Mathematical Background

CSE 373
Data Structures

Mathematical Background

- Today, we will review:
  - Logs and exponents
  - Series
  - Recursion
  - Motivation for Algorithm Analysis

Powers of 2

- Many of the numbers we use in Computer Science are powers of 2
- Binary numbers (base 2) are easily represented in digital computers
  - each "bit" is a 0 or a 1
  - $2^0=1, 2^1=2, 2^2=4, 2^3=8, 2^4=16,\ldots, 2^{10}=1024$ (1K)
  - , an n-bit wide field can hold 2^n positive integers:
    - $0 \le k \le 2^n-1$

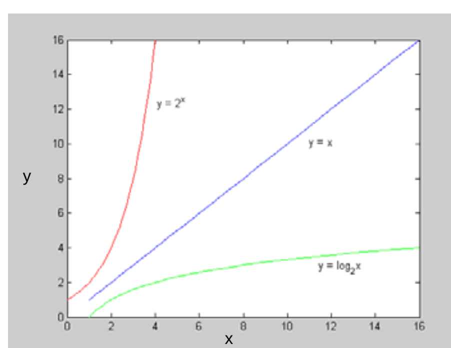
0000000000101011

Unsigned binary numbers

- For unsigned numbers in a fixed width field
  - the minimum value is 0
  - the maximum value is 2^n-1, where n is the number of bits in the field
  - The value is $\sum_{i=0}^{i=n-1} a_i 2^i$
- Each bit position represents a power of 2 with $a_i = 0$ or $a_i = 1$

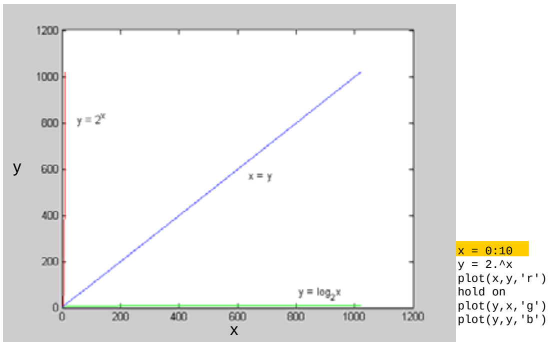
Logs and exponents

- Definition: $\log_2 x = y$ means $x = 2^y$
  - $8 = 2^3$, so $\log_2 8 = 3$
  - $65536 = 2^{16}$, so $\log_2 65536 = 16$
- Notice that $\log_2 x$ tells you how many bits are needed to hold x values
  - 8 bits holds 256 numbers: 0 to 2^8-1 = 0 to 255
  - $\log_2 256 = 8$




```
x = 0:.1:4
y = 2.^x
plot(x,y,'r')
hold on
plot(y,x,'g')
plot(y,y,'b')
```

x, 2^x and $\log_2 x$



2^x and $\log_2 x$

Floor and Ceiling

$\lfloor X \rfloor$ Floor function: the largest integer $\leq X$

$\lfloor 2.7 \rfloor = 2 \qquad \lfloor -2.7 \rfloor = -3 \qquad \lfloor 2 \rfloor = 2$

$\lceil X \rceil$ Ceiling function: the smallest integer $\geq X$

$\lceil 2.3 \rceil = 3 \qquad \lceil -2.3 \rceil = -2 \qquad \lceil 2 \rceil = 2$

Facts about Floor and Ceiling

1. $X - 1 < \lfloor X \rfloor \leq X$
2. $X \leq \lceil X \rceil < X + 1$
3. $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$ if n is an integer

Properties of logs (of the mathematical kind)

- We will assume logs to base 2 unless specified otherwise
- log AB = log A + log B
  - $A=2^{\log_2 A}$ and $B=2^{\log_2 B}$
  - $AB = 2^{\log_2 A} \bullet 2^{\log_2 B} = 2^{\log_2 A + \log_2 B}$
  - so $\log_2 AB = \log_2 A + \log_2 B$
  - [note: log AB $\neq$ log A•log B]

Other log properties

- log A/B = log A – log B
- log (A^B) = B log A
- log log X < log X < X for all X > 0
  - log log X = Y means $2^{2^Y} = X$
  - log X grows slower than X
    - called a "sub-linear" function

