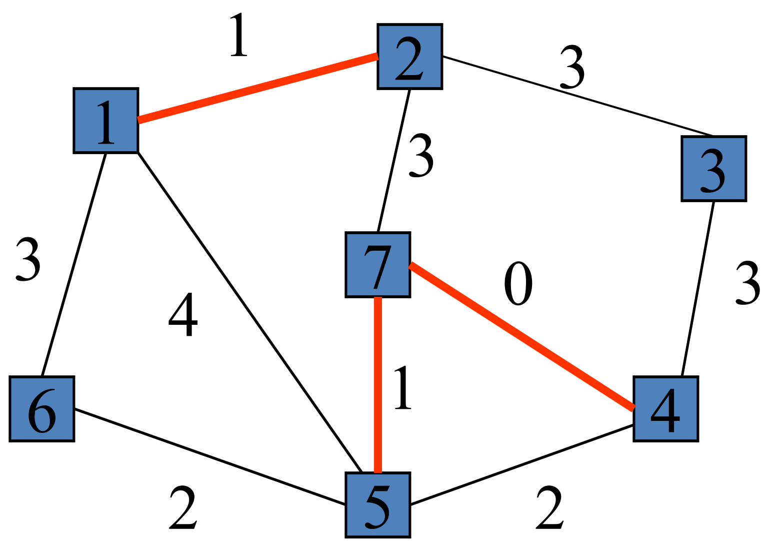
~~{7,4}~~	{2,1}	{7,5}	{5,6}	{5,4}	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
~~0~~	1	1	2	2	3	3	3	3	4

Example of Kruskal 2



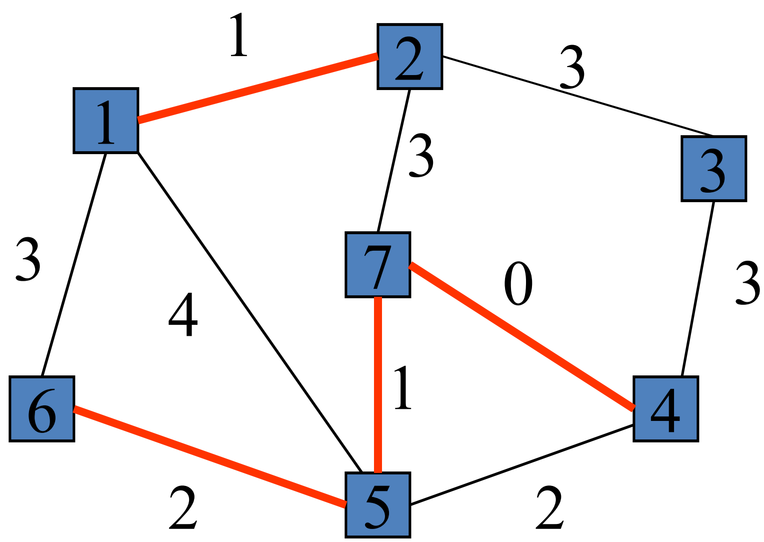
~~{7,4}~~	~~{2,1}~~	{7,5}	{5,6}	{5,4}	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
~~0~~	~~1~~	1	2	2	3	3	3	3	4

Example of Kruskal 3



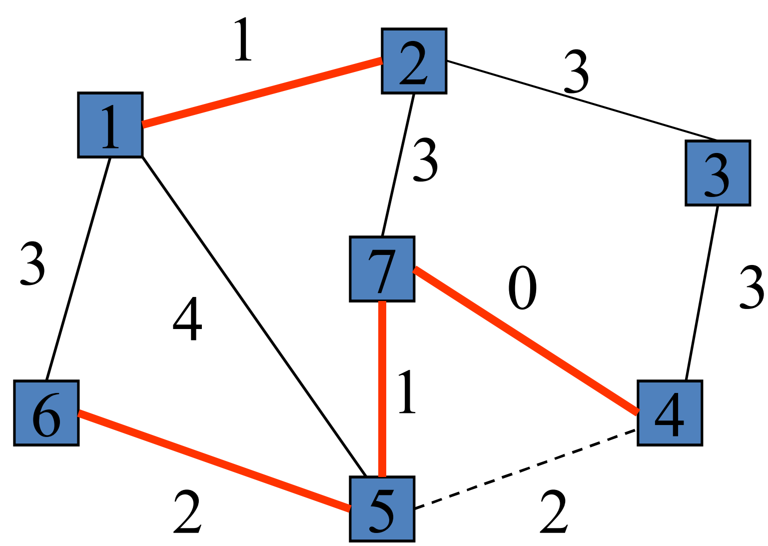
~~{7,4}~~	~~{2,1}~~	~~{7,5}~~	{5,6}	{5,4}	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
~~0~~	~~1~~	~~1~~	2	2	3	3	3	3	4

