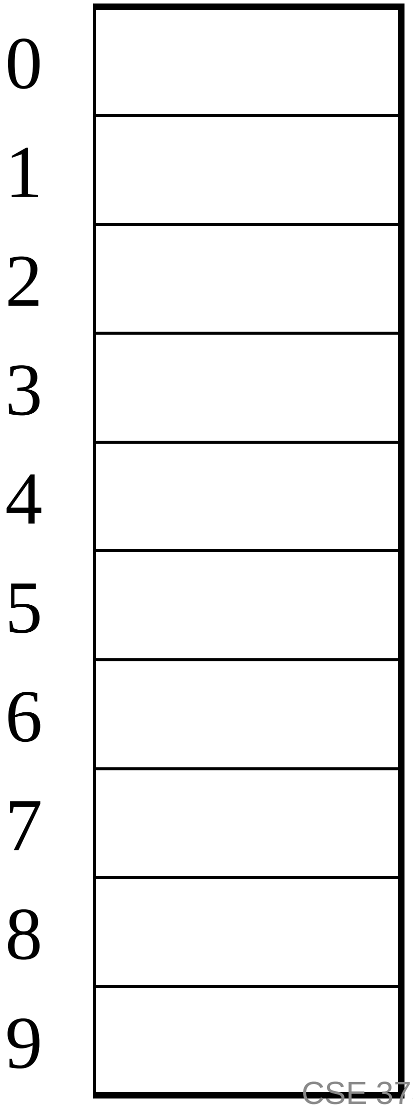
# CSE 373
# Data Structures & Algorithms

## Lecture 11
## Hashing (II)

# Alternative: Use Empty Space in  the Table



**Insert**:
38
19
8
109
10

Try h(K).

If full, try h(K)+1.

If full, try h(K)+2.

If full, try h(K)+3.

Etc…

# Alternative: Use Empty Space in the Table

| | |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | 38 |
| 9 | |

**Insert**:
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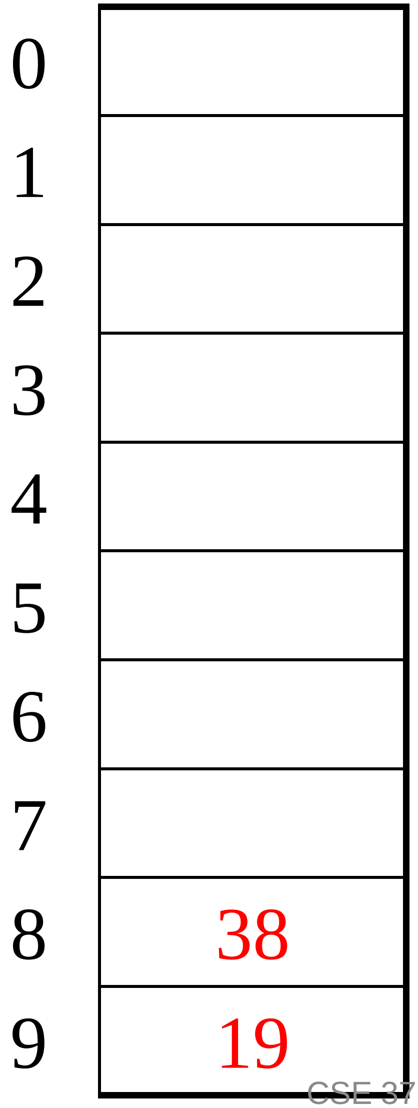

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19
8
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| | |
|---|---|
| 0 | 8 |
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**Insert**:

38

19

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109

10

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Etc…

# Alternative: Use Empty Space in the Table

|   |   |
|---|---|
| 0 | 8 |
| 1 | 109 |
| 2 | 10 |
| 3 |  |
| 4 |  |
| 5 |  |
| 6 |  |
| 7 |  |
| 8 | 38 |
| 9 | 19 |

**Insert**:
38
19
8
109
10

Try h(K).

