Directed Graph Algorithms – Topological Sort

CSE 373
Data Structures
Winter 2007

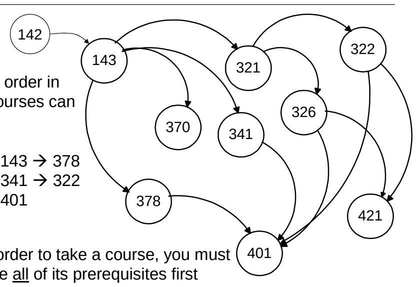
Readings

- Reading Chapter 9
  - › Section 9.2

Topological Sort

Problem: Find an order in which all these courses can be taken.

Example: 142 → 143 → 378 → 370 → 321 → 341 → 322 → 326 → 421 → 401

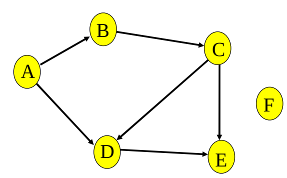


In order to take a course, you must take all of its prerequisites first

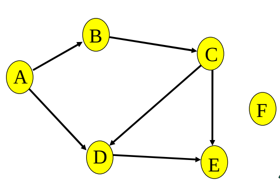
Topological Sort

Given a digraph $G = (V, E)$, find a linear ordering of its vertices such that:

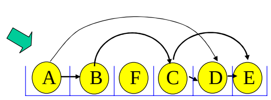
for any edge (v, w) in E, v precedes w in the ordering



Topo sort – valid solution

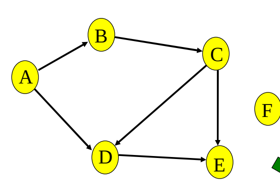


Any linear ordering in which all the arrows go to the right is a valid solution

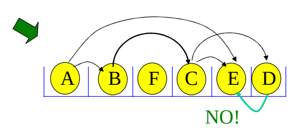


Note that F can go anywhere in this list because it is not connected.
Also the solution is not unique.

Topo sort – invalid solution



Any linear ordering in which an arrow goes to the left is not a valid solution

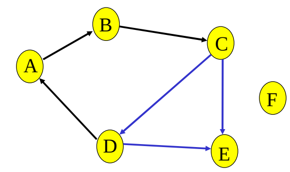


Paths and Cycles

- Given a digraph G = (V,E), a path is a sequence of vertices $v_1, v_2, \ldots, v_k$ such that:
  - › (v_i, v_{i+1}) in E for $1 \leq i < k$
  - › path length = number of edges in the path
  - › path cost = sum of costs of each edge
- A path is a cycle if :
  - › $k > 1$; $v_1 = v_k$
- G is acyclic if it has no cycles.

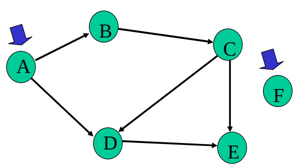
Only acyclic graphs can be topologically sorted

- A directed graph with a cycle cannot be topologically sorted.



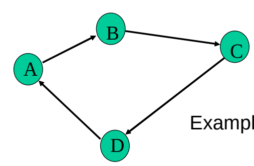
Topo sort algorithm - 1

Step 1: Identify vertices that have no incoming edges
- The "in-degree" of these vertices is zero



Topo sort algorithm - 1a

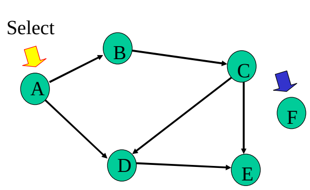
Step 1: Identify vertices that have no incoming edges
- If *no such vertices*, graph has only cycle(s) (cyclic graph)
- Topological sort not possible – Halt.



Example of a cyclic graph

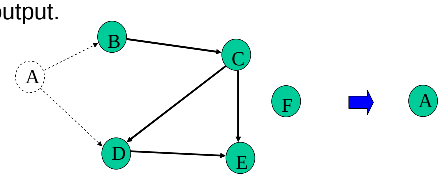
Topo sort algorithm - 1b

Step 1: Identify vertices that have no incoming edges
- Select one such vertex



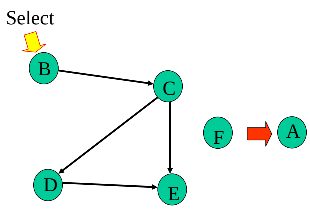
Topo sort algorithm - 2

Step 2: Delete this vertex of in-degree 0 and all its outgoing edges from the graph. Place it in the output.



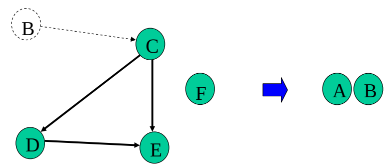
Continue until done

Repeat <u>Step 1</u> and <u>Step 2</u> until graph is empty



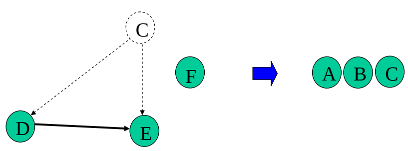
B

Select B. Copy to sorted list. Delete B and its edges.