Example of Kruskal 4



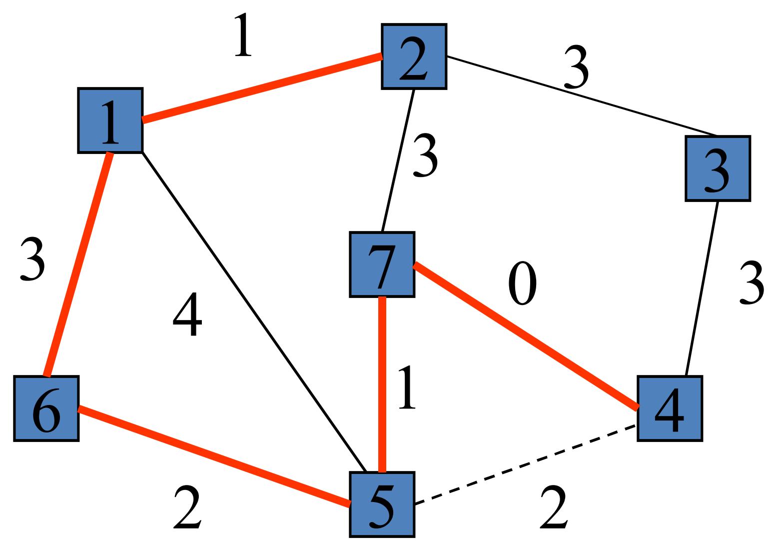
~~{7,4}~~	~~{2,1}~~	~~{7,5}~~	~~{5,6}~~	{5,4}	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
~~0~~	~~1~~	~~1~~	~~2~~	2	3	3	3	3	4

Example of Kruskal 5



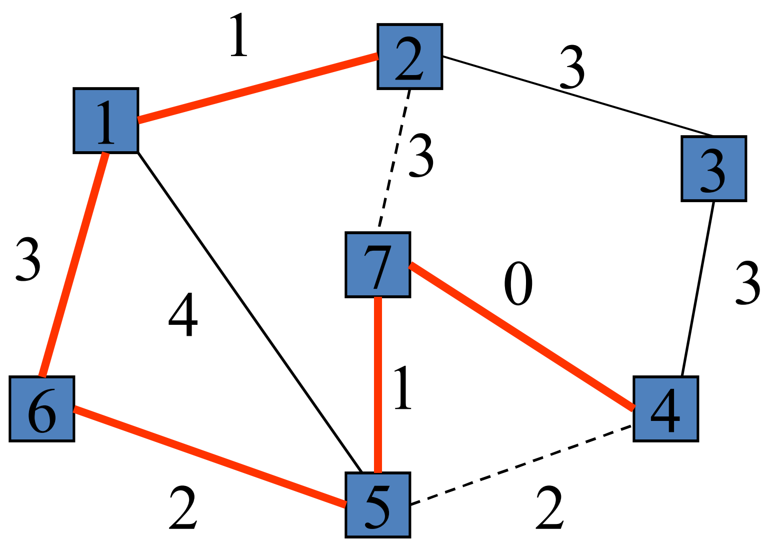
~~{7,4}~~	~~{2,1}~~	~~{7,5}~~	~~{5,6}~~	~~{5,4}~~	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
~~0~~	~~1~~	~~1~~	~~2~~	~~2~~	3	3	3	3	4