If full, try h(K)+1.

If full, try h(K)+2.

If full, try h(K)+3.

Etc…

# Open Addressing

This approach is an example of **open addressing**:

After a collision, try “next” spot.
If there’s another collision, try another, etc.

Finding the next available spot is called **probing**:

$0^{th}$ probe = h(k) % TableSize
$1^{th}$ probe = (h(k) + f(1)) % TableSize
$2^{th}$ probe = (h(k) + f(2)) % TableSize
. . .
$i^{th}$ probe = (h(k) + f(i)) % TableSize

Apply probing for both insert and find…

Need consistency to find where you previously inserted

# Terminology Alert!

**Separate chaining**
is sometimes called
**open hashing**.

**Open addressing**
is sometimes called
**closed hashing.**

# Open Addressing Example, Revisited

| | |
|---|---|
| 0 | 8 |
| 1 | 109 |
| 2 | 10 |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | 38 |
| 9 | 19 |

**Insert**:
38
19
8
109
10

Try h(K)

If full, try h(K)+1.

If full, try h(K)+2.

If full, try h(K)+3.

Etc…

**What is f(i)?**

# Linear Probing

$f(i) = i$

- Probe sequence:

  $0^{th}$ probe = $h(K) \% TableSize$

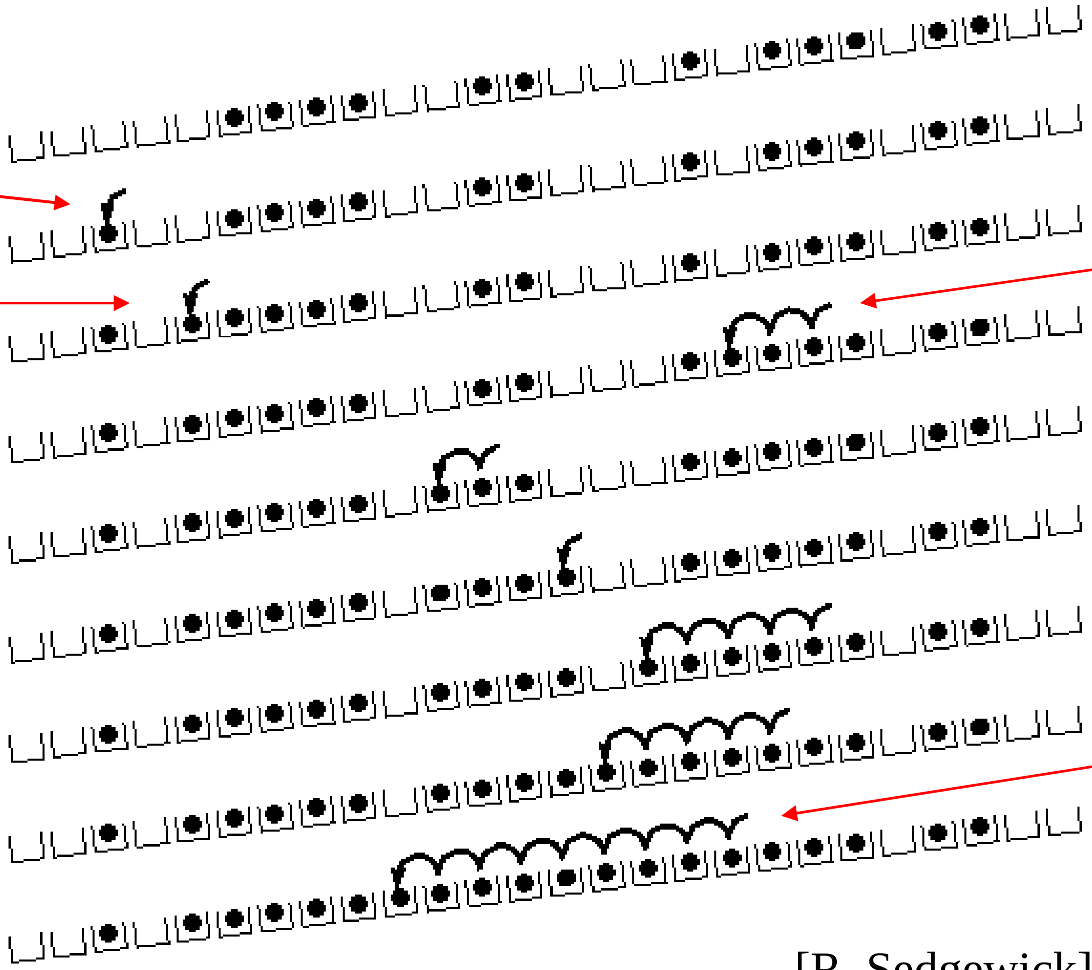
  $1^{th}$ probe = $(h(K) + 1) \% TableSize$

