AVL Trees

CSE 373
Data Structures
Winter 2007

Readings

- Reading Sec. 4.4

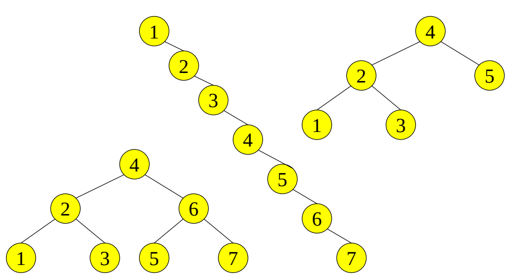
Binary Search Tree - Best Time

- All BST operations are O(d), where d is tree depth
- minimum d is $d = \lfloor \log_2 N \rfloor$ for a binary tree with N nodes
  - › What is the best case tree?
  - › What is the worst case tree?
- So, best case running time of BST operations is O(log N)

Binary Search Tree - Worst Time

- Worst case running time is O(N)
  - › What happens when you Insert elements in ascending order?
    - Insert: 2, 4, 6, 8, 10, 12 into an empty BST
  - › Problem: Lack of "balance":
    - compare depths of left and right subtree
  - › Unbalanced degenerate tree

Balanced and unbalanced BST



Approaches to balancing trees

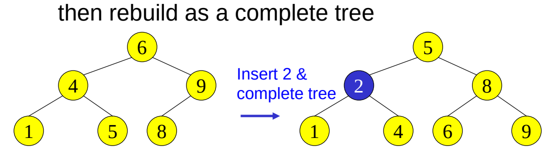
- Don't balance
  - › May end up with some nodes very deep
- Strict balance
  - › The tree must always be balanced perfectly
- Pretty good balance
  - › Only allow a little out of balance
- Adjust on access
  - › Self-adjusting

Balancing Binary Search Trees

- Many algorithms exist for keeping binary search trees balanced
  - › Adelson-Velskii and Landis (AVL) trees (height-balanced trees)
  - › Weight-balanced trees
  - › Red-black trees;
  - › Splay trees and other self-adjusting trees
  - › B-trees and other (e.g. 2-4 trees) multiway search trees

Perfect Balance

- Want a complete tree after every operation
  - › tree is full except possibly in the lower right
- This is expensive
  - › For example, insert 2 in the tree on the left and then rebuild as a complete tree



AVL Trees (1962)

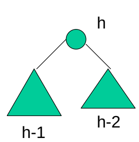
- Named after 2 Russian mathematicians
- Georgii Adelson-Velsky (1922 - ?)
- Evgenii Mikhailovich Landis (1921-1997)

AVL - Good but not Perfect Balance

- AVL trees are height-balanced binary search trees
- Balance factor of a node
  - › height(left subtree) - height(right subtree)
- An AVL tree has balance factor calculated at every node
  - › For every node, heights of left and right subtree can differ by no more than 1
  - › Store current heights in each node

Height of an AVL Tree

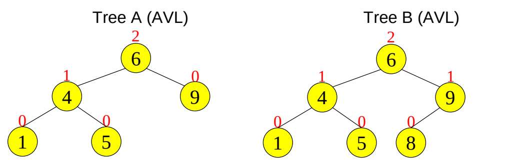
- N(h) = minimum number of nodes in an AVL tree of height h.
- Basis
  - › N(0) = 1, N(1) = 2
- Induction
  - › N(h) = N(h-1) + N(h-2) + 1
- Solution (recall Fibonacci analysis)
  - › $N(h) \geq \phi^h$ $(\phi \approx 1.62)$



Height of an AVL Tree

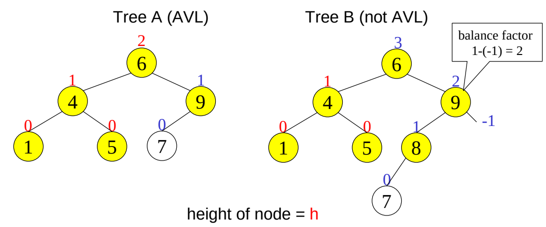
- $N(h) \geq \phi^h$ $(\phi \approx 1.62)$
- Suppose we have n nodes in an AVL tree of height h.
  - › $n \geq N(h)$
  - › $n \geq \phi^h$ hence $\log_\phi n \geq h$ (relatively well balanced tree!!)
  - › $h \leq 1.44 \log_2 n$ (i.e., Find takes O(logn))

