Priority Queues & Binary Heaps

CSE 373
Data Structures
Winter 2007

Readings

- Chapter 6
  - › Section 6.1-6.4

FindMin Problem

- Quickly find the smallest (or highest priority) item in a set
- Applications:
  - › Operating system needs to schedule jobs according to priority instead of FIFO
  - › Event simulation (bank customers arriving and departing, ordered according to when the event happened)
  - › Find student with highest grade, employee with highest salary etc.
  - › Find "most important" customer waiting in line

Priority Queue ADT

- Priority Queue can efficiently do:
  - › FindMin()
    - Returns minimum value but does not delete it
  - › DeleteMin( )
    - Returns minimum value and deletes it
  - › Insert (k)
    - In GT Insert (k,x) where k is the key and x the value. In all algorithms the important part is the key, a "comparable" item. We'll skip the value.
  - › size() and isEmpty()

List implementation of a Priority Queue

- What if we use unsorted lists:
  - › FindMin and DeleteMin are O(n)
    - In fact you have to go through the whole list
  - › Insert(k) is O(1)
- What if we used sorted lists
  - › FindMin and DeleteMin are O(1)
    - Be careful if we want both Min and Max (circular array or doubly linked list)
  - › Insert(k) is O(n)

BST implementation of a Priority Queue

- Worst case (degenerate tree)
  - › FindMin, DeleteMin and Insert (k) are all O(n)
- Best case (completely balanced BST)
  - › FindMin, DeleteMin and Insert (k) are all O(logn)
- Balanced BSTs
  - › FindMin, DeleteMin and Insert (k) are all O(logn)

Better than a speeding BST

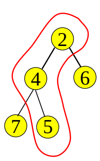
- Can we do better than Balanced Binary Search Trees?
- Very limited requirements: Insert, FindMin, DeleteMin. The goals are:
  - FindMin is O(1)
  - Insert is O(log N)
  - DeleteMin is O(log N)

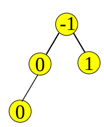
Binary Heaps

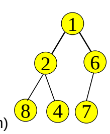
- A binary heap is a binary tree (***NOT*** a BST) that is:
  - Complete: the tree is completely filled except possibly the bottom level, which is filled from left to right
  - Satisfies the heap order property
    - every node is less than or equal to its children
    - or every node is greater than or equal to its children
- The root node is always the smallest node
  - or the largest, depending on the heap order

Heap order property

- A heap provides limited ordering information
- Each *path* is sorted, but the subtrees are not sorted relative to each other
  - A binary heap is NOT a binary search tree



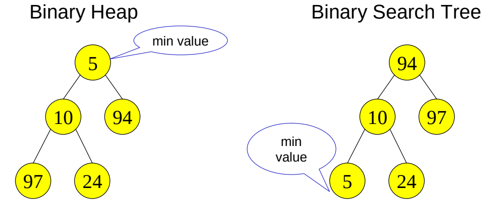




These are all valid binary heaps (minimum)

Binary Heap vs Binary Search Tree

Binary Heap | Binary Search Tree

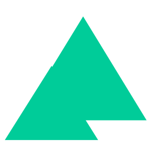


Parent is less than both left and right children

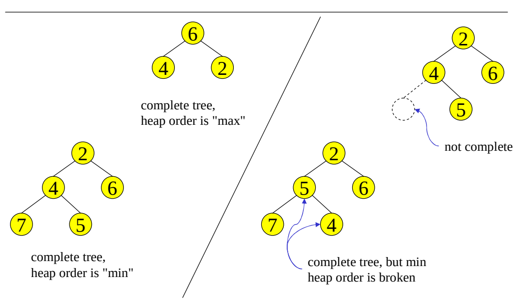
Parent is greater than left child, less than right child

Structure property

- A binary heap is a complete tree
  - All nodes are in use except for possibly the right end of the bottom row

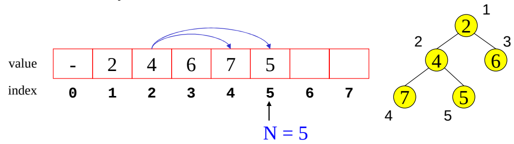


Examples



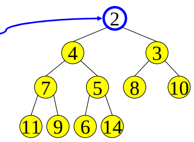
Array Implementation of Heaps

- Root node = A[1]
- Children of A[i] = A[2i], A[2i + 1]
- Keep track of current size N (number of nodes)



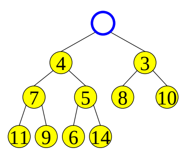
FindMin and DeleteMin

- FindMin: Easy!
  - › Return root value A[1]
  - › Run time = ?
- DeleteMin:
  - › Delete (and return) value at root node