  $2^{th}$ probe = $(h(K) + 2) \% TableSize$

  . . .

  $i^{th}$ probe = $(h(K) + i) \% TableSize$

# Linear Probing – Clustering



no collision

no collision

collision in small cluster

collision in large cluster

[R. Sedgewick]

# Analysis of Linear Probing

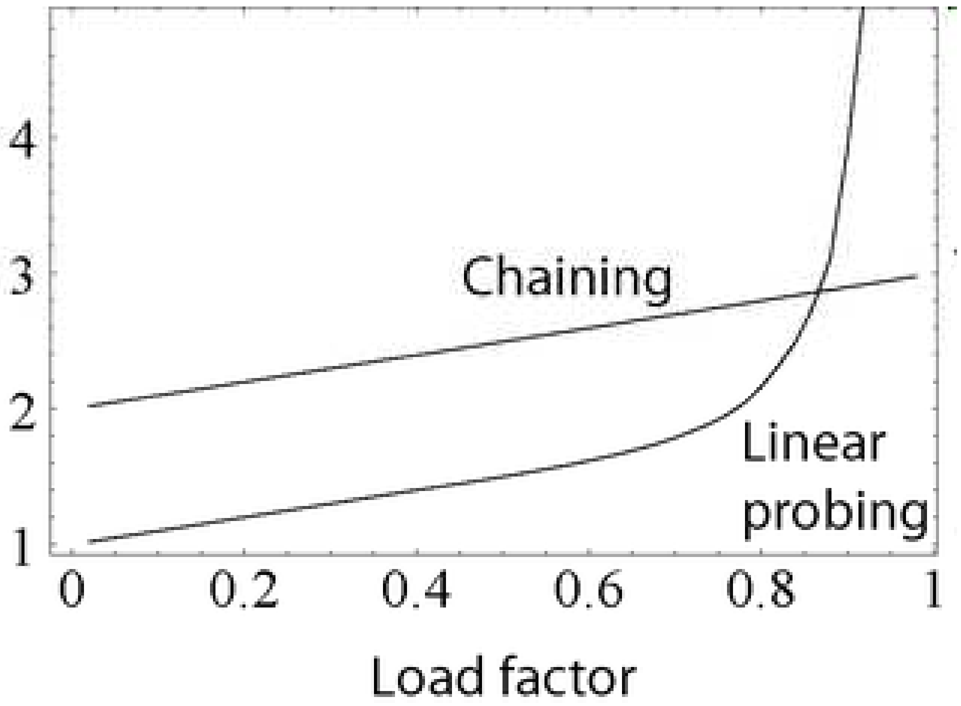
- For *any* $\lambda < 1$, linear probing *will* find an empty slot
- Expected # of probes (for large table sizes)
  - unsuccessful search:

$$\frac{1}{2}\left(1+\frac{1}{(1-\lambda)^2}\right)$$

  - successful search:

$$\frac{1}{2}\left(1+\frac{1}{(1-\lambda)}\right)$$

- Linear probing suffers from ***primary clustering***
- Performance quickly degrades for $\lambda > 1/2$


4
3
2
1
0
0.2
0.4
0.6
0.8
1
Chaining
Linear probing
Load factor

# Quadratic Probing

Less likely to encounter Primary Clustering

$f(i) = i^2$

- Probe sequence:

  $0^{th}$ probe = h(K) % TableSize

  $1^{th}$ probe = (h(K) + 1) % TableSize

  $2^{th}$ probe = (h(K) + 4) % TableSize

  $3^{th}$ probe = (h(K) + 9) % TableSize

  . . .