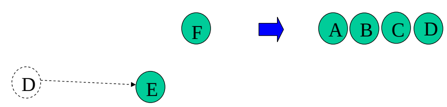
C

Select C. Copy to sorted list. Delete C and its edges.



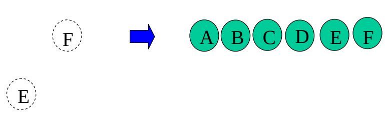
D

Select D. Copy to sorted list. Delete D and its edges.

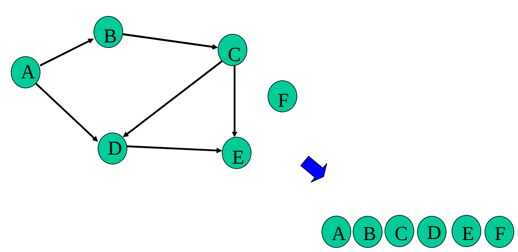


E, F

Select E. Copy to sorted list. Delete E and its edges.
Select F. Copy to sorted list. Delete F and its edges.

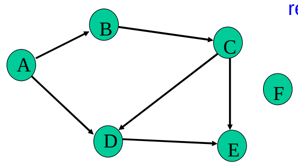


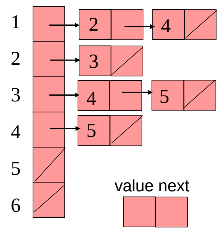
Done



Implementation

Assume adjacency list representation



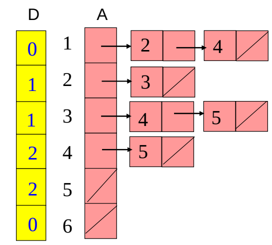


Translation array

1	2	3	4	5	6
A	B	C	D	E	F

Calculate In-degrees

In-Degree array; or add a field to array A



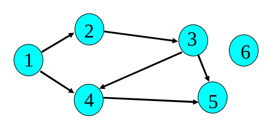
Calculate In-degrees

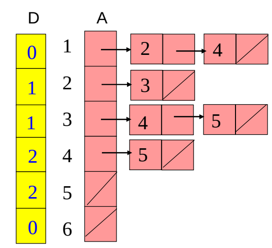
```
for i = 1 to n do D[i] := 0; endfor
for i = 1 to n do
  x := A[i];
  while x ≠ null do
    D[x.value] := D[x.value] + 1;
    x := x.next;
  endwhile
endfor
```


Maintaining Degree 0 Vertices

Key idea: Initialize and maintain a *queue (or stack)* of vertices with In-Degree 0

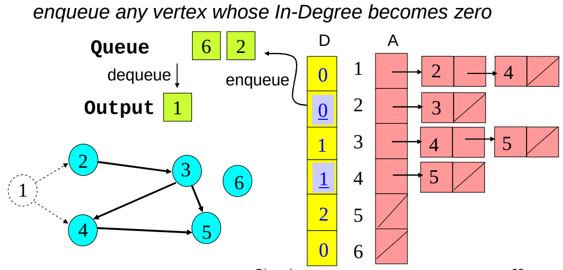
Queue 1 6





Topo Sort using a Queue (breadth-first)

After each vertex is output, when updating In-Degree array, *enqueue any vertex whose In-Degree becomes zero*



Topological Sort Algorithm

1. Store each vertex's In-Degree in an array D
2. Initialize queue with all "in-degree=0" vertices
3. While there are vertices remaining in the queue:
   (a) Dequeue and output a vertex
   (b) Reduce In-Degree of all vertices adjacent to it by 1
   (c) Enqueue any of these vertices whose In-Degree became zero
4. If all vertices are output then success, otherwise there is a cycle.

Some Detail

```
Main Loop
while notEmpty(Q) do
  x := Dequeue(Q)
  Output(x)
  y := A[x];
  while y ≠ null do
    D[y.value] := D[y.value] – 1;
    if D[y.value] = 0 then Enqueue(Q,y.value);
    y := y.next;
  endwhile
endwhile
```

Digraphs 25

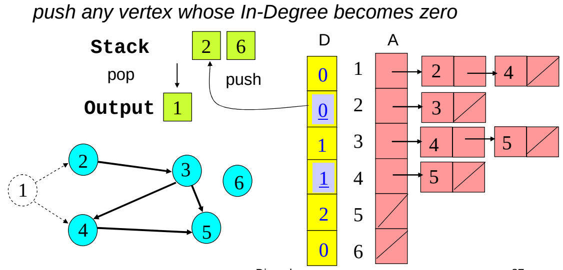
Topological Sort Analysis

- Initialize In-Degree array: O(|V| + |E|)
- Initialize Queue with In-Degree 0 vertices: O(|V|)
- Dequeue and output vertex:
  - › |V| vertices, each takes only O(1) to dequeue and output: O(|V|)
- Reduce In-Degree of all vertices adjacent to a vertex and Enqueue any In-Degree 0 vertices:
  - › O(|E|)
- For input graph G=(V,E) run time = O(|V| + |E|)
  - › Linear time!

Digraphs 26

Topo Sort using a Stack (depth-first)

After each vertex is output, when updating In-Degree array, *push any vertex whose In-Degree becomes zero*



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