A log is a log is a log

- Any base x log is equivalent to base 2 log within a constant factor

$B = 2^{\log_2 B}$

$x = 2^{\log_2 x}$

$$\log_x B = \log_x B$$

$$x^{\log_x B} = B$$

substitution

$x^{\log_x B} = B$ by def. of logs

$$(2^{\log_2 x})^{\log_x B} = 2^{\log_2 B}$$

$$2^{\log_2 x \log_x B} = 2^{\log_2 B}$$

$$\log_2 x \log_x B = \log_2 B$$

$$\boxed{\log_x B = \frac{\log_2 B}{\log_2 x}}$$

Arithmetic Series

- $S(N) = 1 + 2 + \ldots + N = \sum_{i=1}^{N} i$
- The sum is
  - › S(1) = 1
  - › S(2) = 1+2 = 3
  - › S(3) = 1+2+3 = 6
- $\sum_{i=1}^{N} i = \frac{N(N+1)}{2}$

Why is this formula useful when you analyze algorithms?

Algorithm Analysis

- Consider the following program segment:

```
x:= 0;
for i = 1 to N do
  for j = 1 to i do
    x := x + 1;
```

- What is the value of x at the end?

Analyzing the Loop

- Total number of times x is incremented is the number of "instructions" executed = $1 + 2 + 3 + \ldots = \sum_{i=1}^{N} i = \frac{N(N+1)}{2}$
- You've just analyzed the program!
  - › Running time of the program is proportional to N(N+1)/2 for all N
  - › $O(N^2)$

Analyzing Mergesort

```
Mergesort(p : node pointer) : node pointer {
Case {
  p = null : return p; //no elements
  p.next = null : return p; //one element
  else
    d : duo pointer; // duo has two fields first,second
    d := Split(p);
    return Merge(Mergesort(d.first),Mergesort(d.second));
}
}
```

T(n) is the time to sort n items.

$T(0), T(1) \le c$

$T(n) \le T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + dn$

Mergesort Analysis Upper Bound

$T(n) \le 2T(n/2) + dn$ Assuming n is a power of 2

$$
\begin{aligned}
T(n) &\le 2T(n/2) + dn \\
&\le 2(2T(n/4) + dn/2) + dn \\
&= 4T(n/4) + 2dn \\
&\le 4(2T(n/8) + dn/4) + 2dn \\
&= 8T(n/8) + 3dn \\
&\vdots \\
&\le 2^k T(n/2^k) + kdn \\
&= nT(1) + kdn \quad \text{if } n = 2^k \qquad n = 2^k,\ k = \log n \\
&\le cn + dn \log_2 n \\
&= O(n \log n)
\end{aligned}
$$

Recursion Used Badly

- Classic example: Fibonacci numbers F_n

0,1, 1, 2, 3, 5, 8, 13, 21, …

Leonardo Pisano Fibonacci (1170-1250)

- › $F_0 = 0$, $F_1 = 1$ (Base Cases)
- › Rest are sum of preceding two $F_n = F_{n-1} + F_{n-2}$ (n > 1)

Recursion

- A method calling itself, directly or indirectly
- Works because of how method calls are processed anyway
  - A stack holds parameters and local variables for each invocation

Recursive Method Outline

- One or more base cases
- One or more recursive cases
- Recursive cases must get "closer" to a base case
- The process must eventually terminate

Recursion Practice

```
/** Return base^exp
  exp >= 0
*/
double power(double base, int exp);
```


Recursion vs Iteration

- A recursive algorithm can always be expressed iteratively, and vice versa
- Recursion is often more compact and elegant
- Iteration is often more efficient
- Recursion is natural when the data structure is recursive

Recursion Practice

```
/** Return the largest value in a non-
   empty array
*/
double findMax(double[ ] nums);
```


Kickoff: Common Trick

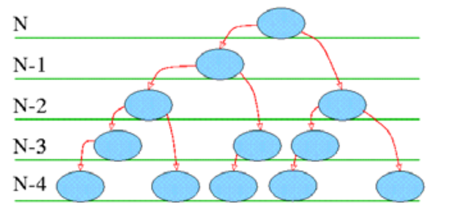
```
double findMax(double[ ] nums) {
    return helper(nums, 0);
}

/** Find the largest value starting at position
"start" of a non-empty array. */
double helper (double[ ] nums, int start);
```

Recursive Procedure for Fibonacci Numbers

```
fib(n : integer): integer {
   Case {
   n ≤ 0 : return 0;
   n = 1 : return 1;
   else : return fib(n-1) + fib(n-2);
   }
   }
```

- Easy to write: looks like the definition of F_n
- But, can you spot the big problem?