  $i^{th}$ probe = (h(K) + $i^2$) % TableSize

# Quadratic Probing Example

| | |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | |
| 9 | |

Insert:
89
18
49
58
79

# Quadratic Probing Example

| | |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | |
| 9 | 89 |

Insert:
89
18
49
58
79

# Quadratic Probing Example

| | |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | 18 |
| 9 | 89 |

Insert:
89
18
49
58
79

# Quadratic Probing Example

| | |
|---|---|
| 0 | 49 |
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | 18 |
| 9 | 89 |

Insert:
89
18
49
58
79

# Quadratic Probing Example

| | |
|---|---|
| 0 | 49 |
| 1 | |
| 2 | 58 |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | 18 |
| 9 | 89 |

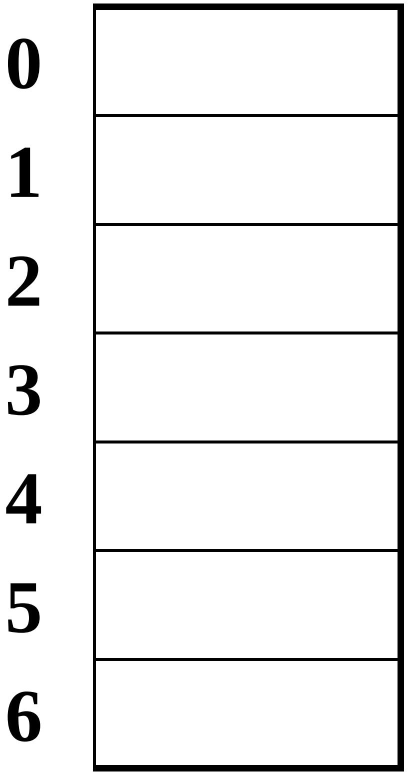
Insert:
89
18
49
58
79

# Quadratic Probing Example

| | |
|---|---|
| 0 | 49 |
| 1 | |
| 2 | 58 |
| 3 | 79 |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | 18 |
| 9 | 89 |

Insert:
89
18
49
58
79

# Another Quadratic Probing Example



TableSize = 7
h(K) = K % 7

insert(76) 76 % 7 =6
insert(40) 40 % 7 =5
insert(48) 48 % 7 =6
insert(5) 5 % 7 =5
insert(55) 55 % 7 =6
insert(47) 47 % 7 =5

# Another Quadratic Probing Example

| | |
|---|---|
| **0** | |
| **1** | |
| **2** | |
| **3** | |
| **4** | |
| **5** | |
| **6** | 76 |

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# Another Quadratic Probing Example

| | |
|---|---|
| **0** | |
| **1** | |
| **2** | |
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# Another Quadratic Probing Example

| | |
|---|---|
| **0** | 48 |
| **1** | |
| **2** | |
| **3** | |
| **4** | |
| **5** | 40 |
| **6** | 76 |

TableSize = 7
h(K) = K % 7

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insert(48) 48 % 7 =6
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insert(55) 55 % 7 =6
insert(47) 47 % 7 =5

# Another Quadratic Probing Example

| | |
|---|---|
| **0** | 48 |
| **1** | |
| **2** | 5 |
| **3** | |
| **4** | |
| **5** | 40 |
| **6** | 76 |

TableSize = 7
h(K) = K % 7

| | |
|---|---|
| insert(76) | 76 % 7 =6 |
| insert(40) | 40 % 7 =5 |
| insert(48) | 48 % 7 =6 |
| insert(5) | 5 % 7 =5 |
| insert(55) | 55 % 7 =6 |
| insert(47) | 47 % 7 =5 |

