CSE 373
Data Structures & Algorithms

Lecture 14

Sorting (II)

Chapter 7 in Weiss

Announcement

- Homework 3 due Thursday, 11:45pm

- Points for each problem are posted on the Website

“Divide and Conquer”

- Very important strategy applied to many computer science problems:
  - Divide problem into smaller parts
  - Independently solve the parts
  - Combine solutions to get overall solution

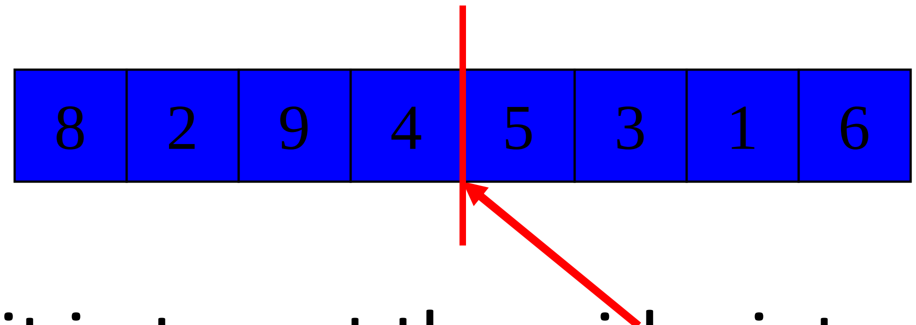
“Divide and Conquer”

- Two divide and conquer sorting methods:

- **Idea 1**: Divide array into two halves, *recursively* sort left and right halves, then *merge* two halves → known as Mergesort

- **Idea 2 :** Partition array into small items and large items, then recursively sort the two smaller portions → known as Quicksort

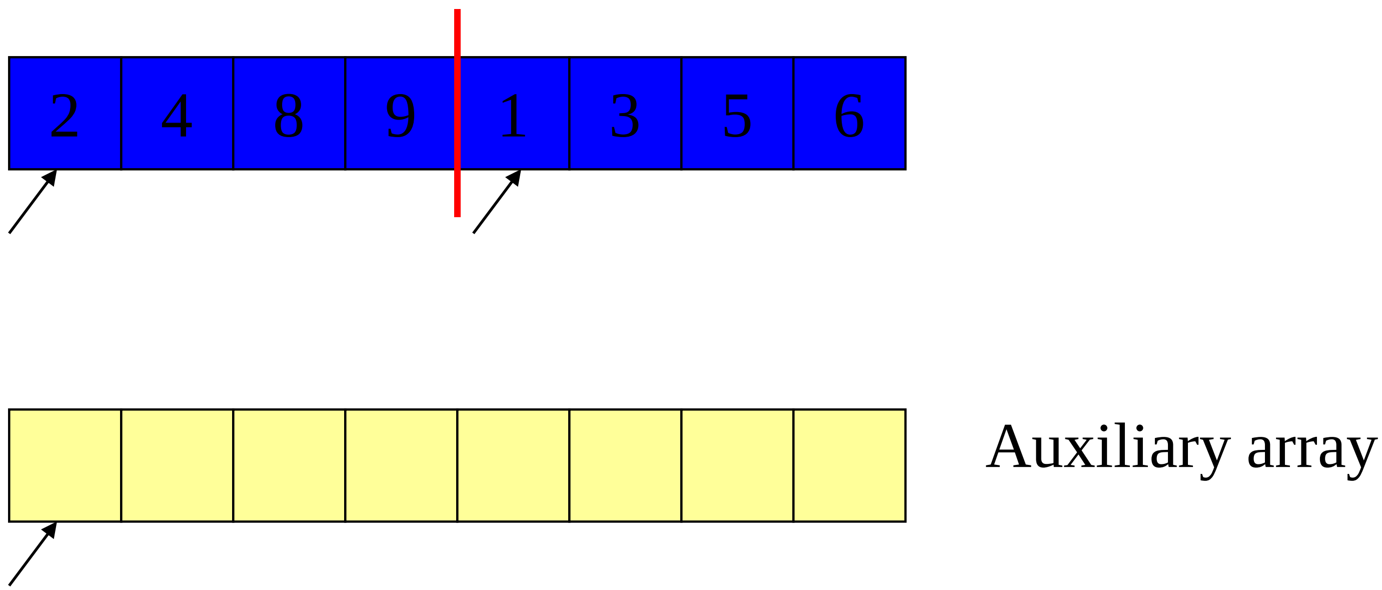
Mergesort



- Divide it in two at the midpoint
- Conquer each side in turn
  (by recursively sorting)
- Merge two halves together

Auxiliary Array

- The merging requires an auxiliary array.



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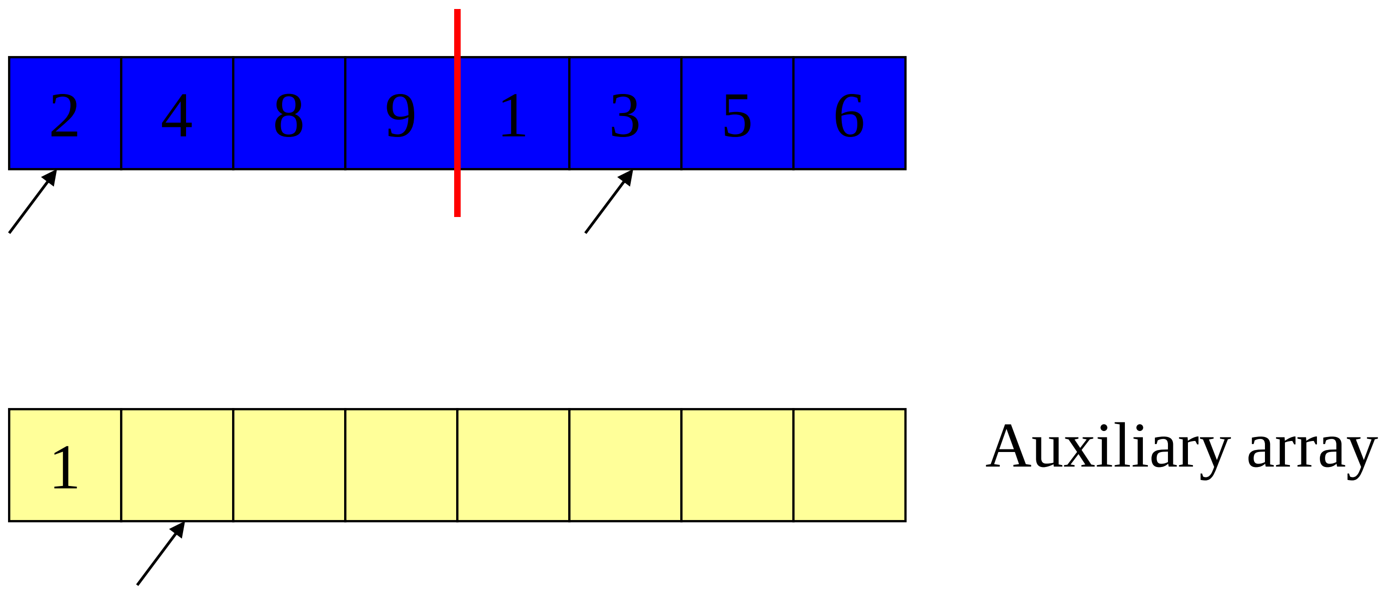