Node Heights



height of node = h
balance factor = h_{left}-h_{right}
empty height = -1

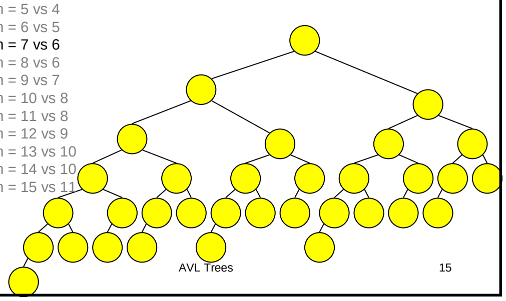
Node Heights after Insert 7



height of node = h
balance factor = h_{left}-h_{right}
empty height = -1

Worst-case AVL Trees

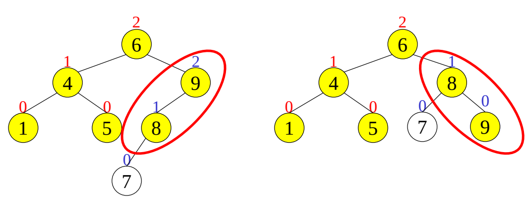
4 nodes: h = 3 vs 3
7 nodes: h = 4 vs 3
12 nodes: h = 5 vs 4
20 nodes: h = 6 vs 5
33 nodes: h = 7 vs 6
54 nodes: h = 8 vs 6
88 nodes: h = 9 vs 7
143 nodes: h = 10 vs 8
232 nodes: h = 11 vs 8
376 nodes: h = 12 vs 9
609 nodes: h = 13 vs 10
986 nodes: h = 14 vs 10
1596 nodes: h = 15 vs 11



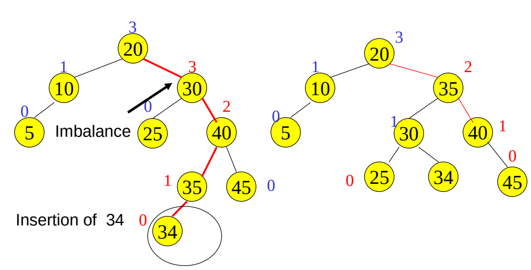
Insert and Rotation in AVL Trees

- Insert operation may cause balance factor to become 2 or –2 for some node
  - only nodes on the path from insertion point to root node have possibly changed in height
  - So after the Insert, go back up to the root node by node, updating heights
  - If a new balance factor (the difference h_{left}-h_{right}) is 2 or –2, adjust tree by *rotation* around the node

Single Rotation in an AVL Tree



Double rotation



Insertions in AVL Trees

Let the node that needs rebalancing be α.

There are 4 cases:

Outside Cases (require single rotation) :

1. Insertion into left subtree of left child of α.
2. Insertion into right subtree of right child of α.

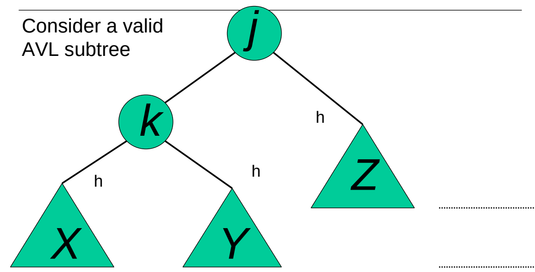
Inside Cases (require double rotation) :

3. Insertion into right subtree of left child of α.
4. Insertion into left subtree of right child of α.

The rebalancing is performed through four separate rotation algorithms.