# Another Quadratic Probing Example

| | |
|---|---|
| **0** | 48 |
| **1** | |
| **2** | 5 |
| **3** | 55 |
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# Another Quadratic Probing Example

| | |
|---|---|
| 0 | 48 |
| 1 | |
| 2 | 5 |
| 3 | 55 |
| 4 | |
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TableSize = 7
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insert(76) 76 % 7 =6
insert(40) 40 % 7 =5
insert(48) 48 % 7 =6
insert(5) 5 % 7 =5
insert(55) 55 % 7 =6
insert(47) 47 % 7 =5

0 + 5 = 5 % 7 = 5
1 + 5 = 6 % 7 = 6
4 + 5 = 9 % 7 = 2
9 + 5 = 14 % 7 = 0
16 + 5 = 21 % 7 = 0
25 + 5 = 30 % 7 = 2
36 + 5 = 41 % 7 = 6
49 + 5 = 54 % 7 = 5
64 + 5 = 69 % 7 = 6
……

Can’t insert 47, even though the table is not full.

# Quadratic Probing:
# Success guarantee for $\lambda < ½$

Assertion #1:

If T = TableSize is **prime** and $\lambda < ½$, then quadratic probing will find an empty slot in T/2 probes or fewer.

Assertion #2:

For prime T and all $0 \leq i,j \leq T/2$ and $i \neq j$,

$$(h(K) + i^2) \% T \neq (h(K) + j^2) \% T$$

(prove this by contradiction if you'd like)

Assertion #3: Assertion #2 proves assertion #1.

# Quadratic Probing: Properties

- For *any* $\lambda < ½$, quadratic probing will find an empty slot; for bigger $\lambda$, quadratic probing *may* find a slot.

- Quadratic probing does not suffer from *primary* clustering: keys hashing to the same *area* are not bad.

- But what about keys that hash to the same *spot*?
  - ***Secondary Clustering!***

# Double Hashing

Idea: given two different (good) hash functions h(K) and g(K), it is unlikely two keys to collide with both.

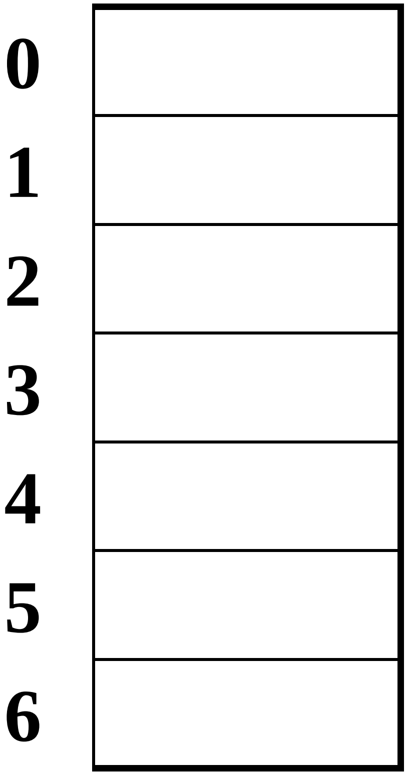
So…let's try probing with a second hash function:

$$f(i) = i * g(K)$$

- Probe sequence:

  $0^{th}$ probe = h(K) % TableSize
  $1^{th}$ probe = (h(K) + g(K)) % TableSize
  $2^{th}$ probe = (h(K) + 2*g(K)) % TableSize
  $3^{th}$ probe = (h(K) + 3*g(K)) % TableSize
  . . .
  $i^{th}$ probe = (h(K) + i*g(K)) % TableSize

# Double Hashing Example



TableSize = 7
h(K) = K % 7
g(K) = 5 – (K % 5)