Example of Kruskal 6



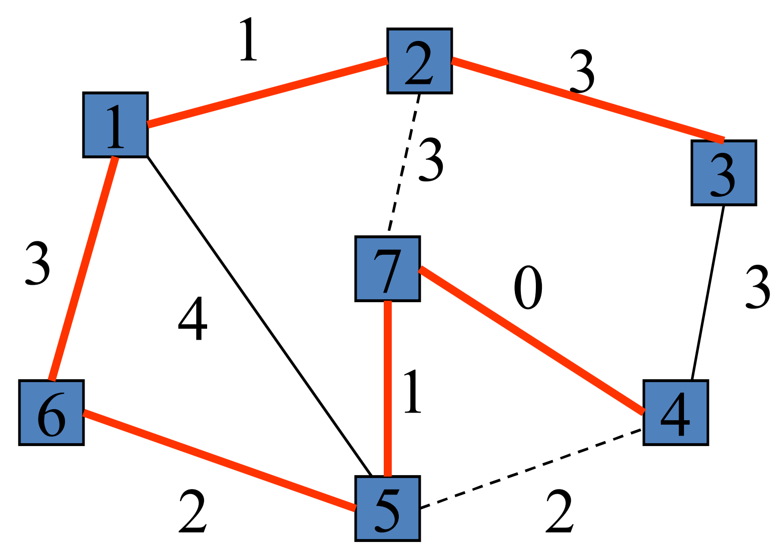
~~{7,4}~~	~~{2,1}~~	~~{7,5}~~	~~{5,6}~~	~~{5,4}~~	~~{1,6}~~	{2,7}	{2,3}	{3,4}	{1,5}
~~0~~	~~1~~	~~1~~	~~2~~	~~2~~	~~3~~	3	3	3	4

Example of Kruskal 7



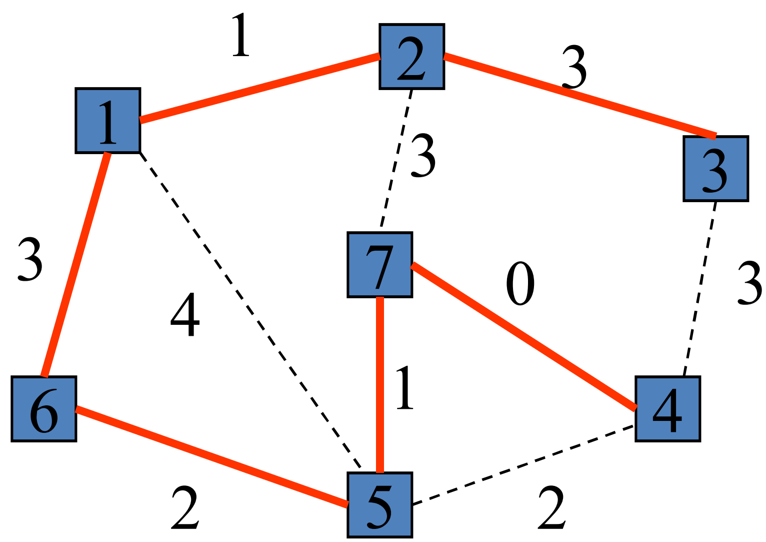
~~{7,4}~~	~~{2,1}~~	~~{7,5}~~	~~{5,6}~~	~~{5,4}~~	~~{1,6}~~	~~{2,7}~~	{2,3}	{3,4}	{1,5}
~~0~~	~~1~~	~~1~~	~~2~~	~~2~~	~~3~~	~~3~~	3	3	4

Example of Kruskal 7



~~{7,4}~~	~~{2,1}~~	~~{7,5}~~	~~{5,6}~~	~~{5,4}~~	~~{1,6}~~	~~{2,7}~~	~~{2,3}~~	{3,4}	{1,5}
~~0~~	~~1~~	~~1~~	~~2~~	~~2~~	~~3~~	~~3~~	~~3~~	3	4

Example of Kruskal 8,9



~~{7,4}~~	~~{2,1}~~	~~{7,5}~~	~~{5,6}~~	~~{5,4}~~	~~{1,6}~~	~~{2,7}~~	~~{2,3}~~	~~{3,4}~~	~~{1,5}~~
~~0~~	~~1~~	~~1~~	~~2~~	~~2~~	~~3~~	~~3~~	~~3~~	~~3~~	~~4~~