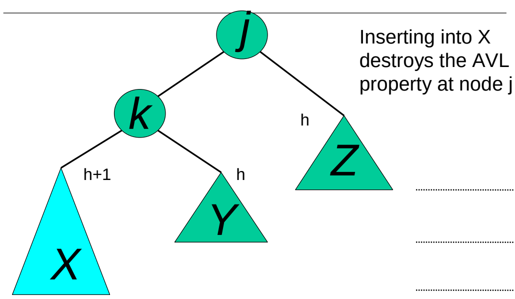
AVL Insertion: Outside Case

Consider a valid AVL subtree



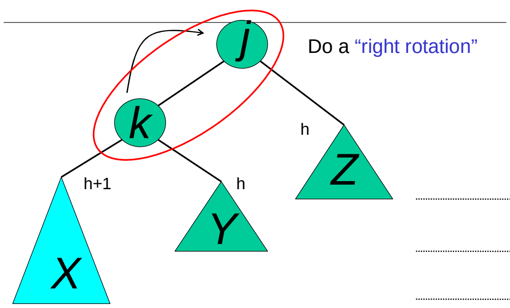
AVL Insertion: Outside Case

Inserting into X destroys the AVL property at node j



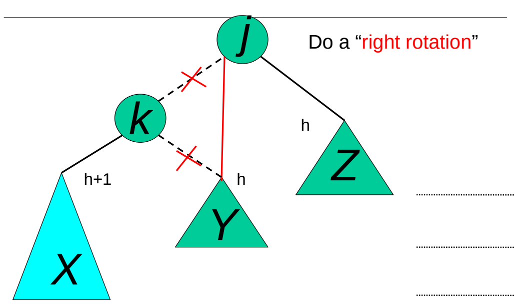
AVL Insertion: Outside Case

Do a "right rotation"



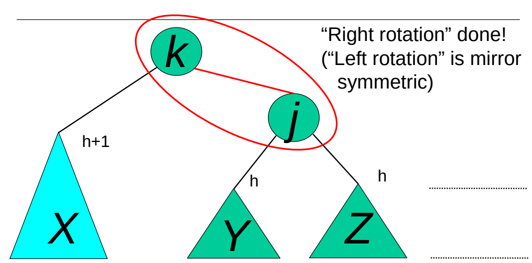
Single right rotation

Do a "right rotation"

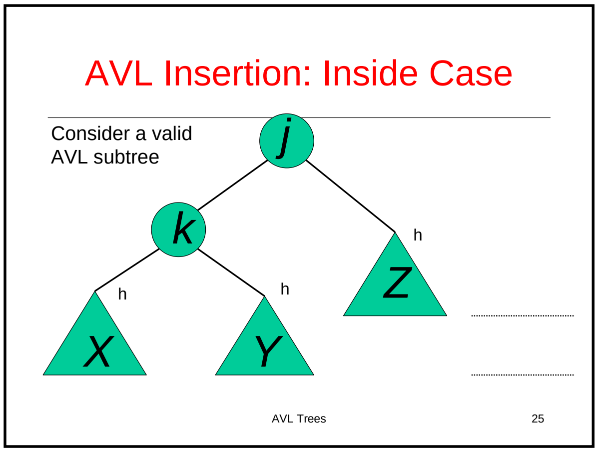


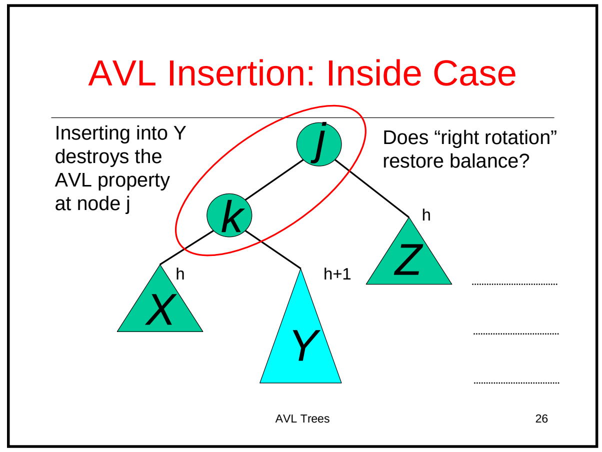
Outside Case Completed

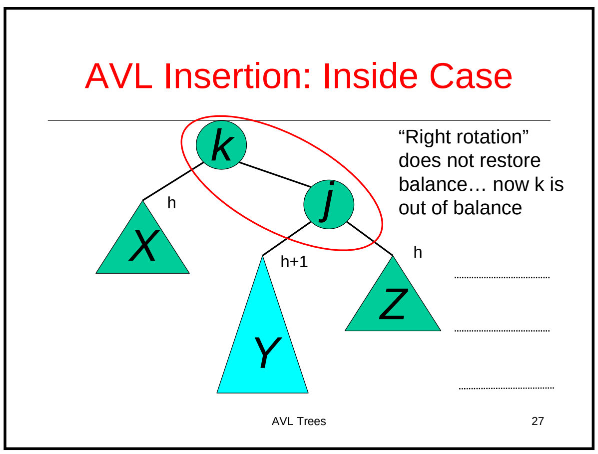
"Right rotation" done!
("Left rotation" is mirror symmetric)