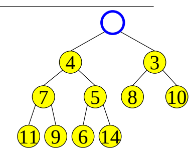


DeleteMin

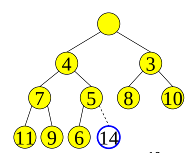
- Delete (and return) value at root node



Maintain the Structure Property

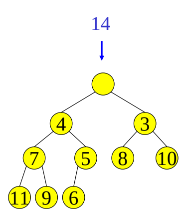


- We now have a “Hole” at the root
  - › Need to fill the hole with another value
- When we get done, the tree will have one less node and must still be complete

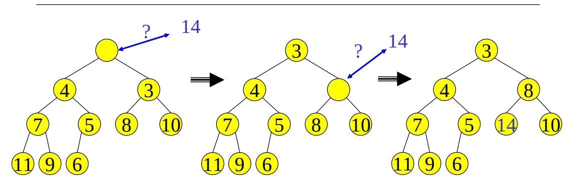


Maintain the Heap Property

- The last value has lost its node
  - › we need to find a new place for it
- We can do a simple insertion sort - like operation to find the correct place for it in the tree



DeleteMin: Percolate Down



- Keep comparing with children A[2i] and A[2i + 1]
- Copy smaller child up and go down one level
- Done if both children are ≥ item or reached a leaf node
- What is the run time?

Percolate Down

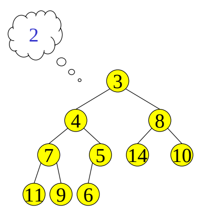
```
PercDown(i:integer, x :integer): {
// N is the number of entries in heap//
j : integer;
Case{
  2i > N : A[i] := x; //at bottom//
  2i = N : if A[2i] < x then
              A[i] := A[2i]; A[2i] := x;
           else A[i] := x;
  2i < N : if A[2i] < A[2i+1] then j := 2i;
           else j := 2i+1;
           if A[j] < x then
              A[i] := A[j]; PercDown(j,x);
           else A[i] := x;
}}
```


DeleteMin: Run Time Analysis

- Run time is O(depth of heap)
- A heap is a complete binary tree
- Depth of a complete binary tree of N nodes?
  - › depth = $\lfloor \log_2(N) \rfloor$
- Run time of DeleteMin is O(log N)

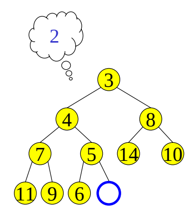
Insert

- Add a value to the tree
- Structure and heap order properties must still be correct when we are done



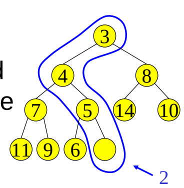
Maintain the Structure Property

- The only valid place for a new node in a complete tree is at the end of the array
- We need to decide on the correct value for the new node, and adjust the heap accordingly

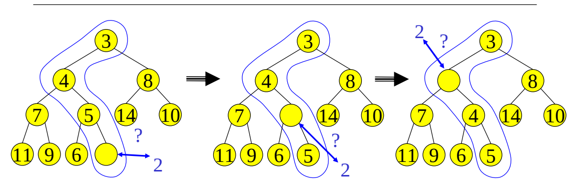


Maintain the Heap Property

- The new value goes where?
- We can do a simple insertion sort operation on the path from the new place to the root to find the correct place for it in the tree

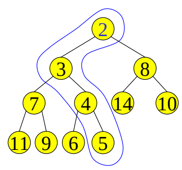


Insert: Percolate Up



- Start at last node and keep comparing with parent A[i/2]
- If parent larger, copy parent down and go up one level
- Done if parent ≤ item or reached top node A[1]
- Run time?

Insert: Done



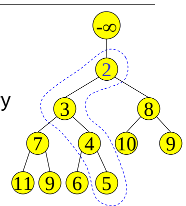
- Run time?

PercUp

```
PercUp(i : integer, x : integer): {
if i = 1 then A[1] := x
else if A[i/2] < x then
       A[i] := x;
     else
       A[i] := A[i/2];
       Percup(i/2,x);
}
```


Sentinel Values

- Every iteration of Insert needs to test:
  - if it has reached the top node A[1]
  - if parent ≤ item
- Can avoid first test if A[0] contains a very large negative value
  - sentinel -∞ < item, for all items
- Second test alone always stops at top



value	-∞	2	3	8	7	4	10	9	11	9	6	5		
index	0	1	2	3	4	5	6	7	8	9	10	11	12	13

Binary Heap Analysis

- Space needed for heap of N nodes: O(MaxN)
  - An array of size MaxN, plus a variable to store the size N, plus an array slot to hold the sentinel
- Time
  - FindMin: O(1)
  - DeleteMin and Insert: O(log N)
  - BuildHeap from N inputs : O(N) (forthcoming)