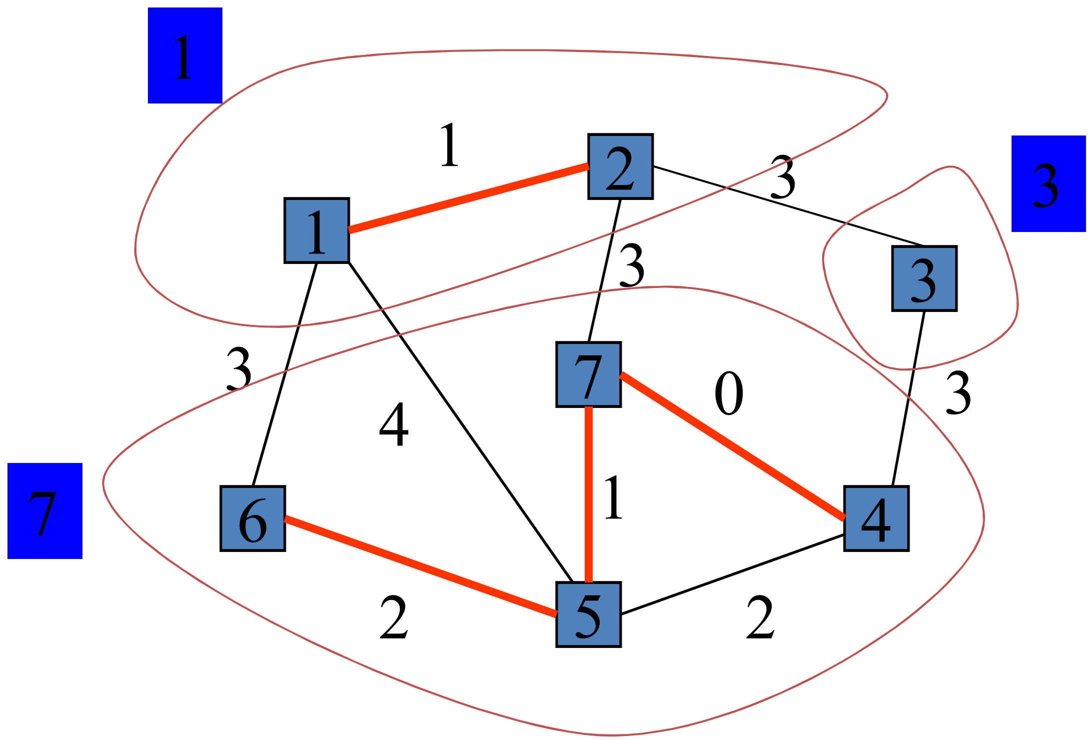
Data Structures for Kruskal

- Sorted edge list

{7,4}	{2,1}	{7,5}	{5,6}	{5,4}	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
0	1	1	2	2	3	3	3	3	4

- Disjoint Union / Find
  - Union(a,b) - union the disjoint sets named by a and b
  - Find(a) returns the name of the set containing a