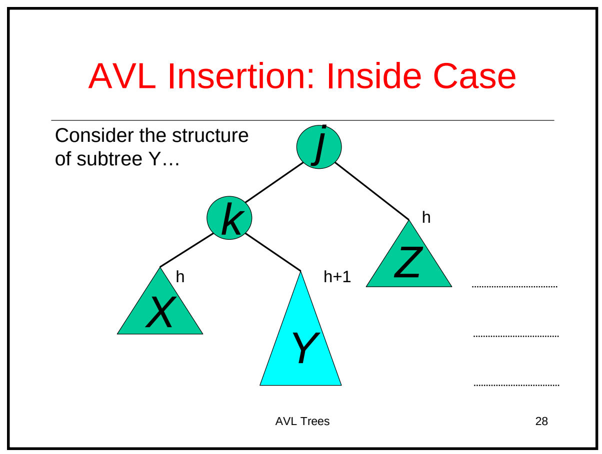


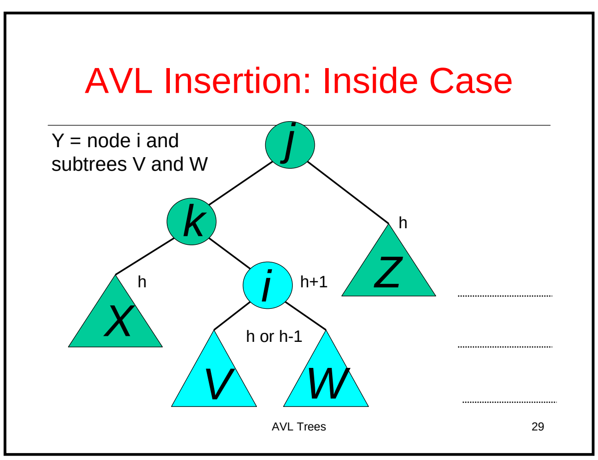
AVL property has been restored

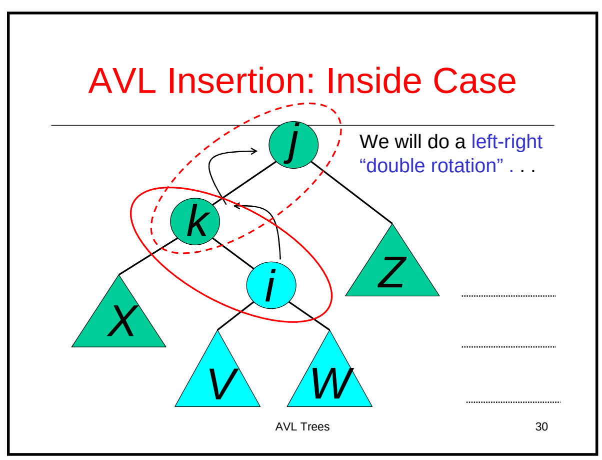

AVL Insertion: Inside Case
Consider a valid AVL subtree
j
k
h
Z
h
h
X
Y
AVL Trees
25


AVL Insertion: Inside Case
Inserting into Y destroys the AVL property at node j
j
Does “right rotation” restore balance?
k
h
h
h+1
Z
X
Y
AVL Trees
26


AVL Insertion: Inside Case
k
“Right rotation” does not restore balance… now k is out of balance
h
j
X
h
h+1
Z
Y
AVL Trees
27