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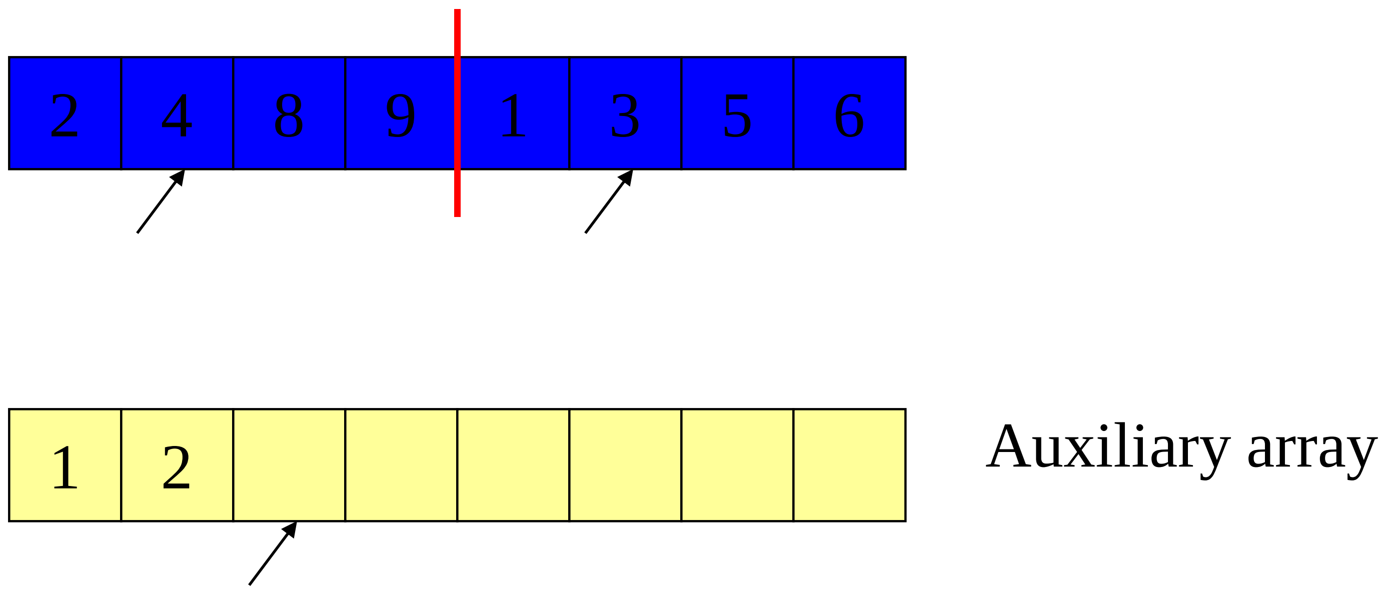

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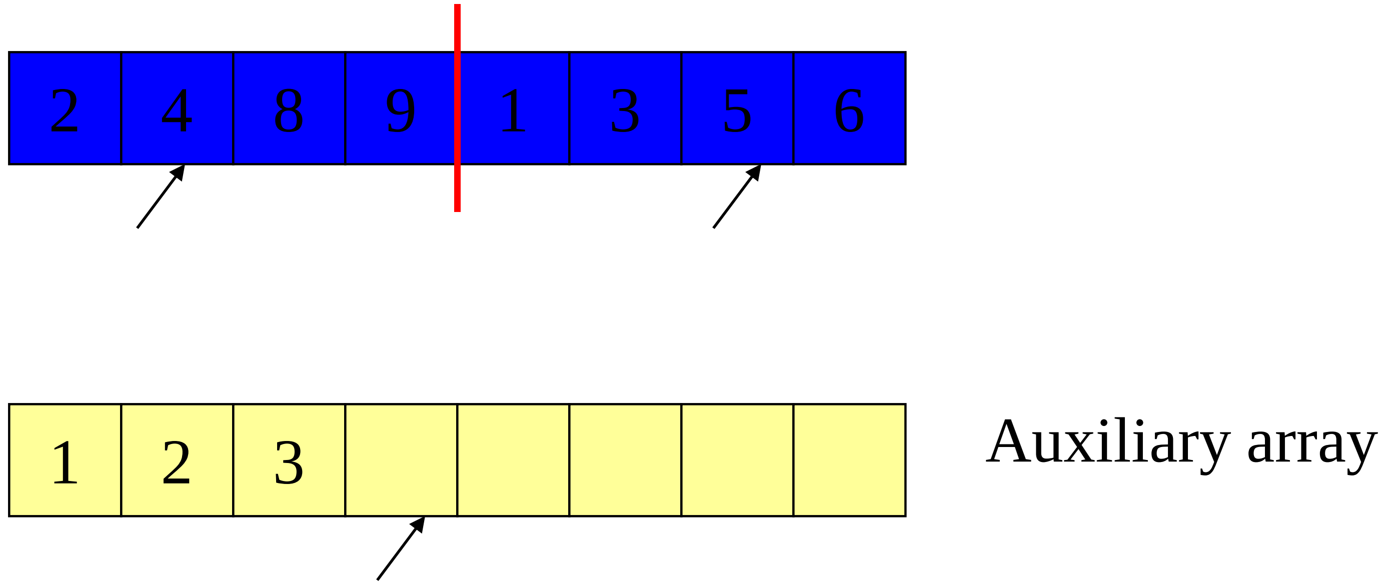

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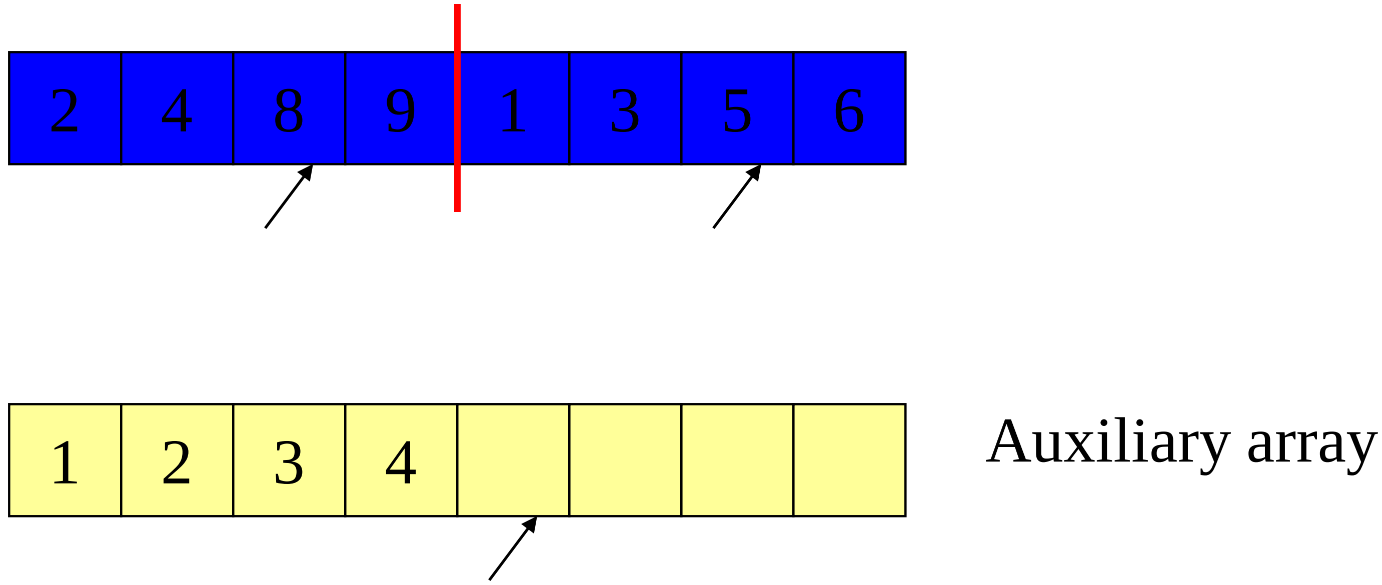