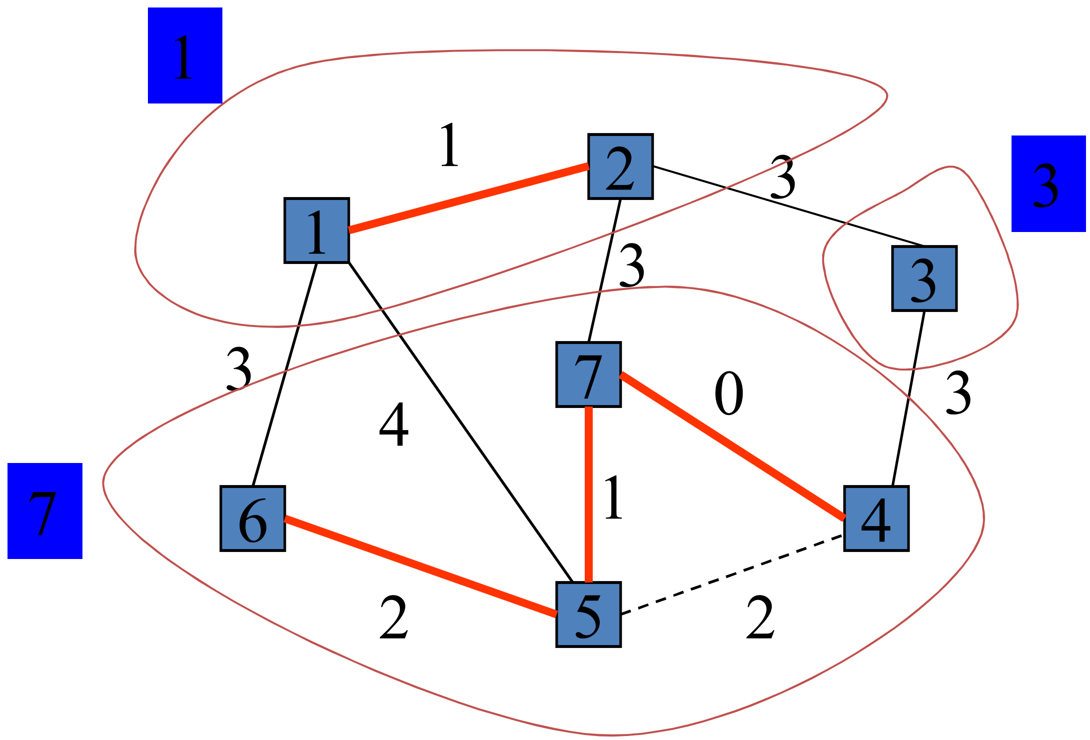
Example of DU/F 1



Find(5) = 7
Find(4) = 7

~~{7,4}~~	~~{2,1}~~	~~{7,5}~~	~~{5,6}~~	{5,4}	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
~~0~~	~~1~~	~~1~~	~~2~~	2	3	3	3	3	4