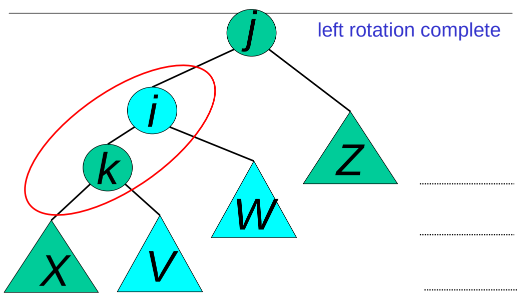

AVL Insertion: Inside Case
Consider the structure of subtree Y…
j
k
h
h
h+1
Z
X
Y
AVL Trees
28


AVL Insertion: Inside Case
Y = node i and subtrees V and W
j
k
h
Z
h
i
h+1
X
h or h-1
V
W
AVL Trees
29


AVL Insertion: Inside Case
j
We will do a left-right “double rotation” . . .
k
Z
i
X
V
W
AVL Trees
30

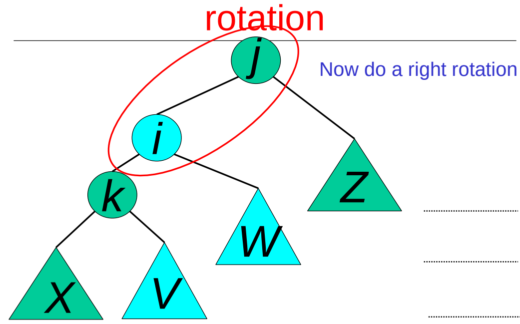
Double rotation : first rotation

left rotation complete



Double rotation : second rotation

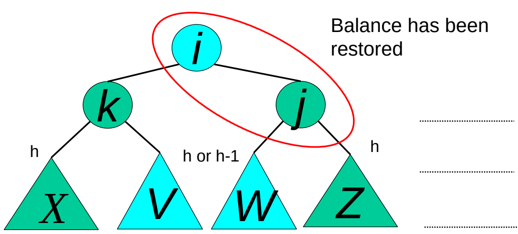
Now do a right rotation



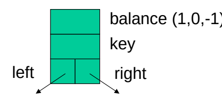
Double rotation : second rotation

right rotation complete

Balance has been restored



Implementation

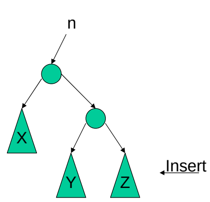


You can either keep the height or just the difference in height, i.e. the balance factor; this has to be modified on the path of insertion even if you don't perform rotations

Once you have performed a rotation (single or double) you won't need to go back up the tree

Single Rotation

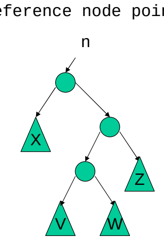
```
RotateFromRight(n : reference node pointer) {
p : node pointer;
p := n.right;
n.right := p.left;
p.left := n;
n := p
}
```



You also need to modify the heights or balance factors of n and p

Double Rotation

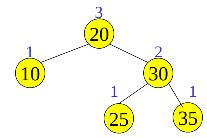
```
DoubleRotateFromRight(n : reference node pointer) {
  RotateFromLeft(n.right);
  RotateFromRight(n);
}
```



Insert in AVL trees

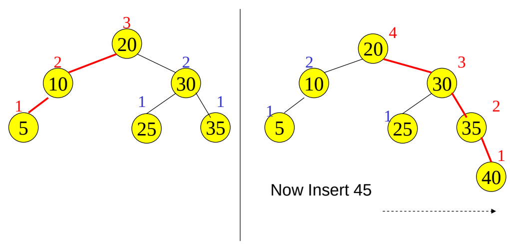
```
Insert(T : tree pointer, x : element) : {
if T = null then
  T := new tree; T.data := x; height := 0;
case
  T.data = x : return ; //Duplicate do nothing
  T.data > x : return Insert(T.left, x);
               if ((height(T.left)- height(T.right)) = 2){
                  if (T.left.data > x ) then //outside case
                         T = RotatefromLeft (T);
                  else                       //inside case
                         T = DoubleRotatefromLeft (T);}
  T.data < x : return Insert(T.right, x);
                code similar to the left case
Endcase
  T.height := max(height(T.left),height(T.right)) +1;
  return;
}
```