Insert(76) 76 % 7 = 6 and 5 - (76 % 5) = 4
Insert(93) 93 % 7 = 2 and 5 - (93 % 5) = 2
Insert(40) 40 % 7 = 5 and 5 - (40 % 5) = 5
Insert(47) 47 % 7 = 5 and 5 - (47 % 5) = 3
Insert(10) 10 % 7 = 3 and 5 - (10 % 5) = 5
Insert(55) 55 % 7 = 6 and 5 - (55 % 5) = 5

# Double Hashing Example

| | |
|---|---|
| **0** | |
| **1** | |
| **2** | |
| **3** | |
| **4** | |
| **5** | |
| **6** | 76 |

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# Double Hashing Example

| | |
|---|---|
| **0** | |
| **1** | |
| **2** | 93 |
| **3** | |
| **4** | |
| **5** | |
| **6** | 76 |

TableSize = 7
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# Double Hashing Example

| | |
|---|---|
| **0** | |
| **1** | |
| **2** | 93 |
| **3** | |
| **4** | |
| **5** | 40 |
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# Double Hashing Example

| | |
|---|---|
| **0** | |
| **1** | 47 |
| **2** | 93 |
| **3** | |
| **4** | |
| **5** | 40 |
| **6** | 76 |

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# Double Hashing Example

| | |
|---|---|
| **0** | |
| **1** | 47 |
| **2** | 93 |
| **3** | 10 |
| **4** | |
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# Double Hashing Example

| | |
|---|---|
| **0** | |
| **1** | 47 |
| **2** | 93 |
| **3** | 10 |
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# Analysis of Double Hashing

- Double hashing is safe for $\lambda < 1$ for at least one case:
  - h(k) = k % p
  - g(k) = q – (k % q)
  - 2 < q < p, and p, q are primes
- Expected # of probes (for large table sizes)
  - unsuccessful search:

$$\frac{1}{1-\lambda}$$

  - successful search:

$$\frac{1}{\lambda}\log_e\left(\frac{1}{1-\lambda}\right)$$

# Deletion in Separate Chaining

How do we delete an element with separate chaining?

Easy, just delete the item from the bucket

# Deletion in Open Addressing

Can we do something similar for open addressing?

- Delete
- Find
- Insert

| | |
|---|---|
| 0 | |
| 1 | |
| 2 | 16 |
| 3 | 23 |
| 4 | 59 |
| 5 | |
| 6 | 76 |

h(k) = k % 7
Linear probing

Delete(23)
Find(59)
Insert(30)

# Deletion in Open Addressing

Can we do something similar for open addressing?

- Delete
- Find
- Insert

| | |
|---|---|
| 0 | |
| 1 | |
| 2 | 16 |
| 3 | |
| 4 | 59 |
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Delete(23)
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# Deletion in Open Addressing

Can we do something similar for open addressing?

- Delete
- Find
- Insert

| | |
|---|---|
| 0 | |
| 1 | |
| 2 | 16 |
| 3 | X |
| 4 | 59 |
| 5 | |
| 6 | 76 |

h(k) = k % 7
Linear probing

Delete(23)
Find(59)
Insert(30)

Need to leave a marker of a deletion

# Deletion in Open Addressing

Can we do something similar for open addressing?

- Delete
- Find
- Insert

| | |
|---|---|
| 0 | |
| 1 | |
| 2 | 16 |
| 3 | 30 |
| 4 | 59 |
| 5 | |
| 6 | 76 |

h(k) = k % 7
Linear probing

Delete(23)
Find(59)
Insert(30)