Recursive Calls of Fibonacci Procedure



- Re-computes fib(N-i) multiple times!

Fibonacci Analysis Lower Bound

T(n) is the time to compute fib(n).

$T(0), T(1) \geq 1$

$T(n) \geq T(n-1) + T(n-2)$

It can be shown by induction that $T(n) \geq \phi^{n-2}$ where

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.62$$

Iterative Algorithm for Fibonacci Numbers

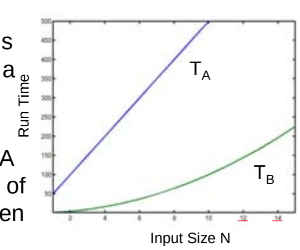
```
fib_iter(n : integer): integer {
fib0, fib1, fibresult, i : integer;
fib0 := 0; fib1 := 1;
case {_
  n < 0 : fibresult := 0;
  n = 1 : fibresult := 1;
  else :
    for i = 2 to n do {
      fibresult := fib0 + fib1;
      fib0 := fib1;
      fib1 := fibresult;
    }
  }
return fibresult;
}
```


Recursion Summary

- Recursion may simplify programming, but beware of generating large numbers of calls
  - › Function calls can be expensive in terms of time and space
- Be sure to get the base case(s) correct!
- Each step must get you closer to the base case

Motivation for Algorithm Analysis

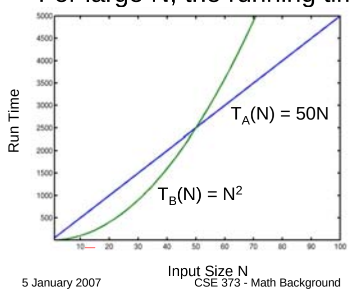
- Suppose you are given two algorithms A and B for solving a problem
- The running times $T_A(N)$ and $T_B(N)$ of A and B as a function of input size N are given



Which is better?

More Motivation

- For large N, the running time of A and B



Now which algorithm would you choose?

Asymptotic Behavior

- The "asymptotic" performance as $N \to \infty$, regardless of what happens for small input sizes N, is generally most important .
- Performance for small input sizes may matter in practice, if you are sure that small N will be common forever .
- We will compare algorithms based on how they scale for large values of N.

Big O

- Mainly used to express upper bounds on time of algorithms. "n" is the size of the input.
- Definition: Let T and f be functions of the natural numbers. ***T(n) is O(f(n)) if there are constants c and n_0 such that***
  $T(n) \leq c\, f(n)$ for all $n \geq n_0$.
- 2*n is O(n)
- 2 + n is O(n)
- 10000n + 10 n $\log_2$ n is O(n log n)
- .00001 n^2 is not O(n log n)

Big O Informality

- Instead of saying "T is O(f)" people often say things like
  - › "T is Big O of f"
  - › "T = O(f)"
  - › "T is bounded by f", etc.
- Be careful how you understand "T = O(f)". This is not an equation!

Why Order Notation

- The difference in performance between two computers is generally a constant multiple (roughly).
  - › The n_0 in our definition takes that into account
- In asymptotic performance ($n \to \infty$) the low order terms are "dominated" by the higher order terms

Some Basic Time Bounds

- In order best to worst:
  - › Logarithmic time is O(log n)
  - › Linear time is O(n)
  - › O(n log n)
  - › Quadratic time is 0(n^2)
  - › Cubic time is O(n^3)
  - › Polynomial time is O(n^k) for some k.
  - › Exponential time is O(c^n) for some c > 1.
- Advice: learn these names and their order!

Kinds of Analysis

- Asymptotic – uses order notation, ignores constant factors and low order terms.
- Upper bound vs. lower bound
- Worst case – time bound valid for all inputs of length n.
- Average case – time bound valid on average – requires a distribution of inputs.
- Amortized – worst case time averaged over a sequence of operations.
- Others – best case, common case (80%-20%) etc.

What to Analyze

- Execution time
  - › Number of instructions executed
  - › Number of some particular operation executed
    - Example: for sorting algorithms, we might just count the number of comparisons made
- Memory
- Disc accesses, network transfer time, power, etc.