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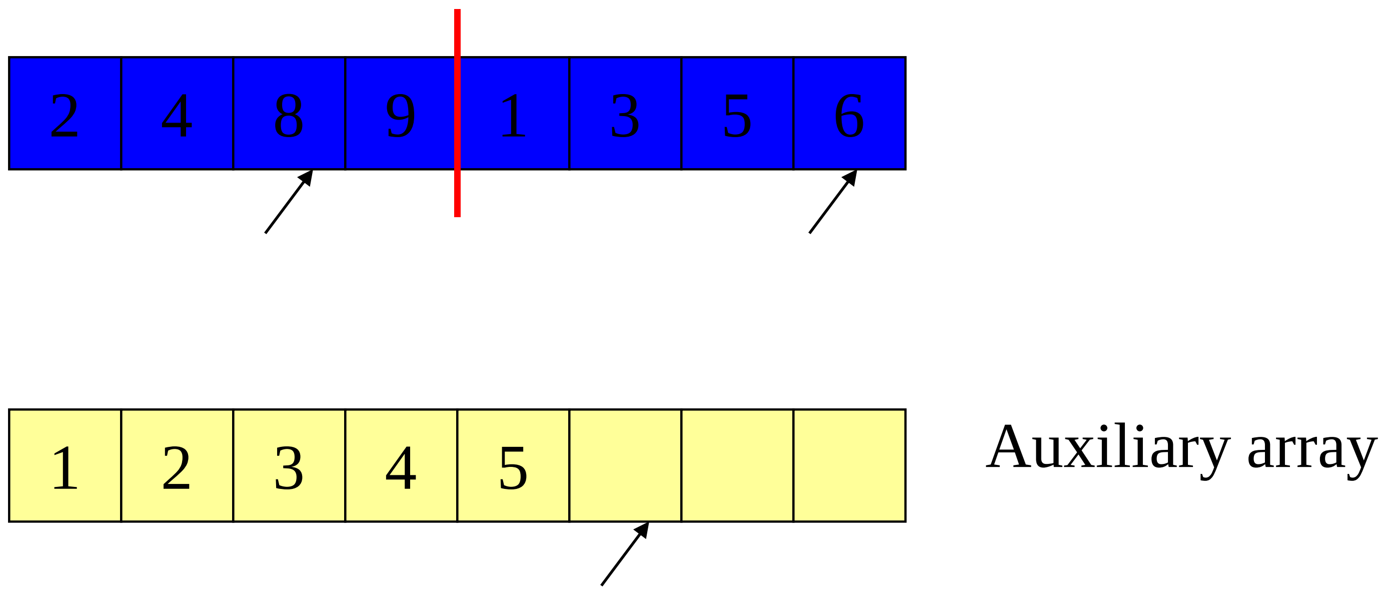

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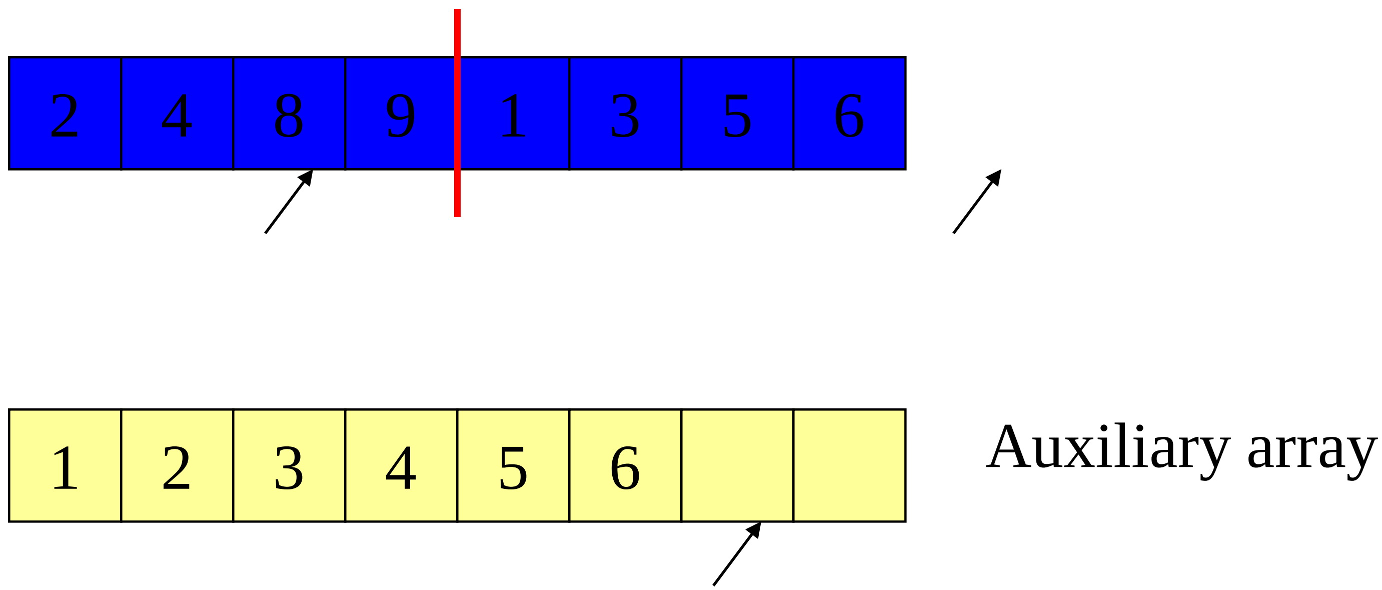

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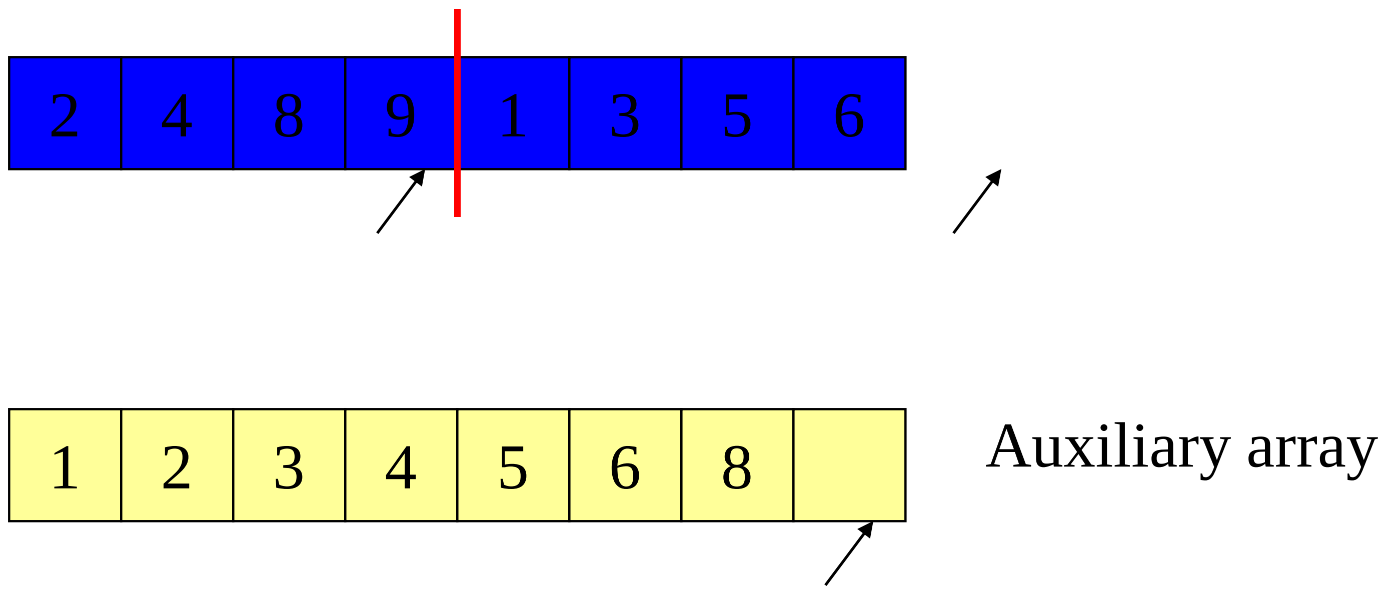