Need to leave a marker of a deletion

# Rehashing

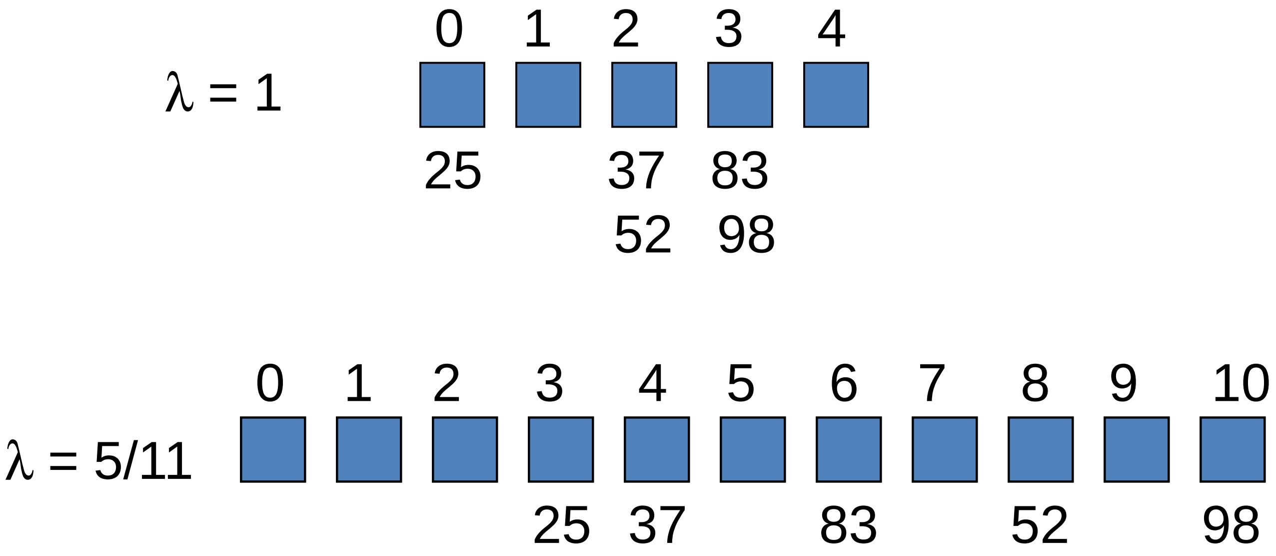
**Idea:** When the table gets too full, create a bigger table (usually 2x as large) and hash all the items from the original table into the new table.

- When to rehash?
  - Separate chaining: full ($\lambda = 1$)
  - Open addressing: half full ($\lambda = 0.5$)
  - When an insertion fails
  - Some other threshold
- Cost of a single rehashing? O(N)

# Rehashing Example

- Separate chaining example:

  $h_1(x) = x \% 5$ rehashes to $h_2(x) = x \% 11$.