Example of DU/F 2

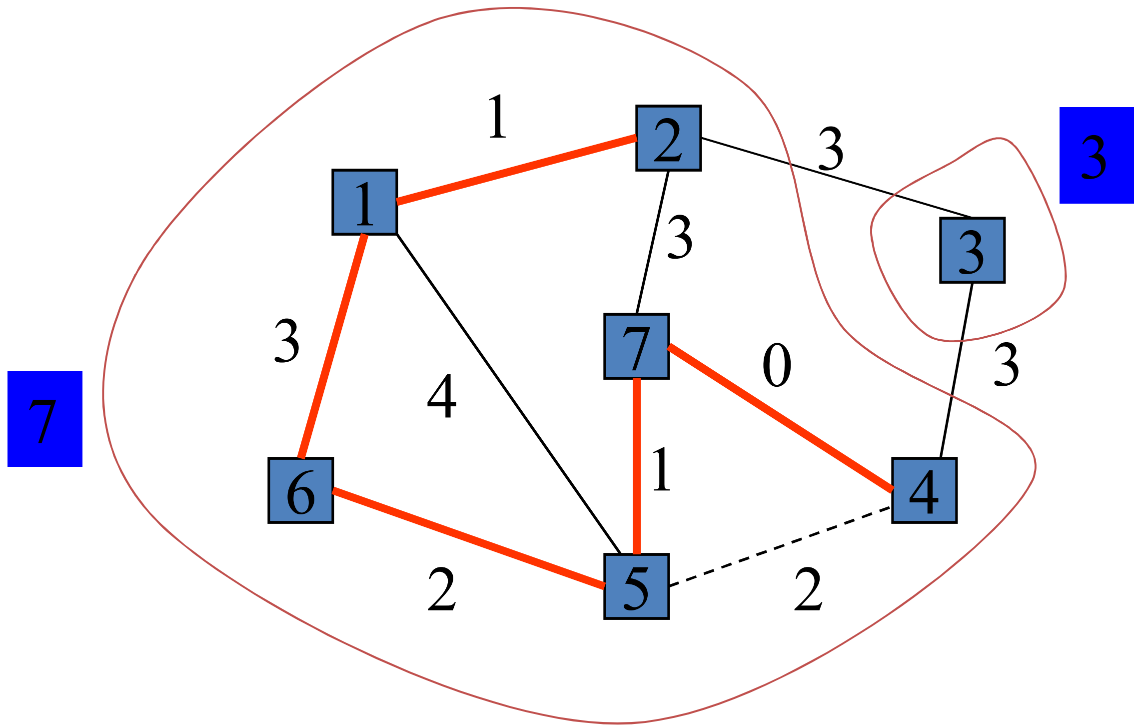


Find(1) = 1
Find(6) = 7

~~{7,4}~~	~~{2,1}~~	~~{7,5}~~	~~{5,6}~~	~~{5,4}~~	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
~~0~~	~~1~~	~~1~~	~~2~~	~~2~~	3	3	3	3	4

Example of DU/F 3

Union(1,7)



~~{7,4}~~	~~{2,1}~~	~~{7,5}~~	~~{5,6}~~	~~{5,4}~~	~~{1,6}~~	{2,7}	{2,3}	{3,4}	{1,5}
~~0~~	~~1~~	~~1~~	~~2~~	~~2~~	~~3~~	3	3	3	4

Kruskal’s Algorithm with DU / F

```
Sort the edges by increasing cost;
Initialize A to be empty;
for each edge {i,j} chosen in increasing order do
    u := Find(i);
    v := Find(j);
    if not(u = v) then
        add {i,j} to A;
        Union(u,v);
```

Kruskal code

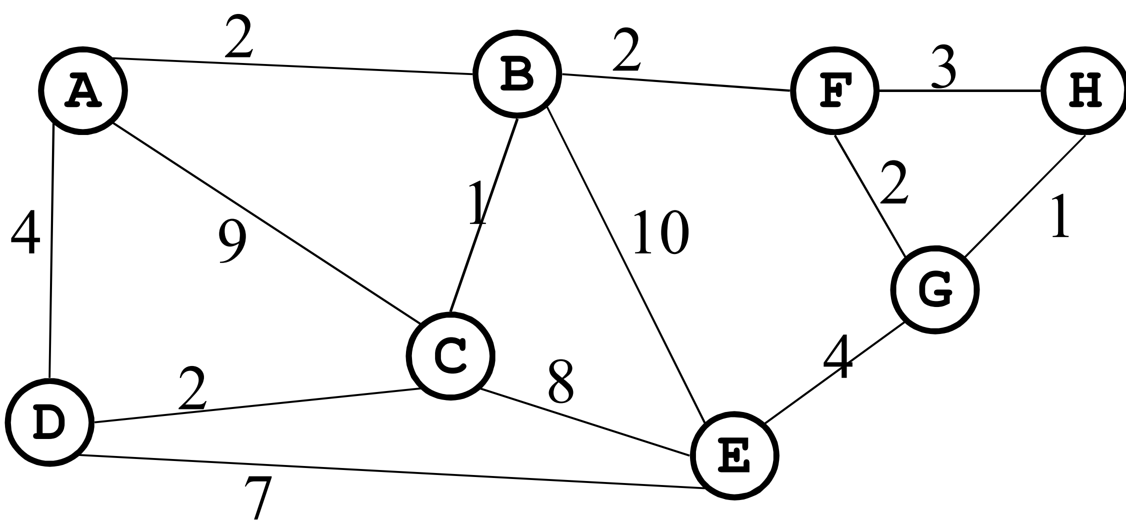
```
void Graph::kruskal(){
  int edgesAccepted = 0;
  DisjSet s(NUM_VERTICES);

  while (edgesAccepted < NUM_VERTICES - 1){
    e = smallest weight edge not deleted yet;
    // edge e = (u, v)
    uset = s.find(u);
    vset = s.find(v);
    if (uset != vset){
      edgesAccepted++;
      s.unionSets(uset, vset);
    }
  }
}
```

|E| heap ops

2|E| finds

|V| unions

Find MST using Kruskal's



Total Cost:

- Now find the MST using Prim's method.
- Under what conditions will these methods give the same result?