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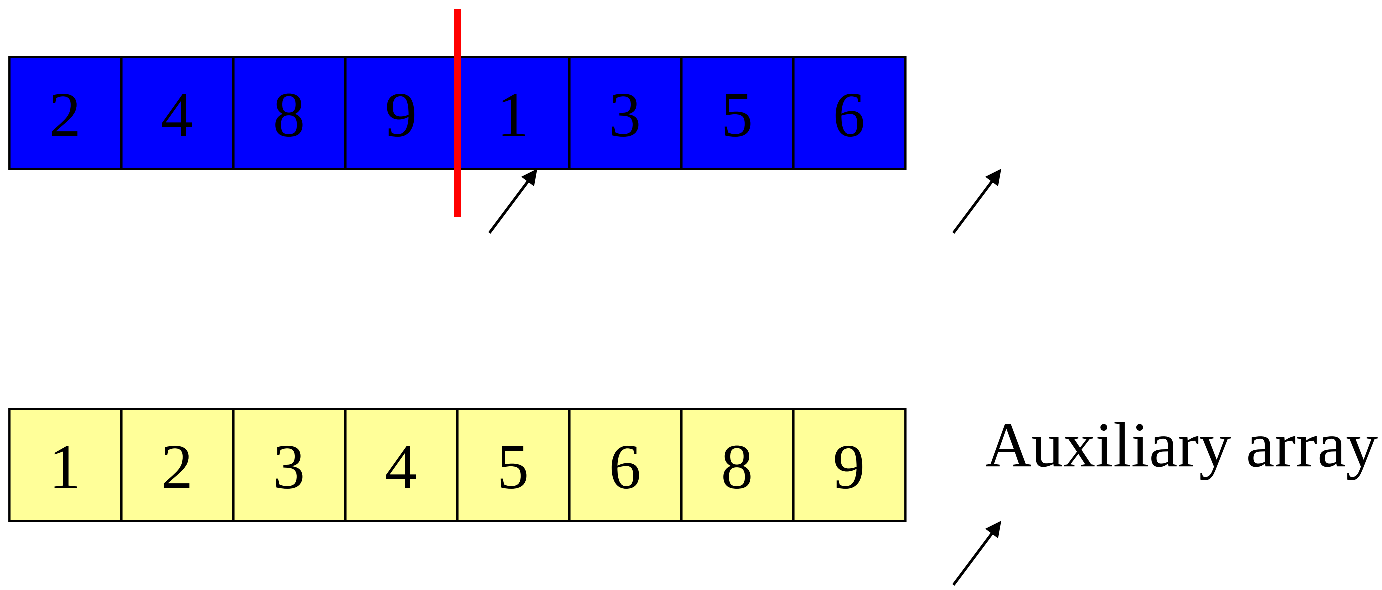
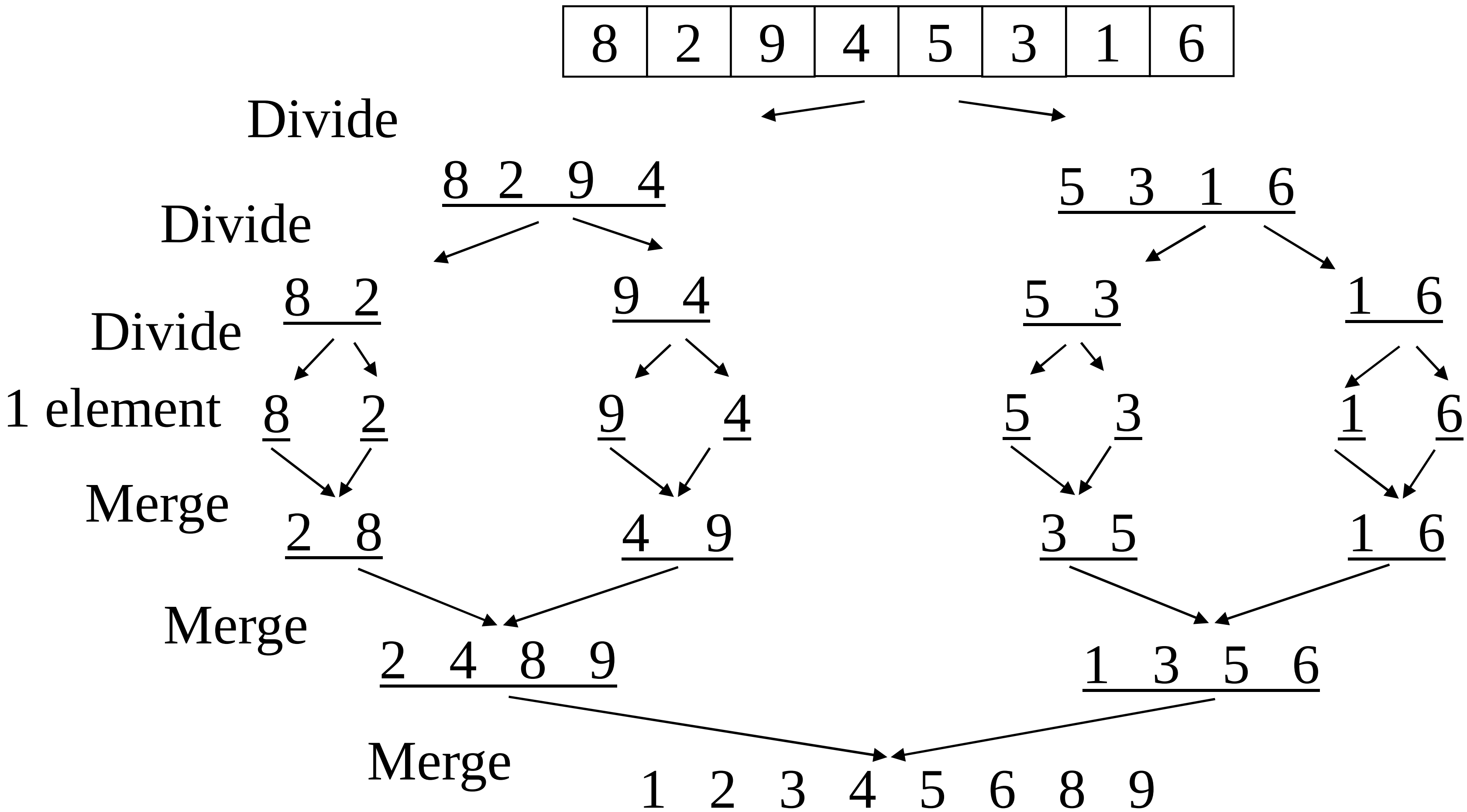
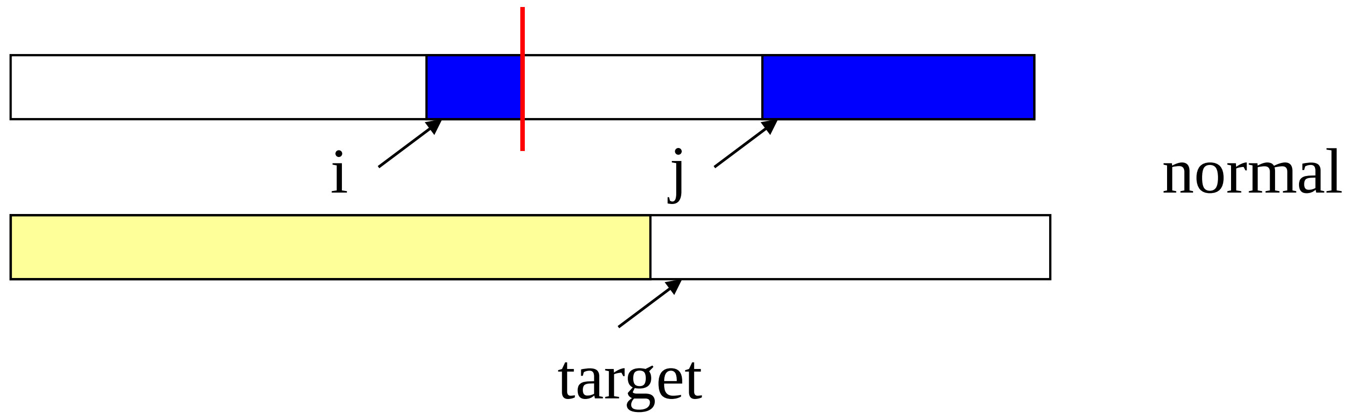