Example of Insertions in an AVL Tree

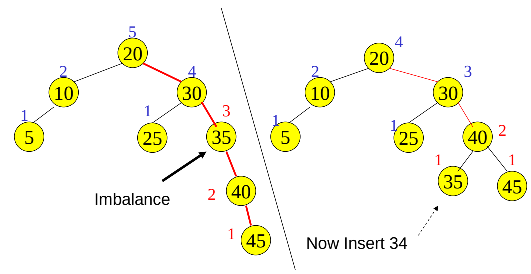


Insert 5, 40

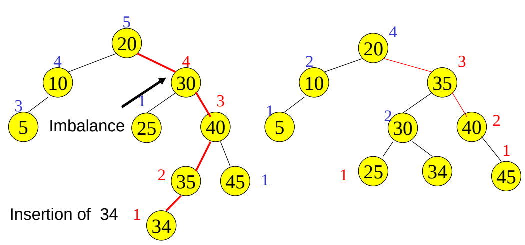
Example of Insertions in an AVL Tree



Single rotation (outside case)



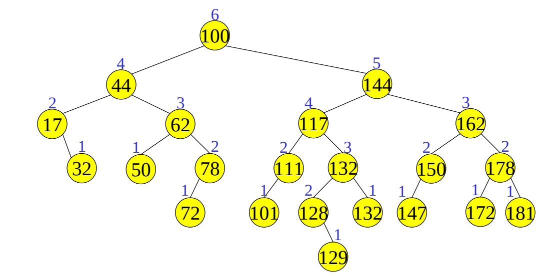
Double rotation (inside case)



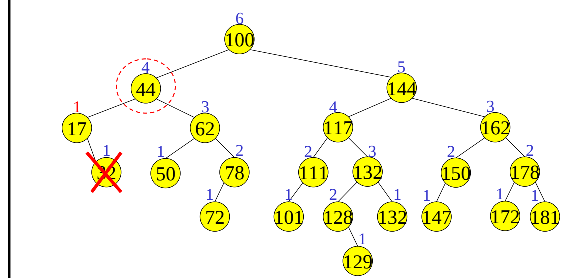
AVL Tree Deletion

- Similar but more complex than insertion
  - Rotations and double rotations needed to rebalance
  - Imbalance may propagate upward so that many rotations may be needed.

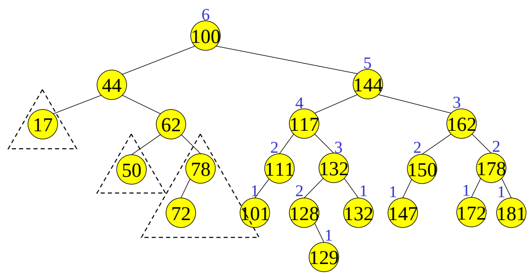
Deletion



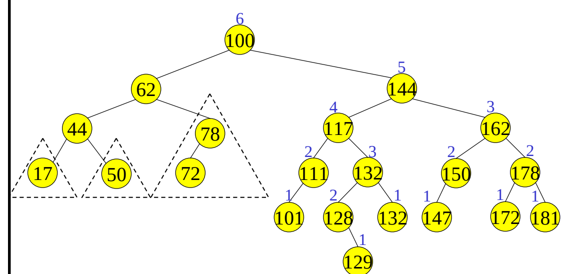
Deletion



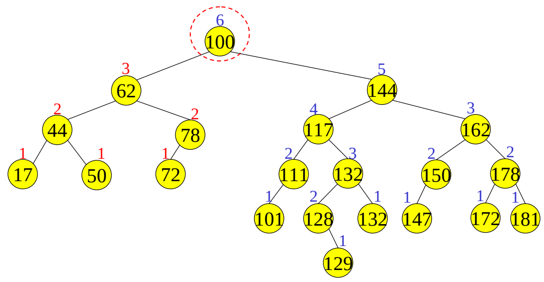
Deletion



Deletion



Deletion



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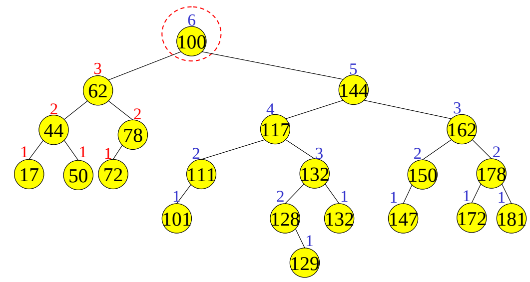