$\lambda = 1$

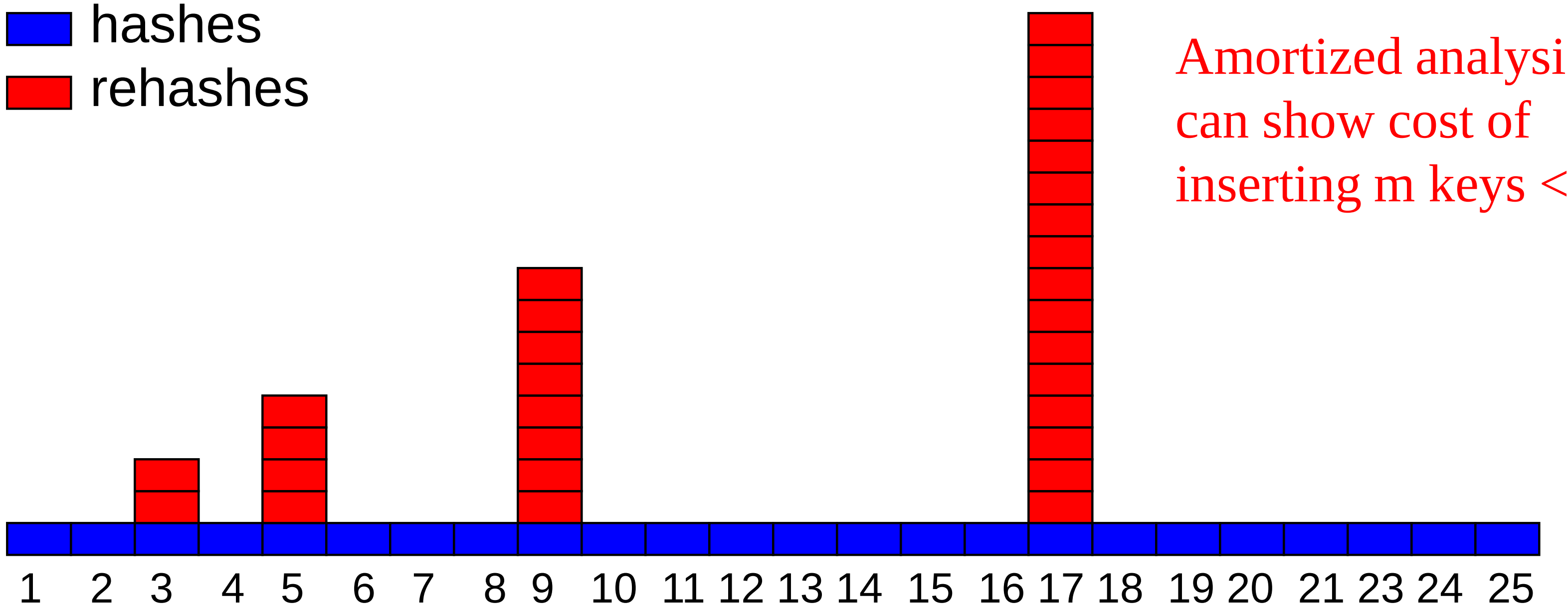
| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 25 |  | 37<br>52 | 83<br>98 |  |

$\lambda = 5/11$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
|  |  |  | 25 | 37 |  | 83 |  | 52 |  | 98 |

# Rehashing Picture

- Starting with table of size 2, double when load factor > 1.



Amortized analysis can show cost of inserting m keys < 3m

# Discussion on Hashing

- Hash tables are one of the most important data structures.
- Hashing has many applications where operations are limited to find, insert, and delete.
- Can use either separate chaining or open hashing
  - Java uses separate chaining
- Rehashing has good amortized complexity.
- Also has a big data version to minimize disk accesses: extendible hashing. (See textbook.)

# Java hashCode() Method

- Class Object defines a hashCode( ) method
  - Intent: returns a suitable hashcode for the object
  - Result is arbitrary int; must scale to fit a hash table (e.g. obj.hashCode() % nBuckets)
  - Used by collection classes like HashMap

- Classes should override with calculation appropriate for instances of the class
  - Calculation should involve semantically "significant" fields of objects

# hashCode() and equals()

- To work correctly, particularly with collection classes (like HashMap), an Object's hashCode() and equals() must obey this rule:

  if a.equals(b) then it must be true that

  a.hashCode() == b.hashCode()

- Why?  Is the reverse also required?

# Aside: Properties of Mod

To keep hashed values within the size of the table, we will generally do:

h(K) = function(K) % TableSize

It’s worth noting a couple properties of the mod function:

- (a + b) % c = [(a % c) + (b % c)] % c
- (a b) % c = [(a % c) (b % c)] % c
- a % c = b % c → (a – b) % c = 0

# Designing Hash Functions

We’ve seen a few possibilities. The simplest is **modular hashing**:

h(K) = K % P

where P is usually just the TableSize.

P is often chosen to be prime:

- Reduces likelihood of collisions due to patterns in data
- Is useful for guarantees on certain hashing strategies (as we’ll see)

But what would be a more convenient value of P?

# Amortized Analysis of Rehashing

- Cost of inserting n keys is $< 3n$
- $2^k + 1 \leq n \leq 2^{k+1}$
  - Hashes $= n$
  - Rehashes $= 2 + 2^2 + \ldots + 2^k = 2^{k+1} - 2$
  - Total $= n + 2^{k+1} - 2 < 3n$
- Example
  - $n = 33$, Total $= 33 + 64 - 2 = 95 < 99$