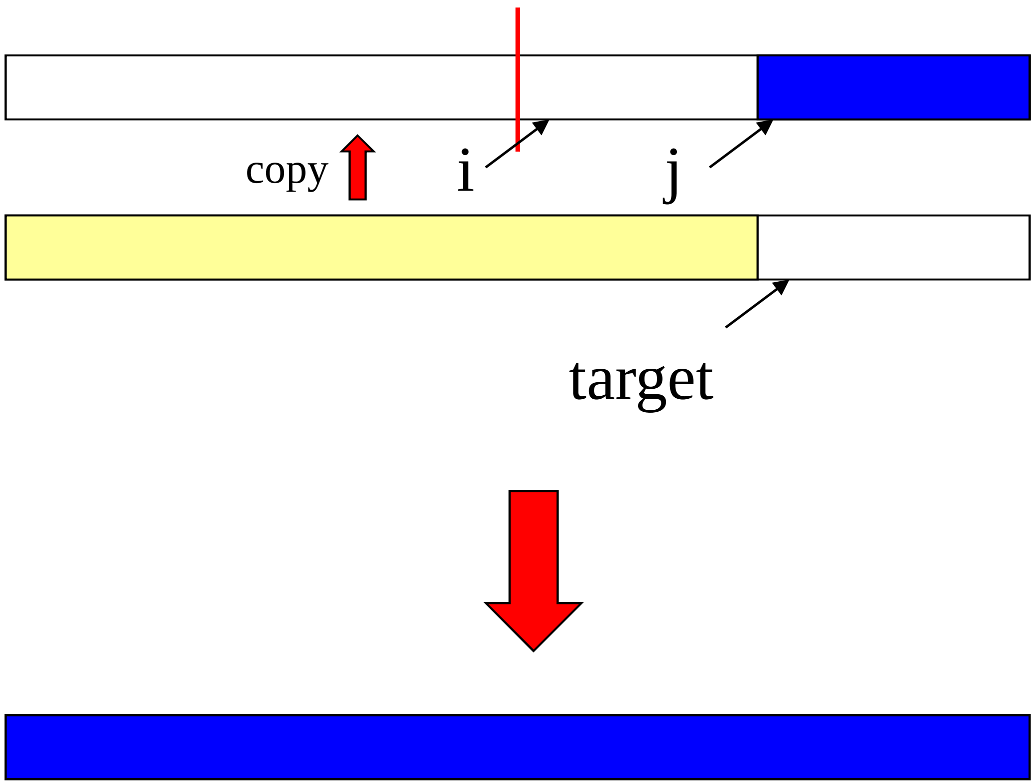
Mergesort Example



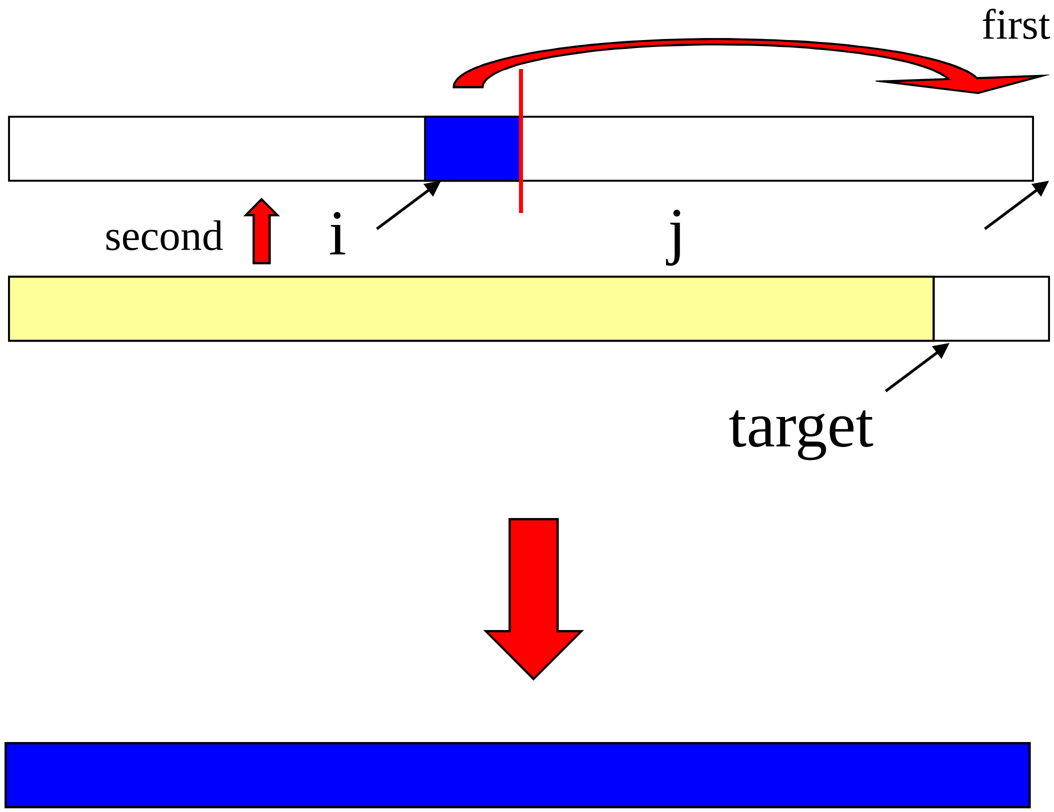
Typical Merging



Left Side Completes First



Right Side Completes First



Right completed first

Recursive Mergesort

```
MainMergesort(A[1..n]: integer array, n : integer) : {
  T[1..n]: integer array;
  Mergesort[A,T,1,n];
}


Mergesort(A[], T[] : integer array, left, right : integer) : {
  if left < right then
    mid := (left + right)/2;
    Mergesort(A,T,left,mid);
    Mergesort(A,T,mid+1,right);
    Merge(A,T,left,right);
}
```