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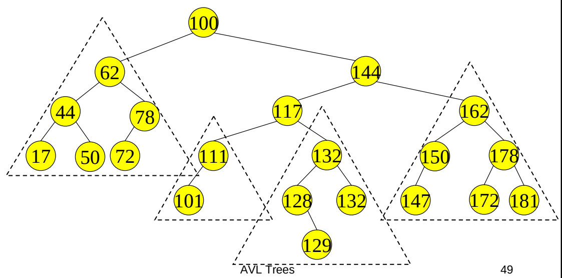

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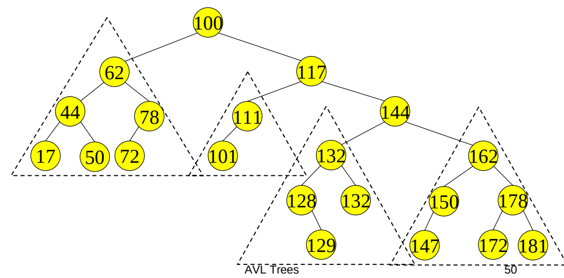

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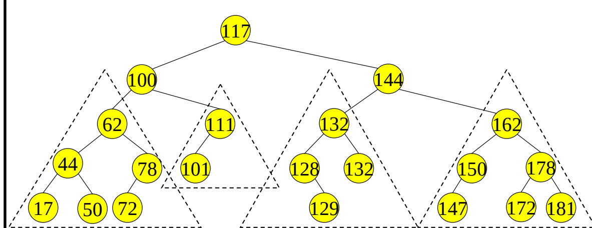

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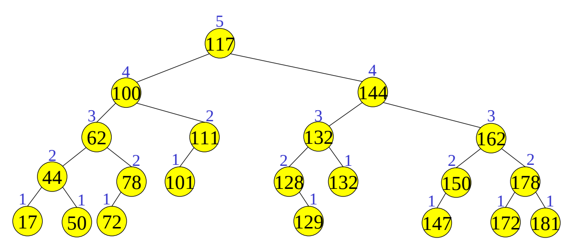

Pros and Cons of AVL Trees

Arguments for AVL trees:

1. Search is O(log n) since AVL trees are always balanced.
2. Insertion and deletions are also O(log n)
3. The height balancing adds no more than a constant factor to the speed of insertion.

Arguments against using AVL trees:

1. Difficult to program & debug; more space for balance factor.
2. Asymptotically faster but rebalancing costs time.
3. Most large searches are done in database systems on disk and use other structures (e.g. B-trees).
4. May be OK to have O(n) for a single operation if total run time for many consecutive operations is fast (e.g. Splay trees).

Non-recursive insertion or the hacker's delight

- Key observations;
  - › At most one rotation
  - › Balance factor: 2 bits are sufficient (-1 left, 0 equal, +1 right)
  - › There is one node on the path of insertion, say S, that is "critical". It is the node where a rotation can occur and nodes above it won't have their balance factors modified

Non-recursive insertion

- Step 1 (Insert and find S):
  - Find the place of insertion and identify the last node S on the path whose BF ≠ 0 (if all BF on the path = 0, S is the root).
  - Insert
- Step 2 (Adjust BF's)
  - Restart from the child of S on the path of insertion. (note: all the nodes from that node on on the path of insertion have BF = 0.)If the path traversed was left (right) set BF to –1 (+1) and repeat until you reach a null link (at the place of insertion)

Non-recursive insertion (ct'd)

- Step 3 (Balance if necessary):
  - If BF(S) = 0 (S was the root) set BF(S) to the direction of insertion (the tree has become higher)
  - If BF(S) = -1 (+1) and we traverse right (left) set BF(S) = 0 (the tree has become more balanced)
  - If BF(S) = -1 (+1) and we traverse left (right), the tree becomes unbalanced. Perform a single rotation or a double rotation depending on whether the path is left-left (right-right) or left-right (right-left)

Non-recursive Insertion with BF's