Merging

```
Merge(A[], T[] : integer array, left, right : integer) : {
  mid, i, j, k, l, target : integer;
  mid := (right + left)/2;
  i := left; j := mid + 1; target := left;
  while i ≤ mid and j ≤ right do
    if A[i] ≤ A[j]
       then T[target] := A[i] ; i:= i + 1;
       else T[target] := A[j]; j := j + 1;
    target := target + 1;
  if i > mid then //left completed//
    for k := left to target-1 do A[k] := T[k];
  if j > right then //right completed//
    k : = mid; l := right;
    while k ≥ i do A[l] := A[k]; k := k-1; l := l-1;
    for k := left to target-1 do A[k] := T[k];
}
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Merging

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```

Mergesort Analysis

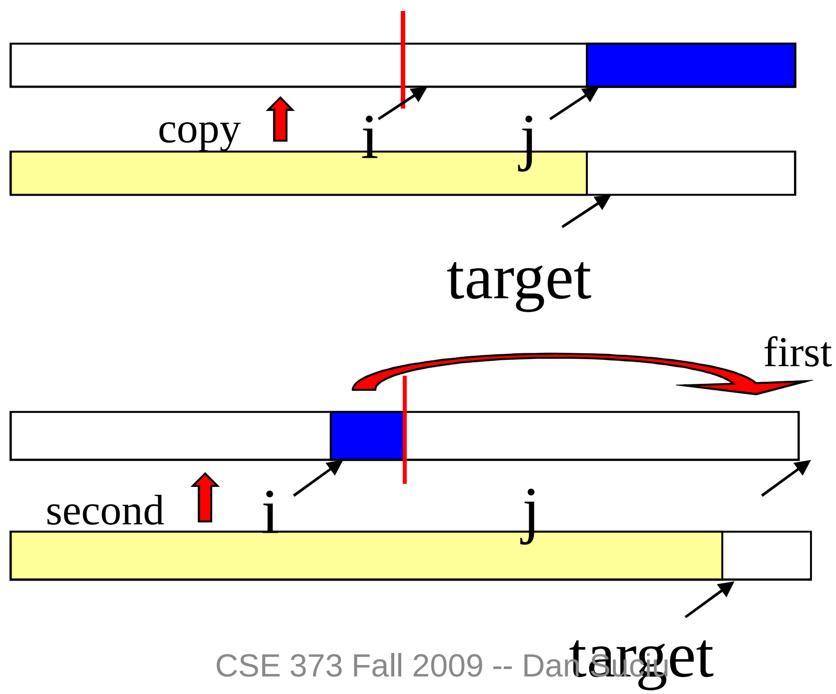
- Let T(N) be the running time for an array of N elements
- Mergesort divides array in half and calls itself on the two halves. After these recursive calls complete, it merges both halves using a temporary array
- Each recursive call takes T(N/2) and merging takes O(N)

Mergesort Recurrence Relation

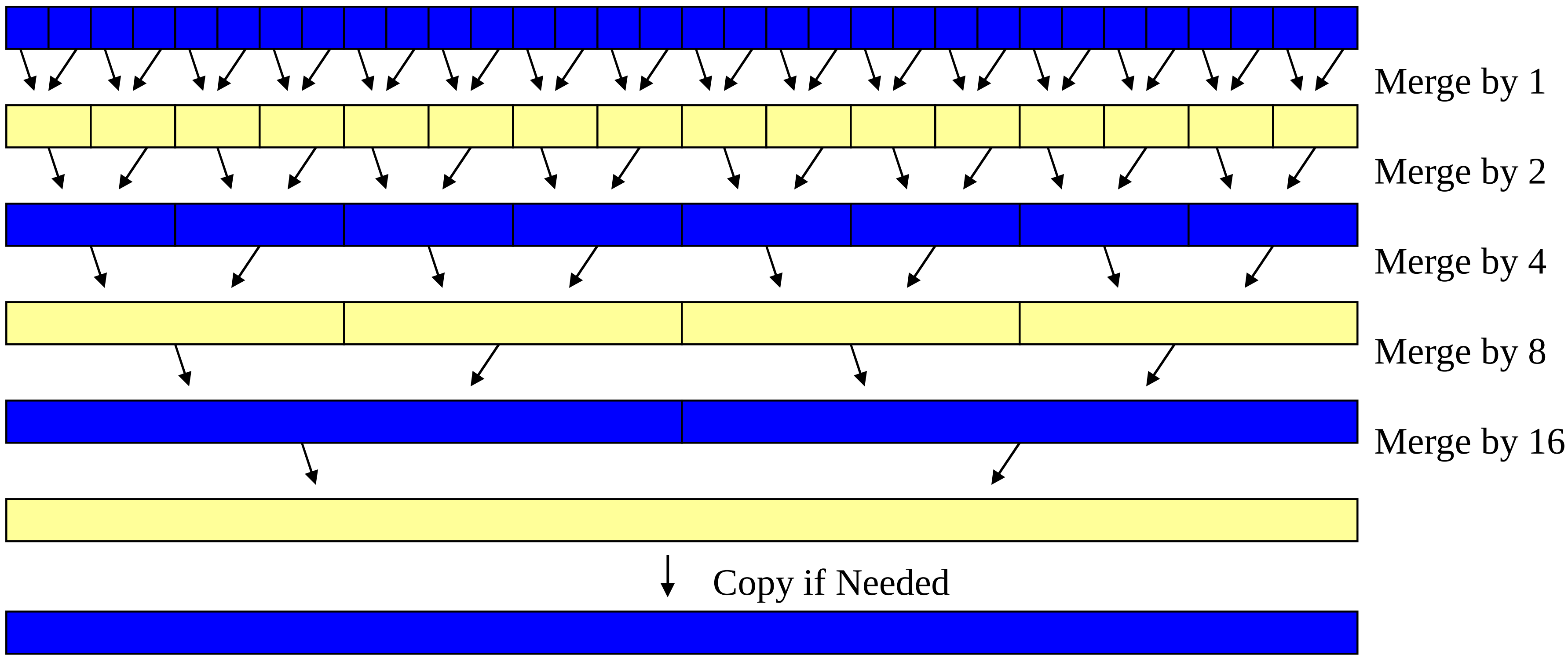
- The recurrence relation for T(N) is:
  - $T(1) \leq c$
    - base case: 1 element array → constant time
  - $T(N) \leq 2T(N/2) + dN$
    - Sorting n elements takes
      - the time to sort the left half
      - plus the time to sort the right half
      - plus an O(N) time to merge the two halves
- $T(N) = O(N \log N)$

Ideas for Obvious Improvement?

- Half our copies are wasted
- Cleaning up the temporary storage:



Iterative Mergesort



Iterative pseudocode

- Sort(array A of length N)
  - Let m = 2, let B be temp array of length N
  - While m<N
    - For i = 1…N in increments of m
      - merge A[i…i+m/2] and A[i+m/2…i+m] into B[i…i+m]
    - Swap role of A and B
    - m=m*2
  - If needed, copy B back to A

Properties of Mergesort

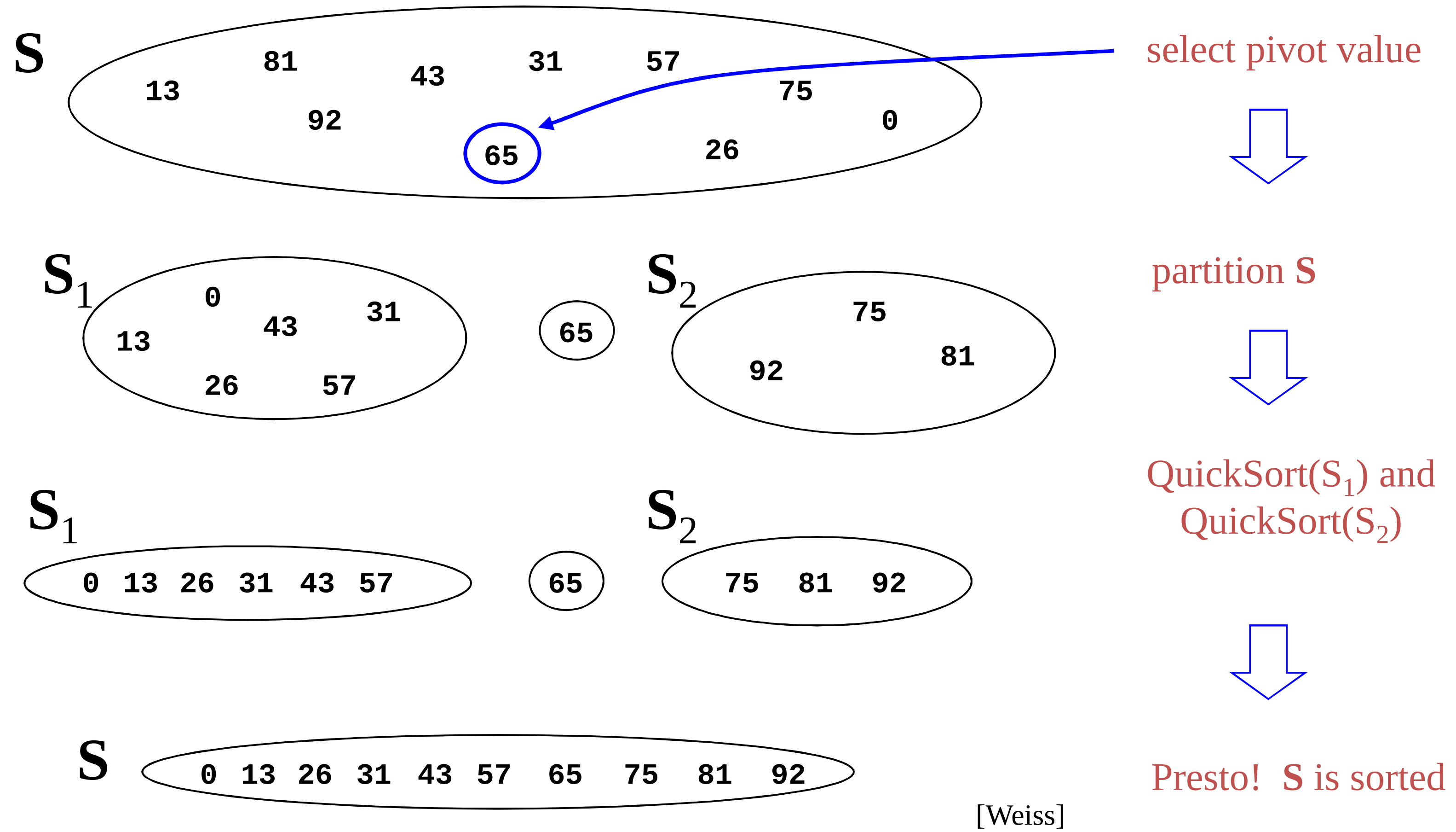
- Not in-place
  - Requires an auxiliary array

- Iterative Mergesort reduces copying

Quicksort

- Quicksort uses a divide and conquer strategy, but does not require the O(N) extra space that MergeSort does
  - Partition array into left and right sub-arrays
    - the elements in left sub-array are all less than pivot
    - elements in right sub-array are all greater than pivot
  - Recursively sort left and right sub-arrays
  - Concatenate left and right sub-arrays in O(1) time

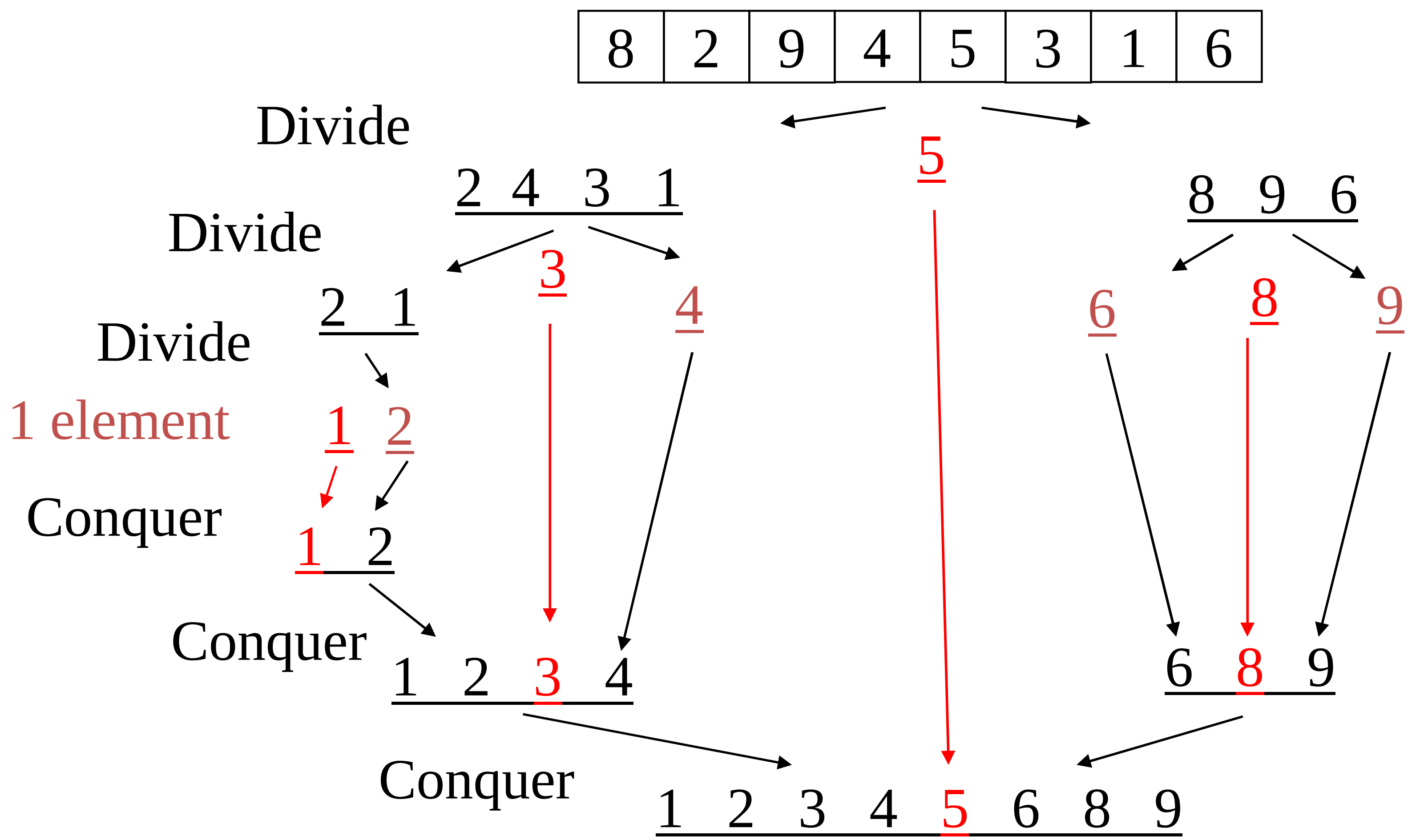
The steps of QuickSort



“Four easy steps”

- To sort an array **S**
  - If the number of elements in **S** is 0 or 1, then return. The array is sorted.
  - Pick an element *v* in **S**.
    This is the *pivot* value.
  - Partition **S**-{*v*} into two disjoint subsets,
    $\mathbf{S}_1$ = {all values $\leq v$}, and $\mathbf{S}_2$ = {all values $\geq v$}.
  - Return QuickSort($\mathbf{S}_1$), *v*, QuickSort($\mathbf{S}_2$)

Quicksort Example



Details, details

- Picking the pivot
  - want a value that causes $|S_1|$ and $|S_2|$ to be non-zero and close to equal in size
- Implementing the actual partitioning
- Dealing with cases where elements are equal to the pivot

Potential Pivot Rules

- Chose A[left]
  - Fast, but too biased, enables worst-case
- Chose A[random], left $\leq$ random $\leq$ right
  - Completely unbiased
  - Will cause relatively even split, but slow
- Median of three, A[left], A[right], A[(left+right)/2]
  - A common approach, tends to be unbiased, and does a little sorting on the side.

Quicksort Partitioning

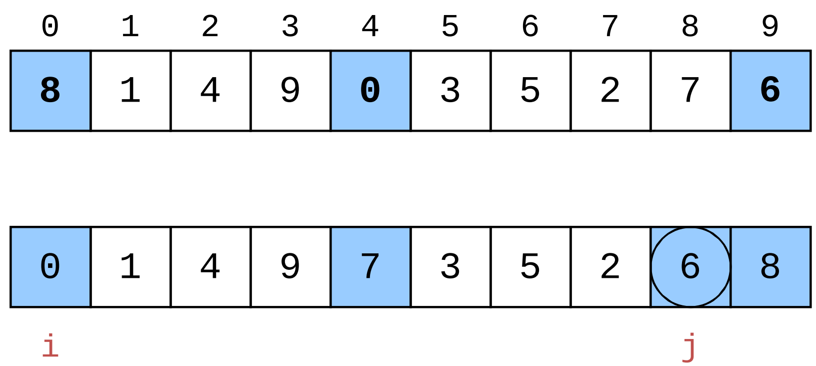
- Need to partition the array into left and right
  - the elements in left sub-array are ≤ pivot
  - elements in right sub-array are ≥ pivot

- How do the elements get to the correct partition?
  - Choose an element from the array as the pivot
  - Make one pass through the rest of the array and swap as needed to put elements in partitions

Partitioning Done In-Place

- One implementation (there are others)
  - median3 finds pivot and sorts left, center, right
  - Swap pivot with next to last element
  - Set pointers i and j to start and end of array
  - Repeat until i and j cross
    - Increment i until you hit element A[i] > pivot
    - Decrement j until you hit element A[j] < pivot
    - Swap A[i] and A[j]
  - Swap pivot (= A[N-2]) with A[i]

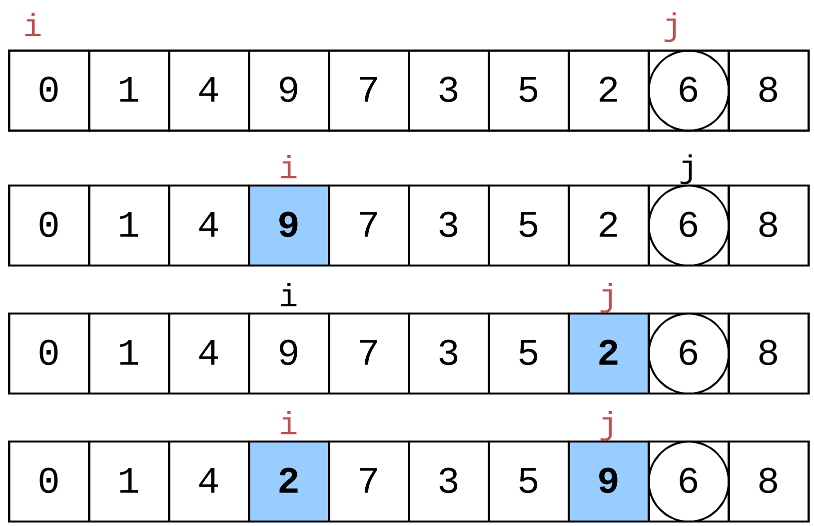
Example



Choose the pivot as the median of three.

Place the pivot and the largest at the right and the smallest at the left

Example

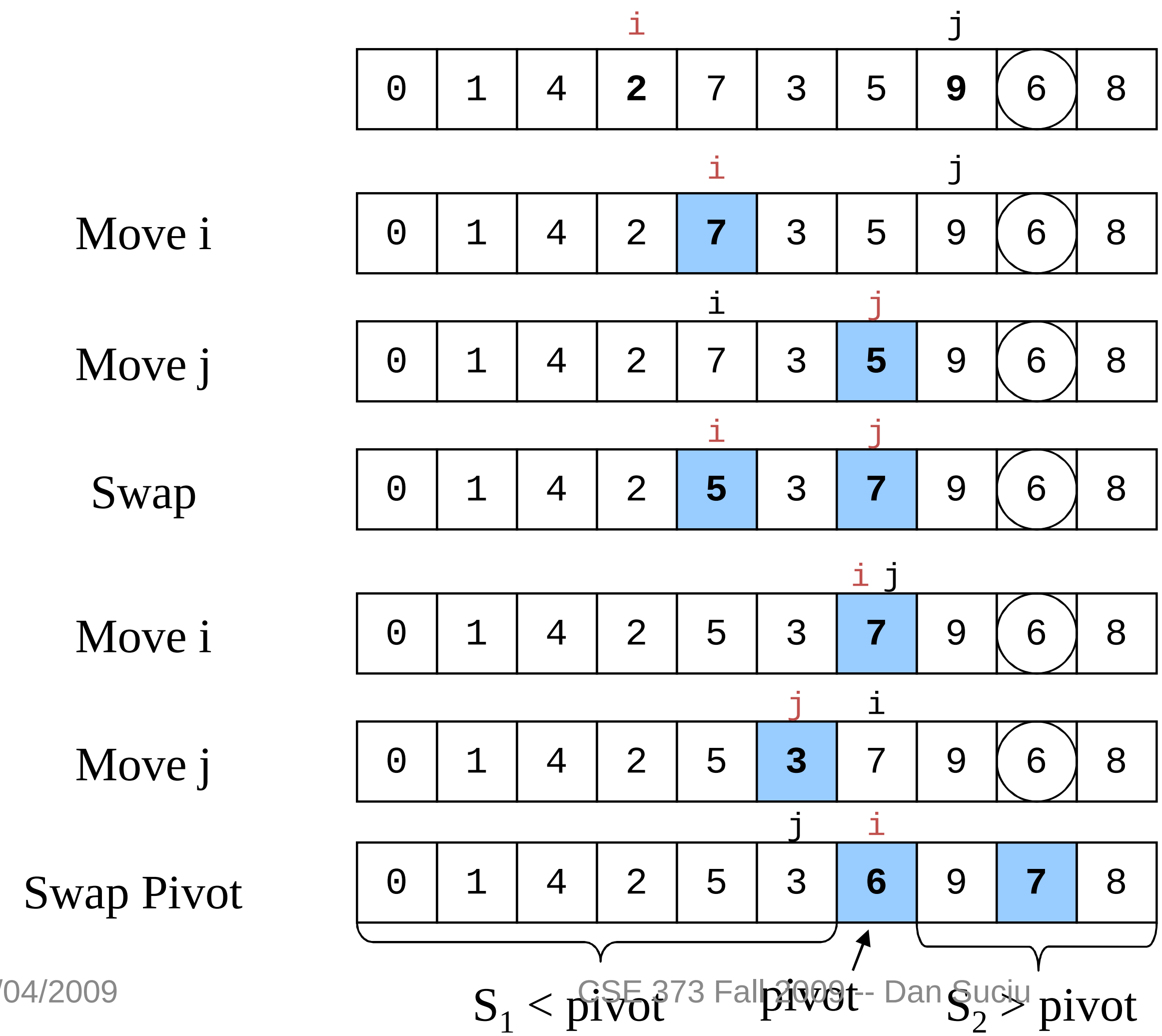


Move i to the right to be larger than pivot.

Move j to the left to be smaller than pivot.

Swap

Example


i j
0 1 4 2 7 3 5 9 6 8
Move i
i j
0 1 4 2 7 3 5 9 6 8
Move j
i j
0 1 4 2 7 3 5 9 6 8
Swap
i j
0 1 4 2 5 3 7 9 6 8
Move i
i j
0 1 4 2 5 3 7 9 6 8
Move j
j i
0 1 4 2 5 3 7 9 6 8
Swap Pivot
j i
0 1 4 2 5 3 6 9 7 8
S_1 < pivot
pivot
S_2 > pivot

Quicksort Best Case Performance

- Algorithm always chooses best pivot and splits sub-arrays in half
  - T(0) = T(1) = O(1)
    - constant time if 0 or 1 element
  - For N > 1,
    2 recursive calls plus linear partitioning
  - T(N) = 2T(N/2) + O(N)
    - Same recurrence relation as Mergesort
  - T(N) = O(N log N)

Quicksort Worst Case Performance

- Algorithm always chooses the worst pivot, one sub-array is empty at each recursion
  - $T(N) \leq a$ for $N \leq C$
  - $T(N) \leq T(N-1) + bN$
  - $\leq T(N-2) + b(N-1) + bN$
  - $\leq T(C) + b(C+1) + \ldots + bN$
  - $\leq a + b(C + C+1 + C+2 + \ldots + N)$
  - $T(N) = O(N^2)$

- Fortunately, *average case* is O(N log N) (see text for proof)

Properties of Quicksort

- No iterative version
  (without using an explicit stack).
- “In-place”, but uses auxiliary storage because of recursive calls.
- O(n log n) average case performance,
  but $O(n^2)$ worst case performance.

Opportunity for Improvement

- Applies for only small N
- Overhead of recursion starts to dominate
- Apply insertion sort for small N

Recursive Quicksort with Cutoff

```
Quicksort(A[]: integer array, left,right : integer): {
   pivotindex : integer;
   if left + CUTOFF ≤ right then
     pivot := median3(A,left,right);
     pivotindex := Partition(A,left,right-1,pivot);
     Quicksort(A, left, pivotindex – 1);
     Quicksort(A, pivotindex + 1, right);
   else
     Insertionsort(A,left,right);
}
```

CUTOFF = 10 is reasonable.

So Which Is Best?

- It’s a trick question, a naïve question
- Myth: “Quicksort is the best in-memory sorting algorithm.”
- Mergesort and Quicksort make different tradeoffs regarding the cost of comparison and the cost of a swap
- Mergesort is also the basis for external sorting algorithms (large N sorting)